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MULTIPLICITY RESULTS FOR SOME QUASILINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper, we study multiplicity of weak solutions for the following class of quasilinear elliptic problems of the form

$$-\Delta_p u - \Delta u = g(u) - \lambda|u|^{q-2}u \quad \text{in } \Omega \text{ with } u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $1 < q < 2 < p \leq n$, λ is a real parameter, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and the nonlinearity $g(u)$ has subcritical growth. The proofs of our results rely on some linking theorems and critical groups estimates.

1. Introduction

In this paper we look for multiple solutions of a class of quasilinear elliptic equations of the form

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta_p u - \Delta u = g(u) - \lambda|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $1 < q < 2 < p \leq n$, λ is a real parameter, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and the nonlinearity $g(t)$ enjoys the following conditions:

- (g_0) $g \in \mathcal{C}^1(\mathbb{R})$, $g(0) = 0$;
- (g_1) there are constants $C_1 > 0$ and α with $p < \alpha < p^*$ such that

$$|g(s)| \leq C_1(1 + |s|^{\alpha-1}) \quad \text{for all } s \in \mathbb{R},$$

  where $p^* = np/(n-p)$ when $p < n$ and $p^* = \infty$ when $p = n$ is the critical Sobolev exponent, and
- (g_2) there are constants $C_2 > 0$ and $\beta > 2$ such that

$$|g'(s)| \leq C_2(1 + |s|^{\beta}) \quad \text{for all } s \in \mathbb{R}.$$

In what follows we will denote by $\lambda_1(p)$ the first eigenvalue of the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda(p)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and $\lambda_k(2)$, $k = 1, 2, \ldots$, the k-th eigenvalue of the laplacian with homogeneous Dirichlet boundary condition, which corresponds to problem (1.1) with $p = 2$.

On problem (P_λ), our main results concern the multiplicity of weak solutions when the nonlinearity $g(t)$ satisfies some additional hypotheses. Our first and second theorems treat the case when the $g(t)$ has "p-sublinear" growth at infinity, more precisely, we assume that

- (g_3) $\limsup_{|s|\to\infty} pG(s)/|s|^p < \lambda_1(p)$, where $G(t) = \int_0^t g(s)\,ds$.

They are formulated as follow.

Theorem 1.1. *Assume that g satisfies* (g_0)–(g_3) *and suppose that $g'(0) > \lambda_1(2)$. Then there exists $\lambda^* > 0$ such that problem* (P_λ) *has at least four nontrivial weak solutions for $\lambda \in (0, \lambda^*)$.*

Next we consider the case when the associated functional of problem (P_λ) has a local linking at origin. This geometric structure implies the existence of another nontrivial weak solution.

Theorem 1.2. *Assume that g satisfies* (g_0)–(g_3)*. Moreover, we assume that $g'(0) \in (\lambda_k(2), \lambda_{k+1}(2)]$, $k \geq 2$, and*

$$|G(s)| \leq \frac{1}{2}\lambda_{k+1}(2)|s|^2 + \frac{1}{p}\lambda_1(p)|s|^p \quad \textit{for all } s \in \mathbb{R}. \tag{1.2}$$

Then there exists $\lambda^* > 0$ such that problem (P_λ) *has at least five nontrivial weak solutions for $\lambda \in (0, \lambda^*)$.*

In our next results we consider the case when $G(u)$ has "p-superquadratic" growth at infinity, that is, we assume the following version of the Ambrosetti–Rabinowitz condition:

(g_4) there are constants $\theta > p$ and $s_0 > 0$ such that for $|s| \geq s_0$,

$$0 \leq \theta G(s) \leq s g(s).$$

In the "p-superquadratic" case our main results are formulated as follow.

Theorem 1.3. *Assume that g satisfies* (g_0)–(g_2), (g_4) *and* $g'(0) > \lambda_1(2)$. *Then there exists $\lambda^* > 0$ such that problem* (P_λ) *has at least two nontrivial solutions for $\lambda \in (0, \lambda^*)$.*

Finally, we consider the case $\lambda < 0$. This case is similar to the concave-convex problems studied in [2].

Theorem 1.4. *Assume that g satisfies* (g_0)–(g_2), (g_4) *and, in addition suppose that,* $g'(0) < \lambda_1(2)$. *Then there exists $\lambda_* < 0$ such that problem* (P_λ) *has at least two positive solutions for $\lambda \in (\lambda_*, 0)$.*

There has been recently a good amount of work on quasilinear elliptic problems. Some of these problems come from a variety of different areas of applied mathematics and physics. For example, they can be found in the study of non-Newtonian fluids, nonlinear elasticity and reaction-diffusions, for discussions about problems modelled by these boundary value problems see for example [15].

The study of multiple solutions for elliptic problems has received considerable attention in recent years. First, we would like to mention the progress involving the following class of semilinear elliptic problems $-\Delta u = \lambda |u|^{q-2}u + g(u)$ in Ω and $u = 0$ in $\partial\Omega$, where $1 < q < 2$. Ambrosetti *at al.* in [2], studied the case $g(u) = \lambda |u|^{r-2}u$, $2 < r < 2^*$. Among others results, they proved the existence of two positive solutions for small positive λ. Perera in [21] proved the existence of multiple solutions when $g(u)$ is sublinear at infinity and λ is small and negative (see also [14] for assymptotically linear and superlinear cases). Multiplicity results involving the p-Laplacian problems of the form $-\Delta_p u = \lambda |u|^{s-2}u + g(u)$ in Ω and $u = 0$ in $\partial\Omega$, where $1 < s < p$, has been studied in [3] and [17], when $g(u) = |u|^{r-2}u$, $p < r < p^*$.

Recently, critical groups computations via Morse theory for a functional like

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(u)\,dx, \quad u \in W_0^{1,p}(\Omega) \tag{1.3}$$

has been studied in [11], where the authors obtained a version of Shifting Theorem in the case $|g'(u)| \leq C(1 + |u|^r)$, with $0 \leq r < p^* - 2$. Cingolani and Degiovanni [10], has proved a existence result for the functional (1.3) when $g(u)$ has p-linear growth at infinity, that is, $\lim_{|t|\to\infty} g(u)/|u|^{p-2}u = \mu$. In fact, they proved a version of the classical existence theorem by Amann and Zehnder [1] for the semilinear problems. Benci *at al.* [6], [7] have studied a problem that involves operator like in left-hand side of (P_λ), which are motivated by problems from physics, in fact arising in the mathematical description of propagation phenomena of solitary waves. Finally, we refer to [16] and [25] where the authors proved multiple solutions for a problems involving more general class of operator than in left hand side of (P_λ).

The rest of this paper is organized as follows. Section 2 contains preliminary results, including a result of Sobolev versus Hölder local minimizers. Section 3 is devoted to proving ours main results.

2. Preliminary results

In this paper we make use of the following notation: C, C_0, C_1, $C_2,\ldots$ denote positive (possibly different) constants. For $1 \leq p < \infty$, $L^p(\Omega)$ denotes the usual Lebesgue space with norm $|u|_p \doteq [\int_\Omega |u|^p\,dx]^{1/p}$ and $W_0^{1,p}(\Omega)$ denotes the Sobolev space endowed with the usual norm $\|u\|_{1,p} = |\nabla u|_p$.

Here we search for weak solutions of problem (P_λ), that is, functions $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \int_\Omega \nabla u\nabla\varphi\,dx + \lambda\int_\Omega |u|^{q-2}u\varphi\,dx - \int_\Omega g(u)\varphi\,dx = 0,$$

for all $\varphi \in W_0^{1,p}(\Omega)$. It is well known that under conditions (g_0)–(g_1) the associated functional of (P_λ), $I_\lambda\colon W_0^{1,p}(\Omega) \to \mathbb{R}$, given by

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega G(u)\,dx,$$

is well defined, continuously differentiable on $W_0^{1,p}(\Omega)$, and its critical points correspond to weak solutions of (P_λ) and conversely (see [12], [23]).

Remark 2.1. Notice that condition (g_3) implies that the functional I_λ is coercive and therefore satisfies the Palais–Smale condition (see Lemma 3.1 in Section 3). On the other hand, under the hypothesis (g_4), the Palais–Smale condition for the functional I_λ can be proved by standard arguments.

Also, it is well known that there exists a smallest positive eigenvalue $\lambda_1(p)$, and an associated function $\varphi_1 > 0$ in Ω that solves (1.1), and that $\lambda_1(p)$ is a simple eigenvalue (see [5]). We recall that we have the following variational characterization

$$\lambda_1(p) = \inf\bigg\{\int_\Omega |\nabla u|^p\,dx : u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p\,dx = 1\bigg\}.$$

Next we show that the local minimum of the associated functional I_λ in $\mathcal{C}^1$-topology is still a local minimum in $W_0^{1,p}(\Omega)$. This result was proved by Brezis and Nirenberg for $p = 2$ (see [8]) and for the quasilinear case we referee to [18], [17], we will include here a proof for the sake of completeness.

LEMMA 2.2. *Assume that g satisfies* (g_1) *and $u_0 \in W_0^{1,p}(\Omega) \cap \mathcal{C}_0^1(\overline{\Omega})$ is a local minimizer of I_λ in the $\mathcal{C}^1$-topology, that is, there exits $r > 0$ such that*

$$I_\lambda(u_0) \leq I_\lambda(u_0 + v), \quad \textit{for all } v \in \mathcal{C}_0^1(\overline{\Omega}) \textit{ with } \|v\|_{C_0^1(\overline{\Omega})} \leq r. \tag{2.1}$$

Then u_0 is a local minimizer of I_λ in $W_0^{1,p}(\Omega)$, that is, there exists $\alpha > 0$ such that

$$I_\lambda(u_0) \leq I_\lambda(u_0 + v), \quad \textit{for all } v \in W_0^{1,p}(\Omega) \textit{ with } \|v\|_{1,p} \leq \alpha.$$

PROOF. If u_0 is a local minimizer of I_λ in the $\mathcal{C}^1$-topology, we see that it is a weak solution of (P_λ). By regularity results in Tolksdorf [24], $u_0 \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$. Now, suppose that the conclusion does not holds. Then for all $\varepsilon > 0$ there exists $v_\varepsilon \in B_\varepsilon$ such that

$$I_\lambda(u_0 + v_\varepsilon) < I_\lambda(u_0), \tag{2.2}$$

where $B_\varepsilon := \{v \in W_0^{1,p}(\Omega) : \|v\|_{1,p} \leq \varepsilon\}$. It is easy to see that I_λ is lower semicontinuous on the convex set B_ε. Notice that B_ε is weakly sequentially compact and weakly closed in $W_0^{1,p}(\Omega)$. By standard lower semicontinuous argument, we know that I_λ is bounded from below on B_ε and there exists $v_\varepsilon \in B_\varepsilon$ such that

$$I_\lambda(u_0 + v_\varepsilon) = \inf_{v \in B_\varepsilon} I_\lambda(u_0 + v).$$

We shall prove that $v_\varepsilon \to 0$ in $\mathcal{C}^1$ as $\varepsilon \to 0$, which is a contradiction with (2.1) and (2.2). The corresponding Euler equation for v_ε involves a Lagrange multiplier $\mu_\varepsilon \leq 0$, namely, v_ε satisfies

$$I'_\lambda(u_0 + v_\varepsilon)(h) = \mu_\varepsilon \int_\Omega |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \nabla h \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

that is,

$$-\Delta_p(u_0 + v_\varepsilon) - \Delta(u_0 + v_\varepsilon) - g(u_0 + v_\varepsilon) - |u_0 + v_\varepsilon|^{q-2}(u_0 + v_\varepsilon) = -\mu_\varepsilon \Delta_p v_\varepsilon.$$

Thus,

$$\begin{aligned} &- \Delta_p u_0 - \Delta u_0 - [\Delta_p(u_0 + v_\varepsilon) - \Delta_p u_0 + \Delta v_\varepsilon] + \mu_\varepsilon \Delta_p v_\varepsilon \\ &\qquad\qquad = g(u_0 + v_\varepsilon) + |u_0 + v_\varepsilon|^{q-2}(u_0 + v_\varepsilon). \end{aligned}$$

This implies that

$$\begin{aligned} &- [\Delta_p(u_0 + v_\varepsilon) - \Delta_p u_0] + \mu_\varepsilon \Delta_p v_\varepsilon \\ &\qquad\qquad = g(u_0 + v_\varepsilon) - g(u_0) + |u_0 + v_\varepsilon|^{q-2}(u_0 + v_\varepsilon) - |u_0|^{q-2} u_0. \end{aligned} \tag{2.3}$$

Notice that we can write (2.3) as

$$\begin{aligned}-\mathrm{div}(A(v_\varepsilon)) :=& -\mathrm{div}(|\nabla(u_0+v_\varepsilon)|^{p-2}\nabla(u_0+v_\varepsilon)\\ &-|\nabla u_0|^{p-2}\nabla u_0+\nabla v_\varepsilon-\mu_\varepsilon|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon)\\ =&\, g(u_0+v_\varepsilon)-g(u_0)+|u_0+v_\varepsilon|^{q-2}(u_0+v_\varepsilon)-|u_0|^{q-2}u_0\\ =&\, g'(\xi)+|u_0+v_\varepsilon|^{q-2}(u_0+v_\varepsilon)-|u_0|^{q-2}u_0\end{aligned}$$

where $\xi \in (\min\{u_0, u_0+v_\varepsilon\}, \max\{u_0, u_0+v_\varepsilon\})$. We know (see Tolksdorf [24]) that for $p>2$ there exits $\rho>0$ independent of u_0 and v_ε such that

$$[|\nabla(u_0+v_\varepsilon)|^{p-2}\nabla(u_0+v_\varepsilon)-|\nabla u_0|^{p-2}\nabla u_0] \geq \rho|\nabla v_\varepsilon|^p.$$

Thus,

$$A(v_\varepsilon).\nabla v_\varepsilon \geq (\rho-\mu_\varepsilon)|\nabla v_\varepsilon|^p+|\nabla v_\varepsilon|^2 \geq C|\nabla v_\varepsilon|^p,$$

since $\mu_\varepsilon \leq 0$. Using the growth condition (g_2) we have

$$|g'(\xi)| \leq C_1+C_2(|u_0|^{\beta-1}+|v_\varepsilon|^{\beta-1}+|u_0|^{q-1}+|v_\varepsilon|^{q-1}).$$

Since $\beta-1>0$ and $q-1>0$, by regularity results obtained in [19], we have that for some $0<\alpha<1$, there exists $C>0$ independent of ε such that

$$\|v_\varepsilon\|_{C^\alpha(\Omega)} \leq \|v_\varepsilon\|_{1,p} \leq C.$$

By the regularity results in [20], we also have that

$$\|v_\varepsilon\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq C_1.$$

This implies that $v_\varepsilon \to 0$ in $\mathcal{C}^1$ as $\varepsilon \to 0$. Since $\|v_\varepsilon\|_{1,p} \to 0$, we have $v_0 \equiv 0$. This completes the proof. □

Now, for $u \in W_0^{1,p}(\Omega)$ we define

$$I_\lambda^\pm(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{q}\int_\Omega |u^\pm|^q\,dx - \int_\Omega G(u^\pm)\,dx,$$

where $u^+ = \max\{u,0\}$ and $u^- = \min\{u,0\}$. Since $g(0)=0$, $I_\lambda^\pm \in \mathcal{C}^1$ and the critical points $u_\pm$ of $I_\lambda^\pm$ satisfy $\pm u_\pm \geq 0$, we conclude that $u_\pm$ are also critical points of I_λ. In fact, $(I_\lambda^\pm)'(u_\pm)[(u_\pm)^\mp] = \int_\Omega |\nabla(u_\pm)^\mp|^p dx + \int_\Omega |\nabla(u_\pm)^\mp|^2 dx = 0$.

Lemma 2.3. *The origin $u \equiv 0$ is a local minimizer for I_λ and $I_\lambda^\pm$, for any $\lambda>0$.*

Proof. By Lemma 2.2, is sufficient to show that $u=0$ is a local minimizer of I_λ in the $\mathcal{C}^1$ topology. First we observe that from (g_1) and the regularity of g, we have

$$G(s) \leq C|s|^2 + C|s|^\alpha \quad \text{for } s \in \mathbb{R},$$

for some positive constant C. Then, for $u \in C_0^1(\overline{\Omega})$ we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega G(u)\,dx \\ &\geq \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega G(u)\,dx \\ &\geq \frac{\lambda}{q}\int_\Omega |u|^q\,dx - C\int_\Omega |u|^2\,dx - C\int_\Omega |u|^\alpha\,dx \\ &\geq \left(\frac{\lambda}{q} - C|u|_{C^0}^{2-q} - C|u|_{C^0}^{\alpha-q}\right)\int_\Omega |u|^q\,dx \geq 0, \end{aligned}$$

if $C|u|_{C_0^1}^{2-q} + C|u|_{C_0^1}^{\alpha-q} \leq \frac{\lambda}{q}$. The same argument works for $I_\lambda^\pm$. □

We do not include here the proof of next lemma because it follows using the same ideas of [21, Lemma 2.1].

LEMMA 2.4. *If $u_\pm$ is a local minimizer for $I_\lambda^\pm$, then it is a local minimizer of I_λ, for any $\lambda > 0$.*

3. Proofs of main theorems

Proof of Theorem 1.1. We already know that the origin is a local minimum of the functional $I_\lambda^\pm$. In the next two lemmas we prove that $I_\lambda^\pm$ is coercive and $\min_{u\in W_0^1(\Omega)} I_\lambda^\pm(u) < 0$. Thus, $I_\lambda^\pm$ has a global minimum $u_0^\pm$ with negative energy. Finally, by applying the Mountain Pass Theorem, we get critical points $u_1^\pm$ with positive energy.

LEMMA 3.1. *The functional $I_\lambda^\pm$ is coercive, lower bounded and satisfies the* (PS) *condition.*

PROOF. By (g_3) there exists $\varepsilon > 0$, small enough, and a constant C such that

$$pG(s) \leq (\lambda_1(p) - \varepsilon)|s|^p + C, \quad \text{for all } s \in \mathbb{R}.$$

Then

$$\begin{aligned} I_\lambda^\pm(u) &\geq \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega G(u)\,dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{(\lambda_1(p)-\varepsilon)}{p}\int_\Omega |u|^p\,dx - C \\ &\geq \frac{1}{p}\left(1 - \frac{(\lambda_1(p)-\varepsilon)}{\lambda_1(p)}\right)||u||_{1,p}^p - C. \end{aligned}$$

Thus $I_\lambda^\pm(u) \to \infty$ as $||u||_{1,p} \to \infty$. □

Lemma 3.2. *Let φ_1 be the first eigenfunction associated to $\lambda_1(2)$. Then there exists $t_0 > 0$ such that $I_\lambda^{\pm}(\pm t_0\varphi_1) < 0$, for all λ in a limited set.*

Proof. Since $g(0) = 0$, $g'(0) > \lambda_1(2)$ and (g_1) holds, for each $\varepsilon > 0$, there exists $C > 0$ such that

$$G(s) \geq \frac{(\lambda_1(2)+\varepsilon)}{2}s^2 - C|s|^\alpha, \quad \text{for all } s \in \mathbb{R}.$$

Then, for $t > 0$ we obtain

$$\begin{aligned} I_\lambda^{\pm}(\pm t\varphi_1) &\leq \frac{t^p}{p}\int_\Omega |\nabla\varphi_1|^p\,dx + \frac{t^2}{2}\int_\Omega |\nabla\varphi_1|^2\,dx \\ &\quad + \frac{t^q\lambda}{q}\int_\Omega |\varphi_1|^q\,dx - \frac{(\lambda_1(2)+\varepsilon)t^2}{2}\int_\Omega |\varphi_1|^2\,dx + Ct^\alpha\int_\Omega |\varphi_1|^\alpha\,dx \\ &= \frac{t^2}{2}\bigg(1 - \frac{(\lambda_1(2)+\varepsilon)}{\lambda_1(2)}\bigg)||\varphi_1||_{1,2}^2 + \frac{t^p}{p}||\varphi_1||_{1,p}^p + \frac{t^q\lambda}{q}|\varphi_1|_q^q + Ct^\alpha|\varphi_1|_\alpha^\alpha. \end{aligned}$$

Since $\alpha > p > 2$ we can conclude the lemma. □

End the proof of Theorem 1.1. By the Mountain Pass Theorem, $I_\lambda^{\pm}$ has a nontrivial critical point $u_1^{\pm}$ with $I_\lambda^{\pm}(u_1^{\pm}) > 0$. Since $I_\lambda^{\pm}$ is bounded bellow, it also has a global minimizer $u_0^{\pm}$ with $I_\lambda^{\pm}(u_0^{\pm}) < 0$. □

Proof of Theorem 1.2. In what follows we assume that the reader is somewhat familiar with Morse theory (see [9] for necessary prerequisites). In particular, we recall that the critical groups of a real $\mathcal{C}^1$ functional Φ at an isolated critical point u_0 with $\Phi(u_0) = c$, are defined by

$$C_q(\Phi, u_0) = H_q(\Phi_c \cap U, (\Phi_c \cap U) \setminus \{u_0\}) \quad \text{for } q \in \mathbb{N}.$$

Here $\Phi_c := \{u : \Phi(u) \leq c\}$, U is a neighbourhood of u_0 such that u_0 is the only critical point of Φ in $\Phi_c \cap U$, and $H_*(\,\cdot\,,\,\cdot\,)$ denote the singular relative homology groups with coefficients in $\mathbb{Z}$.

By the proof of Theorem 1.1, I_λ has four critical points $u_1^{\pm}$, with $I_\lambda(u_1^{\pm}) > 0$, and $u_0^{\pm}$, with $I_\lambda(u_0^{\pm}) < 0$. By Lemma 2.4, we have that $u_0^{\pm}$ are local minimum of I_λ since they are global minimum of $I_\lambda^{\pm}$. Then

$$C_j(I_\lambda, u_0^{\pm}) = \delta_{j0}\mathbb{Z}.$$

Now, in order to prove the existence of another nontrivial solution we will apply the following abstract theorem due to Perera (see [22, Theorem 3.1]):

Theorem 3.3. *Let $X = X_1 \oplus X_2$ be a Banach space with $0 < k = \dim X_1 < \infty$. Suppose that $\Phi \in \mathcal{C}^1(X, \mathbb{R})$, has a finite number of critical points and satisfies the Palais–Smale compactness condition* (PS). *Moreover, assume the following conditions:*

(a) *there exists $\rho > 0$ such that $\sup_{S_\rho^1} \Phi < 0$, where $S_\rho^1 = \{u \in X_1 : ||u|| = \rho\}$,*

(b) $\Phi \geq 0$ *in* X_2,

(c) *there is* $e \in X_1 \setminus \{0\}$ *such that* Φ *is bounded from below on* $\{se + x_2 : s \geq 0,\ x_2 \in X_2\}$.

Then Φ *has a critical point* u_0 *with* $\Phi(u_0) < 0$ *and* $C_{k-1}(\Phi, u_0) \neq 0$.

First we consider $H_k := \bigoplus_{j=1}^{k} \ker(-\Delta - \lambda_j(2)I)$ and $W_k = W_0^{1,p}(\Omega) \cap H_k^{\perp}$, where $H_k^{\perp}$ denotes the orthogonal subspace of H_k in $H_0^1(\Omega)$. Thus we have

$$\begin{aligned} W_0^{1,p}(\Omega) &= H_k \oplus W_k, \\ ||u||_2^2 &\leq \lambda_k(2)|u|_2^2, \qquad \text{for all } u \in H_k, \\ ||u||_2^2 &\geq \lambda_{k+1}(2)|u|_2^2, \quad \text{for all } u \in W_k. \end{aligned}$$

The next lemma is a verification of the hypotheses of Theorem 3.3.

LEMMA 3.4. *The functional* I_λ *enjoys the following properties:*

(a) *there exist* $\lambda^* > 0$ *and* $\rho > 0$ *such that* $\sup_{S_\rho^k} I_\lambda < 0$, *for* $0 < \lambda < \lambda^*$, *where* $S_\rho^k = \{u \in H_k\ ;\ ||u||_{1,p} = \rho\}$;

(b) $I_\lambda \geq 0$ *in* W_k;

(c) I_λ *is bounded from below.*

PROOF. (a) Since $g(0) = 0$, by (g_1) given $\varepsilon > 0$ there exists $C > 0$ such that

$$G(s) \geq \frac{(g'(0) - \varepsilon)}{2} s^2 - C|s|^\alpha, \quad \text{for all } s \in \mathbb{R}.$$

Since the norm are equivalent in H_k, for each $u \in H_k$ we have

$$\begin{aligned} I_\lambda(u) \leq & \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx \\ & + \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{(g'(0) - \varepsilon)}{2}\int_\Omega u^2\,dx + C\int_\Omega |u|^\alpha\,dx \\ \leq & \frac{1}{2}\bigg(1 - \frac{(g'(0) - \varepsilon)}{\lambda_k(2)}\bigg)||u||_2^2 + C(\lambda||u||_p^q + ||u||_p^p + ||u||_p^\alpha) \\ \leq & \frac{c}{2}\bigg(1 - \frac{(g'(0) - \varepsilon)}{\lambda_k(2)}\bigg)||u||_p^2 + C(\lambda||u||_p^q + ||u||_p^p + ||u||_p^\alpha), \end{aligned}$$

where, in the last inequality, we use that the norms are equivalent in a finite dimensional space. Since, we can choose ε such that $g'(0) - \varepsilon > \lambda_k(2)$, and using that $\alpha > p > 2$ then we can take ρ and λ^*, small enough, such that (a) holds.

(b) If $u \in W_k$, by (1.2), we have

$$\begin{aligned} I_\lambda(u) \geq \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{q}\int_\Omega |u|^q\,dx & \\ - \frac{\lambda_{k+1}(2)}{2}\int_\Omega u^2\,dx - \frac{\lambda_1(p)}{p}\int_\Omega |u|^p\,dx \geq 0. & \end{aligned}$$

We notice that (c) already was proved. □

End the Proof of Theorem 1.2. Thus I_λ has a critical point u_2 with $I_\lambda(u_2) < 0$ and $C_{k-1}(I_\lambda, u_2) \neq 0$. Then we can conclude, using $k \geq 2$, that u_2 is different of $u_0^\pm$ and $u_1^\pm$. □

Proof of Theorem 1.3. This theorem is a direct application of the Mountain Pass Theorem (see [4], [23]).

Lemma 3.5. *Let φ_1 be the first eigenfunction associated to $\lambda_1(2)$. Under the above conditions, there exist $t_0 > 0$ and $\lambda^* > 0$ such that $I_\lambda^\pm(\pm t_0\varphi_1) < 0$, for all $\lambda \in (0, \lambda^*)$.*

Proof. By $g(0) = 0$, $g'(0) > \lambda_1(2)$ and (g_1) we have that, given $\varepsilon > 0$ there exists $C > 0$ such that

$$G(s) \geq \frac{(\lambda_1(2) + \varepsilon)}{2} s^2 - C|s|^\alpha, \quad \text{for all } s \in \mathbb{R}.$$

Then, for $t > 0$,

$$\begin{aligned}
I_\lambda^\pm(\pm t\varphi_1) &= \frac{t^p}{p} \int_\Omega |\nabla \varphi_1|^p \, dx + \frac{t^2}{2} \int_\Omega |\nabla \varphi_1|^2 \, dx \\
&\quad + \frac{t^q \lambda}{q} \int_\Omega |\varphi_1|^q \, dx - \frac{(\lambda_1(2) + \varepsilon) t^2}{2} \int_\Omega |\varphi_1|^2 \, dx + Ct^\alpha \int_\Omega |\varphi_1|^\alpha \, dx \\
&= \frac{t^2}{2} \bigg(1 - \frac{(\lambda_1(2) + \varepsilon)}{\lambda_1(2)}\bigg) ||\varphi_1||_{1,2}^2 + \frac{t^p}{p} ||\varphi_1||_{1,p}^p + \frac{t^q \lambda}{q} |\varphi_1|_q^q + Ct^\alpha |\varphi_1|_\alpha^\alpha \\
&\leq \frac{t^2}{q} \bigg\{ \bigg(1 - \frac{(\lambda_1(2) + \varepsilon)}{\lambda_1(2)}\bigg) ||\varphi_1||_{1,2}^2 \\
&\quad + t^{p-2} ||\varphi_1||_{1,p}^p + t^{q-2} \lambda |\varphi_1|_q^q + C_1 t^{\alpha-2} |\varphi_1|_\alpha^\alpha \bigg\}.
\end{aligned}$$

Since $\alpha > p > 2 > q > 1$, we can choose $\lambda^* > 0$ such that $I_\lambda^\pm(\pm t_0\varphi_1) < 0$, for all $\lambda \in (0, \lambda^*)$. □

Now, the proof of Theorem 1.3 follows from standard argument using the Mountain Pass Theorem. □

Proof of Theorem 1.4. We will look for critical points of the $\mathcal{C}^1$ functional

$$I_\lambda^+(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{\lambda}{q} \int_\Omega |u^+|^q \, dx - \int_\Omega G(u^+) \, dx$$

for $u \in W_0^{1,p}(\Omega)$.

The proof of this theorem is similar in spirit to that of Theorem 2.1 in [13]. We will shown that the functional I_λ^+ has the mountain-pass structure which together with the compactness condition will give us a positive solution in a positive level. After that, we will proof that I_λ^+ has a positive local minimum in a ball around the origin with negative energy.

We made it in three steps.

Step 1. There exist $r > 0$ and $a > 0$ such that $I_\lambda^+(u) \geq a$ if $||u||_{1,p} = r$.

Indeed, using condition (g_1) and $g'(0) < \lambda_1(2)$, for ε small and $s \in \mathbb{R}$ we have

$$G(s) \leq \frac{1}{2}(\lambda_1(2) - \varepsilon)|s|^2 + C|s|^\alpha, \quad \text{for all } s \in \mathbb{R}.$$

Thus, (remember that $\lambda < 0$)

$$\begin{aligned}
I_\lambda^+(u) \geq & \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{q}\int_\Omega |u|^q\,dx \\
& - \frac{(\lambda_1(2) - \varepsilon)}{2}\int_\Omega |u|^2\,dx - C\int_\Omega |u|^\alpha\,dx \\
\geq & \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\left(1 - \frac{(\lambda_1(2) - \varepsilon)}{\lambda_1(2)}\right)\int_\Omega |\nabla u|^2\,dx \\
& + \frac{\lambda}{q}\int_\Omega |u|^q\,dx - C\int_\Omega |u|^\alpha\,dx \\
\geq & A||u||_{1,p}^p + \lambda B||u||_{1,p}^q - C||u||_{1,p}^\alpha.
\end{aligned}$$

By [13, Lemma 3.2] there exists $\lambda_* < 0$ such that if $\lambda \in (\lambda_*, 0)$, I_λ^+ satisfies the property above.

Step 2. There exists $t_M > r$ such that $I_\lambda^+(t_M\varphi_1) \leq 0$.

Indeed, let φ_1 the first eigenfunction associated to $\lambda_1(p)$. For $t > 0$ we have

$$\begin{aligned}
I_\lambda^+(t\varphi_1) = & \frac{t^p}{p}\int_\Omega |\nabla\varphi_1|^p\,dx + \frac{t^2}{2}\int_\Omega |\nabla\varphi_1|^2\,dx + \frac{t^q\lambda}{q}\int_\Omega |\varphi_1|^q\,dx - \int_\Omega G(t\varphi_1)\,dx \\
= & t^p\left\{\frac{1}{p}\int_\Omega |\nabla\varphi_1|^p\,dx + \frac{t^{2-p}}{2}\int_\Omega |\nabla\varphi_1|^2\,dx\right. \\
& \left. + \frac{t^{q-p}\lambda}{q}\int_\Omega |\varphi_1|^q\,dx - \int_\Omega \frac{G(t\varphi_1)}{t^p}\,dx\right\}.
\end{aligned}$$

It follows from condition (g_4) that there exists a positive constant C such that $G(s) \geq Cs^\theta$, for all $s \geq s_0$. Thus

$$\begin{aligned}
I_\lambda^+(t\varphi_1) \leq t^p & \left\{\frac{1}{p}\int_\Omega |\nabla\varphi_1|^p\,dx + \frac{t^{2-p}}{2}\int_\Omega |\nabla\varphi_1|^2\,dx\right. \\
& \left. + \frac{t^{q-p}\lambda}{q}\int_\Omega |\varphi_1|^q\,dx - Ct^{\theta-p}\int_\Omega |\varphi_1|^\theta\,dx\right\}.
\end{aligned}$$

Now, observing that

$$\lim_{t\to+\infty}\frac{t^{2-p}}{2}\int_\Omega |\nabla\varphi_1|^2\,dx = \lim_{t\to+\infty}\frac{t^{q-p}\lambda}{q}\int_\Omega \varphi_1^q\,dx = 0,$$

for all $\lambda \in (\lambda_*, 0)$ and $\theta > p$, we obtain $I_\lambda^+(t_M\varphi_1) \leq 0$ for some $t_M > 0$. Thus, we can apply the Mountain Pass Theorem to obtain a critical point u_1 such that $I_\lambda^+(u_1) > 0$.

Step 3. There exists $0 < t_m < r$ *such that* $I^+_\lambda(t_m\varphi_1) < 0$.

Indeed, let φ_1 the first eigenfunction associated to $\lambda_1(2)$. We have that

$$I^+_\lambda(t\varphi_1) = t^2\bigg\{\frac{t^{p-2}}{p}\int_\Omega |\nabla\varphi_1|^p\,dx + \frac{1}{2}\int_\Omega |\nabla\varphi_1|^2\,dx + \frac{t^{q-2}\lambda}{q}\int_\Omega |\varphi_1|^q\,dx - \int_\Omega \frac{G(t\varphi_1)}{t^2}\,dx\bigg\}.$$

Since $q < 2$, $\lambda < 0$ and $G(t)/t^2$ is bounded next to $t = 0$, our claim follows.

Now, the minimum of the functional I^+_λ in a closed ball of $W^{1,p}_0(\Omega)$ with center in zero and radius r such that

$$I^+_\lambda(u) \geq 0 \quad \text{for all } u \text{ with } ||u||_{1,p} = r,$$

is achieved in the correspondent open ball and thus yields a nontrivial critical point u_2 of I^+_λ, with $I^+_\lambda(u_2) < 0$ and $||u_2||_{1,p} < r$. □

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