Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 34, 2009, 1–20

ON A VARIANT OF THE MAXIMUM PRINCIPLE INVOLVING RADIAL p-LAPLACIAN WITH APPLICATIONS TO NONLINEAR EIGENVALUE PROBLEMS AND NONEXISTENCE RESULTS

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Abstract. We obtain the variant of maximum principle for radial solutions of p-harmonic equation $-a\Delta_p(w) = \phi(w)$. As a consequence of this result we prove monotonicity of constant sign solutions, analyze the support of the solutions and study their oscillations. The results are applied to various type nonlinear eigenvalue problems and nonexistence theorems.

1. Introduction

Problems involving p-Laplace operator are subject of intensive studies as they very well illustrate many of phenomena that occur in nonlinear analysis. Among their applications are singular and nonsingular boundary value problems which appear in various branches of mathematical physics. They arise as a model example in the fluid dynamics ([18], [27], [28], Chapter 2 in [31], [55], [62]);

2000 *Mathematics Subject Classification.* Primary: 35B50; Secondary: 33C45, 35J15, 34C11.

Key words and phrases. Maximum principles, radial solutions, p-Laplace equation, singular PDE's, elliptic equations.

The part of this research has been done when the first author was a graduate student at the Department of Mathematics at Syracuse University.

The work of second author is supported by the KBN grant no. 1/PO3A/008/29 and partially supported by EC FP6 Marie Curie programmes SPADE2 and CODY.

glaciology [6]; stellar dynamics [44]; in the theory of electrostatic fields [36]; in the more general context in quantum physics ([11], [12]); in the nonlinear elasticity theory as a basic model ([8], [30]); and many others (see e.g. [45]). The PDEs involving p-Laplacian are considered in differential geometry in the study of critical points for p-harmonic maps between Riemannian manifolds ([20], [46]) and the eigenvalue problems for p-Laplacian on Riemannian manifolds serve for estimations of the diameter of the manifolds [41]. Eigenvalue problems involving p-Laplacian are applied in functional analysis to derive sharp Poincaré and Writinger type inequalities ([33], [34]), Sobolev embeddings and isoperimetric inequalities ([10]), [23], Chapter II in [45]). Geometric properties of p-harmonic functions play significant role in the theory of Carnot–Caratheodory groups like Heisenberg group (see e.g. [48]) and in the analysis on metric spaces (see [4], [15] and references therein).

One of the problems we encounter when investigating p-harmonic equation is that very few explicit solutions are known – affine, quasiradial, radial. Among them radial solutions form the widest nontrivial class in which many properties of p-harmonic world can be detected. Another motivation to study radial solutions comes from the seminal paper by Gidas, Ni and Nirenberg [39] who extended Serrin's moving plane method from [57] and proved that in some cases only radial solutions are admitted (see also e.g. [16], [22], [29], [47], [58] for some further generalizations).

Moreover, it can happen that among the solutions of the PDE are the radial ones even if the radial solutions are not the only ones (see e.g. [17], [21], [40]).

We shall consider radial solutions of the equation

$$-a(|x|)\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)) \quad \text{a.e. in } B = B(0,R) \subset \mathbb{R}^n. \tag{1.1}$$

The precise meaning of the solution w will be clarified in Section 2. We assume that $p > 1$, $n > 1$, $R \in (0,\infty]$ (for $R = \infty$ the above equation is defined on $\mathbb{R}^n$), $a(\,\cdot\,)$ is nonnegative and belongs to a certain class of functions which will be described later, while ϕ is an arbitrary odd continuous function such that $\tau\phi(\tau)$ is of constant sign for $\mathcal{L}^1$ almost all τ's. In general our equation is given in a nondivergent form.

Such PDEs appear in astrophysics [44] in relation to Matukuma equations (developed in 1930' to describe the dynamics of a globular cluster of stars); as well as in physical phenomena related to equilibria of anisotropic continuous media [25].

Our main result is the variant of the maximum principle which can be explained as follows. For a being in a certain function class $\mathcal{A}$, radial solutions of (1.1) achieve their extrema either at 0 in the case $\tau\phi(\tau) > 0$ almost everywhere or on ∂B (in case $R = \infty$ at ∞) when $\tau\phi(\tau) < 0$ almost everywhere. Moreover, in the second case the solution is either nonpositive or nonnegative and $w(x)$ is monotonic in $|x|$. The class $\mathcal{A}$ can be easily recognized. For example every nonnegative concave (not necessarily strictly concave) function which is sufficiently regular is its member. In particular every positive constant belongs to $\mathcal{A}$.

The cases $\tau\phi(\tau) > 0$ a.e. and $\tau\phi(\tau) < 0$ a.e. are far different. It is well known that if $\tau\phi(\tau) > 0$ when $\tau \neq 0$ then solutions of (1.1) may oscillate. As the model example one considers the radial solutions to the nonlinear eigenvalue problem:

$$\begin{cases} -\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \lambda |w(x)|^{p-2}w(x) & \text{a.e. in } B, \\ w \equiv 0 & \text{on } \partial B, \end{cases}$$

where λ is positive. Here $a \equiv 1 \in \mathcal{A}$, $\phi(\tau) = \lambda|\tau|^{p-2}\tau$, see e.g. [61] and our Remark 3.3. As w solves (1.1) for an arbitrary ball $B(0,r) \subseteq B(0,R)$ with $r < R$, we cannot expect $|w|$ to achieve maximum on $\partial B(0,r)$ for any $r < R$. Therefore, in this case the maximum principle does not hold in the classical sense. However, under assumption that $\tau\phi(\tau) > 0$ almost everywhere we show that the sequence of local maxima for $|w(x)|$ is nonincreasing in $|x|$ (see Proposition 2.2). Therefore, we call it maximum principle even though the solutions of (1.1) may oscillate. On the other hand oscillations for the solutions are not permitted in the case $\tau\phi(\tau) < 0$ almost everywhere (Corollary 2.3). Moreover, in this case the solution must be either nonpositive or nonnegative and problem (1.1) with Dirichlet boundary data possesses only trivial solutions (Proposition 4.1). Let us mention that in both cases: $\tau\phi(\tau) > 0$ almost everywhere and $\tau\phi(\tau) < 0$ almost everywhere the nonpositive or nonnegative solutions are monotone in $|x|$.

This phenomenon has been observed in the celebrated paper by Gidas, Ni and Nirenberg [39] and its generalizations which specialize to p-Laplacian, but all of them deal with the nonnegative solutions and $a \equiv 1$ in (1.1) with Dirichlet boundary data and various assumptions on ϕ. The version with $a \equiv 1$ and the nonnegative solution can be found in [37] with different proof. To our best knowledge this kind of result in the remaining cases: a not necessarily constant and the solution w being not necessarily of constant sign is in general unknown.

As a direct consequence of our maximum principle we obtain new uniqueness and nonexistence results. Some of them follow from our main result directly, but we also derive the new uniqueness result for nonnegative radial solutions of nonlinear eigenvalue problems like $-a(|x|)\Delta_p w(x) = \lambda w(x)^{q-1}$ under Dirichlet boundary data. This one uses a radial variant of Derrick–Pohozhaev identity and our maximum principle (Proposition 4.7).

Our results are obtained by elementary techniques. Similar methods were used earlier in [42, Proposition 3.2], where the authors dealt with an abstract quasilinear equation: $A(\tau, u(\tau), u'(\tau))u''(\tau) + B(\tau, u(\tau), u'(\tau)) = 0$ in dimension one. Our work here is more specialistic and deals with a different assumption on the dimension. This assumption plays an essential role. For example now we can also allow PDEs which are not singular, where by singular equation of type (1.1) we mean such that $a(|x|)$ achieves zero in $\overline{B}$. In dimension $n = 1$ the function $a(\,\cdot\,)$ is supposed to vanish at least at one of the endpoints of interval $[0, R]$.

The paper is organized as follows. In Section 2 we prove maximum principle and its more subtle variant concerning the behavior of critical values of the solution of (1.1). Then we apply them to analyze constant sign solutions and to conclude that in our cases the nonnegative solutions are monotonic. We also observe that for some equations defined on the entire space the solutions are compactly supported. Section 3 is devoted to apply our results to linear and nonlinear eigenvalue problems. We also link them with several eigenvalue problems studied in the literature like oscillatory properties of eigenfunctions for equations defined on balls and on the entire space. Section 4 deals with nonexistence of solutions of (1.1) under various Dirichlet conditions. In the last section we discuss admissible weights $a(|x|)$ in (1.1) as the elements of set $\mathcal{A}$.

2. Main results

Basic notation. We use the standard notation for Sobolev spaces $W^{k,p}(\Omega)$ and $W^{k,p}_{\mathrm{loc}}(\Omega)$ where Ω is a given domain in the Euclidean space. By ∇f we denote the distributional gradient of f. The k–th distributional derivative of a one-variable function is denoted by $f^{(k)}$. We say that $f \in C^k([\alpha, \beta])$ (or $f \in C^k([\alpha, \beta))$, $C^k((\alpha, \beta])$ respectively) if f is continuous together with its first k derivatives including endpoints of an interval. Here we admit also $\beta = \infty$ and in such a case we assume that the limits of $f, f', \ldots, f^{(k)}$ exist at ∞ and are finite. If $k = 0$ we omit it in the notation. By $B(0, R)$ we denote a ball in $\mathbb{R}^n$ centered at 0 with radius R. If $p \in (1, \infty)$, we write $p^* = p/(p-1)$ to denote Hölder conjugate of p. For such p define continuous function $\Phi_p(\lambda) = |\lambda|^{p-2}\lambda$ when $\lambda \neq 0$, $\Phi_p(0) = 0$. Here λ can be either scalar or vector.

Our PDE. We consider the equation

$$-a(|x|)\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)) \quad \text{a.e. in } B = B(0,R) \subset \mathbb{R}^n, \tag{2.1}$$

where $w \in C^1(B) \cap C(\overline{B})$ is a radially symmetric and satisfies some additional assumptions, $p > 1$, $R \in (0, \infty]$ (for $R = \infty$ the equation is defined on the whole $\mathbb{R}^n$). As for $a(\,\cdot\,)$, we assume that it belongs to the given class function $\mathcal{A}$ specified later. Function ϕ is an arbitrary odd continuous and integrable function such that $\tau\phi(\tau)$ is of constant sign for almost all τ's.

The solution of (2.1). In order to define a solution of (2.1) we will use the slightly modified definition of the classical solution of (2.1) from [37]. Let $w(x) =$

$u(|x|) \in C^1(B) \cap C(\overline{B})$ and $\Phi_p(u') \in W^{1,1}((0,R))$. From these assumptions we deduce that $\Phi_p(\nabla w) \in W^{1,1}(B, \mathbb{R}^n)$ and by $\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x))$ we mean the distributional divergence of $\Phi_p(\nabla w)$. In particular equation (2.1) holds almost everywhere and u solves the ODE:

$$
\begin{cases} a(\tau)(\Phi_p(u'(\tau)))' + (n-1)\dfrac{a(\tau)}{\tau}|u'(\tau)|^{p-2}u'(\tau) + \phi(u(\tau)) = 0 \\ \hfill \text{a.e. for } \tau \in (0,R), \\ u'(0) = 0. \end{cases} \tag{2.2}
$$

Moreover, $u \in C^1([0,R)) \cap C([0,R])$ and $\Phi_p(u') \in W^{1,1}((0,R))$.

The governing sets. We start by introducing the special sets, important in our approach. Let $n > 1, p > 1$ be given numbers and $R \in (0,\infty]$. We denote

$$
\begin{aligned} \mathcal{A} = \mathcal{A}(n,p,R) := \{a \in W^{1,1}_{\mathrm{loc}}((0,R)) \cap L^\infty((0,r)) \text{ for every } r < R : a \geq 0, \\ \kappa_{\beta(n,p)}(a,\tau) \geq 0 \text{ for almost every } \tau \in (0,R)\}, \end{aligned}
$$

$$
\kappa_\beta(a,\tau) := \beta\frac{a(\tau)}{\tau} - a'(\tau), \quad \beta(n,p) = \frac{p(n-1)}{p-1} = p^*(n-1). \tag{2.3}
$$

The maximum principle. To abbreviate the notation let us introduce the following set of assumptions.

ASSUMPTIONS B.

(a) $p > 1$, $n > 1$, $R \in (0,\infty]$, $B = B(0,R) \subset \mathbb{R}^n$;
(b) $w \in C^1(B) \cap C(\overline{B})$ is the radial solution to (2.1);
(c) ϕ is an arbitrary odd continuous function such that $\tau\phi(\tau)$ is either positive or negative for almost all τ's;
(d) $a \in \mathcal{A}$.

Our main result reads as follows.

PROPOSITION 2.1. *Suppose that Assumptions* B *are satisfied. Then if* $\tau\phi(\tau)>0$ *almost everywhere, we have*

$$
\sup_{x\in B} |w(x)| = |w(0)|,
$$

while if $\tau\phi(\tau) < 0$ *almost everywhere, then* $|w(x)|$ *is nondecreasing with respect to* $|x|$*, in particular*

$$
\sup_{x \in B}|w(x)| = \begin{cases} \sup\limits_{x\in\partial B} |w(x)| & \text{if } R < \infty, \\ \limsup\limits_{|x|\to\infty} |w(x)| & \text{if } R = \infty. \end{cases}
$$

PROOF. We prove the case $R < \infty$ only, as the remaining part follows the same lines. Let $w(x) = u(|x|)$. Then u solves (2.2).

Case 1. $\tau\phi(\tau) > 0$ almost everywhere. The equation (2.2) implies

$$\phi(u(\tau))u'(\tau) = -a(\tau)(\Phi_p(u'(\tau)))'u'(\tau) - (n-1)\frac{a(\tau)}{\tau}|u'(\tau)|^p. \tag{2.4}$$

Define $\Phi(\tau) := \int_0^\tau \phi(s)\,ds$ and

$$A(x_1, x_2) := \Phi(|u(x_1)|) - \Phi(|u(x_2)|).$$

It suffices to show that for every $x \in (0, R)$ we have $A(x, 0) \le 0$. The same inequality for $A(R, 0)$ follows then by the continuity argument.

We note that under our assumptions $\Phi(|u(x)|) \in W^{1,1}_{\mathrm{loc}}((0,R))$ (Φ is locally Lipshitz, $|u| \in W^{1,1}((0,r))$ for every $r < R$, and we use the Nikodym ACL Characterization Theorem, see e.g. Theorem 1, Chapter 1.1.3 in [49]). Henceforth, for any x and ε such that $R > x > \varepsilon > 0$:

$$\begin{aligned} A(x,\varepsilon) &= \int_\varepsilon^x \frac{d}{d\tau}(\Phi(|u(\tau)|))\,d\tau \\ &= \int_\varepsilon^x \Phi'(|u(\tau)|)\mathrm{sgn}\,u(\tau)u'(\tau)\,d\tau = \int_\varepsilon^x \phi(u(\tau))u'(\tau)\,d\tau. \end{aligned}$$

According to (2.4) and the very definition of class $\mathcal{A}$ we arrive at inequality

$$\phi(u(\tau))u'(\tau) \le -a'(\tau)\left(1 - \frac{1}{p}\right)|u'(\tau)|^p - a(\tau)(\Phi_p(u'(\tau)))'u'(\tau). \tag{2.5}$$

To proceed further we consider expressions

$$\Psi(\tau, \lambda_1) := -\left(1 - \frac{1}{p}\right)a(\tau)|\lambda_1|^p \quad \text{and} \quad h(\lambda_1) := |\lambda_1|^{p/(p-1)}. \tag{2.6}$$

As $p > 1$, the mapping $h(\,\cdot\,)$ is locally Lipshitz. Moreover, $\Phi_p(u') \in W^{1,1}(0,R)$ (so it is also bounded), therefore using the ACL property again we check that

$$|u'|^p = h(\Phi_p(u')) \in W^{1,1}(0,R).$$

As $a(\tau)$ belongs to $W^{1,1}_{\mathrm{loc}}(0,R)$, this implies $\Psi(\tau, u'(\tau)) \in W^{1,1}_{\mathrm{loc}}(0,R)$. By direct computation we obtain:

$$\begin{aligned} (|u'|^p)' &= (h(\Phi_p(u')))' = h'(\Phi_p(u')) \cdot (\Phi_p(u'))' \\ &= \frac{p}{p-1}(|\Phi_p(u')|^{1/(p-1)}\mathrm{sgn}\,(\Phi_p(u')))(\Phi_p(u'))' \\ &= \frac{p}{p-1}u' \cdot (\Phi_p(u'))' \quad \text{a.e.} \end{aligned}$$

Therefore the right hand side in equation (2.5) is exactly $(d/d\tau)(\Psi(\tau,u'(\tau)))$ where $\Psi(\tau,u'(\tau))\in W^{1,1}_{\mathrm{loc}}(0,R)$ and $\Psi(\,\cdot\,,\,\cdot\,)$ is given by (2.6). Hence, we get

$$
\begin{aligned}
A(x,\varepsilon) &\le \int_\varepsilon^x \frac{d}{d\tau}(\Psi(\tau,u'(\tau))) = \Psi(\tau,u'(\tau))|_\varepsilon^x \\
&= -\left(1-\frac{1}{p}\right)a(x)|u'(x)|^p + \left(1-\frac{1}{p}\right)\cdot a(\varepsilon)\cdot|u'(\varepsilon)|^p \\
&\le \left(1-\frac{1}{p}\right)\cdot a(\varepsilon)\cdot |u'(\varepsilon)|^p.
\end{aligned}
\tag{2.7}
$$

Let $\varepsilon\to 0$. As $a(\,\cdot\,)$ is bounded close to 0 and $u'(\varepsilon)\to 0$ as $\varepsilon\to 0$, we arrive at: $A(x,0)\le 0$. Therefore the first assertion follows.

Case 2. $\tau\phi(\tau)<0$ almost everyhere. We substitute $\widetilde{\phi}(\tau)=-\phi(\tau)$ and compute that

$$
\begin{aligned}
\widetilde{\phi}(u(\tau))u'(\tau) &= a(\tau)(\Phi_p(u'(\tau)))'u'(\tau) + (n-1)\frac{a(\tau)}{\tau}|u'(\tau)|^p \\
&\ge a(\tau)(\Phi_p(u'(\tau)))'u'(\tau) + \left(1-\frac{1}{p}\right)a'(\tau)|u'(x)|^p.
\end{aligned}
\tag{2.8}
$$

The last equals $(d/d\tau)(\Psi(\tau,u'(\tau)))$, where $\Psi(\tau,\lambda_1)=(1-1/p)a(\tau)|\lambda_1|^p$ and $\Psi(\tau,u'(\tau))\in W^{1,1}_{\mathrm{loc}}(0,R)$. As $\tau\widetilde{\phi}(\tau)>0$ almost everywhere we observe that the function $\Phi(\tau)=\int_0^\tau\widetilde{\phi}(s)\,ds$ is increasing for $\tau>0$. Integrating (2.8) over $(x,R-\varepsilon)$ with $\varepsilon>0$ sufficiently small we get

$$
\begin{aligned}
A(R-\varepsilon,x) &= \Phi(|u(R-\varepsilon)|)-\Phi(|u(x)|) \\
&\ge \left(1-\frac{1}{p}\right)a(R-\varepsilon)|u'(R-\varepsilon)|^p - \left(1-\frac{1}{p}\right)a(x)|u'(x)|^p \\
&\ge -\left(1-\frac{1}{p}\right)a(x)|u'(x)|^p = \mathcal{L}(x).
\end{aligned}
$$

Therefore, also $A(R,x)\ge\mathcal{L}(x)$. We need to verify the condition $A(R,x)\ge 0$ only for $x\in(0,R)$ such that $u'(x)=0$ or at $x=0$ (the maximum point for $|u|$ and for u^2 are the same). In both cases $\mathcal{L}(x)=0$. As for any $r<R$, u solves (2.2) on $(0,r)$, we get: $\sup\{|u(\tau)| : \tau\in(0,r)\}=|u(r)|$, whenever $r<R$. This gives the monotonicity of $|w|$ with respect to $|x|$ and completes the proof of the proposition. □

Local behavior of solutions. Our next goal is to describe the local behavior of solutions of (2.1) more precisely.

Proposition 2.2. *Suppose that Assumptions* B *are satisfied. Define*

$$
\begin{aligned}
&\Gamma=\{r\in[0,R) : \textit{every } x\in\partial B(r) \textit{ is a critical point for } w\} \\
&M\colon[0,R]\to[0,\infty),\quad M(r)=|w|_{\{|x|=r\}}.
\end{aligned}
\tag{2.9}
$$

If $\tau\phi(\tau) > 0$ almost everywhere then the mapping $M|_\Gamma$ is nonincreasing with respect to r, while if $\tau\phi(\tau) < 0$ almost everywhere then the mapping M is nondecreasing on the whole $[0, R]$.

Proof. Only the case $\tau\phi(\tau) > 0$ almost everywhere has to be proven. Let us take an arbitrary $r_1, r_2 \in \Gamma$ such that $r_1 < r_2$ and use the same notation as in the proof of Proposition 2.1. The same computation as in (2.7) with ε, x substituted by r_1, r_2 implies that:

$$A(r_2, r_1) \leq \left(1 - \frac{1}{p}\right) \cdot a(r_1) \cdot |u'(r_1)|^p = 0.$$

Therefore the first assertion follows. □

Monotonicity of constant sign solutions. As an immediate consequence we obtain the corollary below, which seems to be related to the celebrated result by Gidas, Ni and Nirenberg [39] based on moving plane method discovered by Serrin in [57]. Its simplest variant asserts that the classical positive solution of $\Delta u + f(u) = 0$ in the ball, with the Dirichlet type boundary data and C^1 function f is radially symmetric, nonnegative and monotone (see Theorem 1 in [39]). For some extensions of this result to equations involving p-Laplacian we refer to [16], [22], [29], [47], [58].

Our formulation is essentially weaker because we assume that the solution is radially symmetric.

Corollary 2.3. *Suppose that Assumptions* B *are satisfied. Then we have:*

(a) *If $\tau\phi(\tau) > 0$ almost everywhere and $w \geq 0$ then $w(x)$ is nonincreasing with respect to $|x|$.*
(b) *If $\tau\phi(\tau) < 0$ almost everywhere then w is either nonnegative or nonpositive. Moreover, if $w \geq 0$ then $w(x)$ is nondecreasing with respect to $|x|$.*

Remark 2.4. Obviously the assumption that $\tau\phi(\tau)$ is either positive or negative almost everywhere is purely technical here. In the case $w \geq 0$ it suffices to have ϕ defined on $[0, \infty)$ while in the case $w \leq 0$ it suffices that ϕ is defined on $(-\infty, 0]$. Note also that if $w \leq 0$ solve the PDE then also $v = -w \geq 0$ solves the same PDE.

Vanishment property. Our next corollary allows to deduce that in some cases the solution of (2.1) vanishes close to ∂B or close to the origin.

Corollary 2.5. *Suppose that Assumptions* B *are satisfied. Then we have*

(a) *If $\tau\phi(\tau) > 0$ almost everywhere, $w(x) = 0$ for $|x| = \tau_0 \in (0, R)$ and w does not vanish identically on some neighbourhood of x then either $w \equiv 0$ for every x with $\tau_0 \leq |x| \leq R$ or the function $u(r)$ such that*

$w(x) = u(|x|)$ has its separate zero at τ_0 – in particular must change its sign at τ_0.

(b) *If $\tau\phi(\tau) > 0$ almost everywhere, w is either nonpositive or nonnegative and there exists $\tau_0 \in (0, R)$ such that $w(x) = 0$ for $|x| = \tau_0$ then $w \equiv 0$ for $|x| \in [\tau_0, R)$.*

(c) *If $\tau\phi(\tau) < 0$ almost everywhere and there exists $\tau_0 \in (0, R]$ such that $w(x) = 0$ for $|x| = \tau_0$ ($\lim_{|x|\to\infty} w(x) = 0$ if $\tau_0 = \infty$) then $w \equiv 0$ for $|x| \in [0, \tau_0]$.*

Proof. Only part (a) requires comment as the remaining parts follow easily. We note that under our assumptions τ_0 is the accumulation point of the set of zeroes of function M (see (2.9)) or it is its separate zero. In the first case $\tau_0 \in \Gamma$ and we apply Proposition 2.2. In the second case $u'(\tau_o) \neq 0$, therefore u must change its sign at τ_0. □

Illustrations and remarks. We end this section with several remarks illustrating our results. Futher consequences will be discussed later.

Remark 2.6 (radial ground states and nonnegative solutions). Franci, Lanconelli and Serrin [37] studied quasilinear elliptic PDE

$$\mathrm{div}(A(|\nabla w|)\nabla w) + f(w) = 0, \quad x \in \mathbb{R}^n,\ n \geq 2, \tag{2.10}$$

where $A(s)$ is the real positive continuous function on $(0, \infty)$, $A(0) = 0$ and A satisfies certain additional assumptions. The authors obtained the existence and uniqueness results for radial ground states. Recall that by ground state of the PDE we mean such solution which is non-negative, non-trivial, continuously differentiable, and tends to zero as x approaches infinity. For $A(s) = |s|^{p-2}$ we retrieve p-Laplacian on the left hand side in (2.10). In such a case equation (2.10) takes the form $-\Delta_p w = f(w)$ which is our PDE with the weight function $a \equiv 1$ and $\phi = f$. Proposition 1 on page 182 in [37] shows that under certain assumptions which include the case dealing with p-Laplacian the ground state of (2.10) is decreasing in $|x|$ for f being positive (see also Lemma 1.1.3 on page 189). As noticed by the authors of [37] on page 187 the same kind of result follows for equation defined on $B(0, R)$. Namely, the nonnegative radial solutions of (2.10) with positive f and $x \in B(0, R)$ are monotonic in $|x|$ and achieve their maximum at 0. Corollary 2.3 allows to handle the equation $a(x)\Delta_p w(x) = f(w(x))$ where $a(\cdot)$ is not necessarily a constant function and $w(\cdot)$ is not necessarily nonnegative. The techniques used in [37] were different then ours.

Remark 2.7 (compact support property). For $R = \infty$ and $\tau_0 < \infty$ the first statement in Corollary 2.5 asserts that the solution of the equation (2.1) is compactly supported. Similar result was obtained in [37] for nonnegative radial solutions of equation (2.10), see Proposition 1.3.1 on page 192 in [37]; see also

[52] for the case of the Laplace operator. Our equation (2.1) is specialized to p-Laplacian: $a\Delta_p w = f(w)$ and permits the nonconstant function $a(\,\cdot\,)$ as well as the solutions which can be of the nonconstant sign.

The illustration of Proposition 2.2 and remaining conclusions are postponed to the next sections.

3. Applications to eigenvalue problems

We point out on the following immediate consequences of Propositions 2.1 and 2.2 and Corollary 2.3 for linear and nonlinear eigenvalue problems which seem to be missing in the literature. Dealing with the parameter $q > 1$ we uniformly treat the linear and nonlinear eigenvalue problems. However, in some cases set of solutions to our equations may consist of $w \equiv 0$ only. For discussion in this direction we refer to our next section. For existence theory and some further results dealing with eigenvalue problems we refer to books [32], [38] and their references.

Corollary 3.1. *Suppose that $p > 1$, $n > 1$, $R \in (0,\infty]$, $q > 1$, $B = B(0,R) \subset \mathbb{R}^n$, $a \in \mathcal{A}$ and M,Γ are given by* (2.9)*. Assume that w is a radial solution to the following eigenvalue problem*

$$-a(|x|)(\operatorname{div}|\nabla w(x)|^{p-2}\nabla w(x)) = \lambda |w(x)|^{q-2}w(x), \quad \textit{a.e. in } B.$$

Then we have:

(a) *If $\lambda > 0$, we have $\sup_{x\in B}|w(x)| = |w(0)|$ and the mapping $M|_\Gamma$ is decreasing with respect to r.*

(b) *If $\lambda < 0$, we have*

$$\sup_{x\in B}|w(x)| = \begin{cases} \sup\limits_{x\in\partial B}|w(x)| & \textit{if } R<\infty,\\ \limsup\limits_{|x|\to\infty}|w(x)| & \textit{if } R=\infty,\end{cases}$$

w is either nonnegative or nonpositive and the mapping $|w(x)|$ is nondecreasing with respect to $|x|$.

Our next corollary applies to solutions of constant sign.

Corollary 3.2. *Let $B = B(0,R) \subset \mathbb{R}^n$, $R \in (0,\infty]$, $p > 1, q > 1$, $n > 1$, $a \in \mathcal{A}$, $\lambda > 0$, and let w be a radial solution to the eigenvalue problem*

$$-a(|x|)(\operatorname{div}|\nabla w(x)|^{p-2}\nabla w(x)) = \lambda |w(x)|^{q-2}w(x), \quad \textit{a.e. in } B.$$

If $w \geq 0$ then $w(x)$ is nonincreasing with respect to $|x|$.

Remark 3.3 (eigenvalue problems on the ball). Walter dealt in [61] with the following operator:

$$L_p^\alpha u = r^{-\alpha}(r^\alpha u'^{(p-1)})' = (p-1)|u'|^{p-2}u'' + \frac{\alpha}{r}u'^{(p-1)},$$

where $r \in \mathbb{R}$ is an independent variable, $\alpha \geq 0$, $p > 1$ and $s^{(p)} = |s|^{p-1}s$ (p real). For $\alpha = n-1$ and $r = |x|$ this is the Δ_p Laplacian applied to the radial function $u(|x|)$ with $|x| = r$. Among other results he considered the eigenvalue problem:

$$L_p^\alpha u + (q(r) + \lambda s(r))u^{(p-1)} = 0 \quad \text{in } [0, R], \quad u'(0) = 0, \quad u(R) = 0. \tag{3.1}$$

It is proven in [61] on page 183 that if only the functions $q(r)$ and $s(r)$ are continuous and $s(r)$ is positive on $[0, R]$ then the above eigenvalue problem has a countable number of simple eigenvalues $\lambda_1 < \lambda_2 < \dots$, such that $\lim_{n\to\infty} \lambda_n = \infty$, and no other eigenvalues. Each eigenfunction u_n has $n-1$ simple zeros in $(0, R)$. Between 0 and the first zero of u_n, between two consecutive zeros of u_n and between the last zero of u_n and R there is one and only one zero of u_{n+1}.

Similar result with $s \equiv 1$ and $q \equiv 0$ (except the part that zeroes of u_n and u_{n+1} separate each other) was obtained by del Pino and Manásevich in [26] and by Anane in [5] (when $n = p = 2$ the solution is the Bessel function) and by Binding and Volkmer in [14], for $p = 2$.

Let us comment the link between the above contribution and our results. Suppose that for a given n the function $1/a_n(r) := q(r) + \lambda_n s(r)$ is positive and $a_n \in \mathcal{A}$. Corollary 3.1 applied with $p-1$ instead of p in (3.1) and $\lambda \equiv 1$ reveals, in addition to the mentioned results, that each $|u_n|$ attains its maximum at 0. Moreover, the sequence of maximas of $|u_n|$ is nonincreasing in r. Another conclusion can be deduced in the case: $q \equiv 0$, $a := 1/s \in \mathcal{A}$, $\lambda_n < 0$. In such a case as a consequence of Corollary 3.1 the equation (3.1) admits no nontrivial solutions. If $s \in L^1(0, R)$ this follows also directly by integration by parts.

REMARK 3.4 (oscillatory properties: the case of entire space). Bartušek, Cecchi, Došlá and Marini in the paper [9] considered the quasilinear ODE:

$$a(t)\Phi_p(x'(t))' + b(t)\Phi_q(x(t)) = 0, \quad t \in \mathbb{R}_+ = [0, \infty), \tag{3.2}$$

including Emden–Fowler equation:

$$(a(t)x'(t))' + b(t)|x(t)|^\mu \operatorname{sgn} x(t) = 0, \quad \mu \neq 1.$$

With the notation of [9] functions a, b are continuous positive defined on $\mathbb{R}_+$, $\Phi_s(u) = |u|^{s-2}u$, $s > 1$, while function $a^{1/(p-1)}b$ is continuously differentiable on $\mathbb{R}_+$. Authors investigated the existence theory for oscillatory solutions of the above ODE. These are solutions with infinite number of zeroes and with the property that there exists the sequence $\{\tau_n\}$ converging to ∞ such that $u(\tau_n) = 0$. Otherwise the solution is called nonoscillatory. The ODE (3.2) is called oscillatory (nonoscillatory) if every nontrivial solution of (3.2) is oscillatory (nonoscillatory), respectively. The main results of that paper are the necessary and sufficient conditions for equation (3.2) to be oscillatory. In the case $a \equiv \text{const} > 0$ and $1/b \in \mathcal{A}$ one can apply similar techniques as we used in

Corollary 3.1 to verify that if oscillatory solutions exist, the sequence of maxima for $|x|$ is decreasing with respect to t. Results in this direction and systematic theory can be found e.g. in monographs [1], [19], see also [2], [3], [59], [63] and references therein.

4. Applications to nonexistence and uniqueness results

We start with several nonexistence results which follow directly from our techniques.

Proposition 4.1. *Let $B = B(0,R) \subset \mathbb{R}^n$, $R \in (0,\infty]$, $p > 1$, $n > 1$, ϕ is an arbitrary odd continuous function such that $\tau\phi(\tau)$ is either positive or negative $\mathcal{L}^1$-almost all τ and let $a \in \mathcal{A}$. Then the following problems admit only trivial solutions:*

$$\text{(a)}\ \begin{cases} -a(|x|)\operatorname{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)), & \textit{a.e. in } B = B(0,R) \subset \mathbb{R}^n, \\ w(0) = 0, \end{cases}$$

in the case $\tau\phi(\tau) > 0$ almost everywhere;

$$\text{(b)}\ \begin{cases} -a(|x|)\operatorname{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)) & \textit{a.e. in } B = B(0,R) \subset \mathbb{R}^n, \\ w \equiv 0 \quad \textit{on } \partial B(0,R) \textit{ for } R < \infty \quad \textit{or} \quad \lim_{|x|\to\infty} w(x) = 0 \quad \textit{for } R = \infty, \end{cases}$$

in the case $\tau\phi(\tau) < 0$ almost everywhere.

Remark 4.2. The second statement in Proposition 4.1 shows that in the case $\tau\phi(\tau) < 0$ a.e. under our assumptions problem: $-a\Delta_p w = \phi(w)$ has only trivial solution under Dirichlet boundary condition for $R < \infty$. If $R = \infty$ there are no nontrivial solutions in any $W^{1,r}(\mathbb{R}^n)$ with $r \in [1,\infty)$ as such functions must vanish at ∞.

Our next results provide the reader with examples of PDEs involving p-Laplacian which admit no radial solutions at all.

Proposition 4.3. *Let $B = B(0,R) \subset \mathbb{R}^n$, $R \in (0,\infty]$, $p > 1$, $n > 1$, ϕ is an arbitrary odd continuous function such that $\tau\phi(\tau)$ is either positive or negative $\mathcal{L}^1$-almost all τ, and let $a \in \mathcal{A}$. The following problem*

$$\begin{cases} -a(|x|)\operatorname{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)), & \textit{a.e. in } B = B(0,R) \subset \mathbb{R}^n, \\ w(0) = c, \\ w \equiv b \quad \textit{on } \partial B \textit{ for } R < \infty \quad \textit{or} \quad \lim_{|x|\to\infty} w(x) = b \quad \textit{for } R = \infty, \end{cases}$$

admits no radial solutions when either: $\tau\phi(\tau) > 0$ almost everywhere and $|c| < |b|$ or: $\tau\phi(\tau) < 0$ almost everywhere and $|c| > |b|$.

Proposition 4.4. *Let $B = B(0,R) \subset \mathbb{R}^n$, $R \in (0,\infty]$, $p > 1$, $n > 1$, ϕ is an arbitrary odd continuous function such that $\tau\phi(\tau)$ is either positive or negative $\mathcal{L}^1$-almost everywhere, and let $a \in \mathcal{A}$. The following problems admit no radial solutions*

$$
\text{(a)}\quad \begin{cases} -a(|x|)\,\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)), & \text{a.e. in } B = B(0,R) \subset \mathbb{R}^n, \\ w \geq 0, \\ w(x_0) = c, \quad x_0 \in B, \\ w \equiv b \quad \text{on } \partial B \text{ for } R < \infty \quad \text{or} & \lim_{|x|\to\infty} w(x) = b \quad \text{for } R = \infty, \end{cases}
$$

when either $\tau\phi(\tau) > 0$ almost everywhere and $c < b$ or $\tau\phi(\tau) < 0$ almost everywhere and $c > b$.

$$
\text{(b)}\quad \begin{cases} -a(|x|)\,\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)), & \text{a.e. in } B = B(0,R) \subset \mathbb{R}^n, \\ w \leq 0, \\ w(x_0) = c, \quad x_0 \in B, \\ w \equiv b \quad \text{on } \partial B \text{ for } R < \infty \quad \text{or} & \lim_{|x|\to\infty} w(x) = b \quad \text{for } R = \infty, \end{cases}
$$

in the case when either $\tau\phi(\tau) > 0$ almost everywhere and $c > b$ or $\tau\phi(\tau) < 0$ almost everywhere and $c < b$.

As a direct consequence of the first equation we obtain the following result (see Remark 2.6 for the definition of ground state).

Proposition 4.5. *Let $p > 1$, $n > 1$, ϕ is an arbitrary continuous function defined on $[0,\infty)$ such that $\phi(0) = 0$, $\phi(\tau) < 0$ for almost all τ, and let $a \in \mathcal{A}$. Then the problem*

$$
-a(|x|)\,\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \phi(w(x)), \quad \text{a.e in } \mathbb{R}^n,
$$

admits no radial ground states.

Remark 4.6 (links with Gidas–Ni–Nirenberg type results). The above result corresponds to the series of results of Gidas–Ni–Nirenberg type ([7], [24], [58]), where authors proved radial symmetry of solutions to equations of type

$$
-\Delta_p w(x) = \phi(w), \quad w > 0,\ x \in \mathbb{R}^n, \tag{4.1}
$$

with the ground state conditions at ∞ (i.e. $\lim_{|x|\to\infty} w(x) = 0$), under various constraints imposed on p, n and ϕ. Proposition 4.5 with $a \equiv 1$ and ϕ being locally Lipshitz, decreasing and such that $\phi(0) = 0$ (in particular $\phi < 0$) shows that radial ground state solutions may not exist. As a consequence of results obtained in [24] restricted to this special case every positive classical solution of (4.1) is radially symmetric. Therefore, when ϕ is locally Lipshitz, decreasing and such that $\phi(0) = 0$ the equation (4.1) admits no classical ground state solutions: neither radial nor other. Notice that the definition of the classical solution in [24] is slightly stronger than ours.

Next result contributes to formulations of Corollaries 3.1 and 3.2. It shows that in some cases the only solution is trivial. The proof of the first statement is based on the radial variant of Derrick–Pohozhaev identity (see e.g. [35, Section 9.4.2] for the nonradial variant dealing with $p = 2$) and our maximum principle. For uniqueness results dealing with nonnegative solutions to nonlinear eigenvalue problems like $-a(|x|)\Delta_p w(x) = \lambda w(x)^{q-1}$ with Dirichlet boundary data we refer to the Serrin's paper [56], later contributions, e.g. [43], [50], [60], [13] and literature therein. The second statement is known to the specialists even in the more general context of equations defined on a bounded starshaped domains, [51], [53], [54]. It is independent of our maximum principles dealing with general a, but for readers convenience we include the proof of second statement as well.

Proposition 4.7. *Assume that $p, q > 1$, $\lambda \in \mathbb{R}$, $0 < R < \infty$, $B = B(0, R) \subseteq \mathbb{R}^n$, $a \in \mathcal{A}$, and consider the following PDE:*

$$\begin{cases} -a(|x|)\Delta_p w(x) = \lambda w(x)^{q-2} w(x) & \text{in } B, \\ w \equiv 0 & \text{on } \partial B. \end{cases} \tag{4.2}$$

Then we have:

(a) *If $a \in \mathcal{B}$, where*

$$\mathcal{B} = \mathcal{B}(n,p,q,R) := \bigg\{ a \in \mathcal{A} : a' \geq 0 \ \text{a.e.}\ \kappa_s(a,\tau) > 0 \ \text{a.e.} \ \text{with } s := \left(\frac{n}{p} - \frac{n}{q} - 1\right)p^* > 0 \bigg\},$$

(see (2.3)), and $w \in C^2(B) \cap C^1(\overline{B})$ is radial and nonnegative then $w \equiv 0$.

(b) *If $a \equiv 1$ and ($\lambda \leq 0$ or $\lambda > 0$ and $n/p - n/q - 1 > 0$) and $w \in C^2(B) \cap C^1(\overline{B})$ is an arbitrary solution of (4.2) then $w \equiv 0$.*

Remark 4.8. On the contrary to the first statement we do not impose radiality or sign conditions on w in the second statement.

Remark 4.9. Note that $s = n(1/p' - 1/q)p^*$ where $p' = np/(n-p)$ is Sobolev conjugate to p in case $p < n$. Therefore the condition $s > 0$ implies in particular that $p < n$ and $q > p'$.

Proof of Proposition 4.7. (a) If $\lambda < 0$ the assertion follows directly from Proposition 3.1.

Let $\lambda = 0$. The very definition of $\mathcal{B}$ implies that set $\{\tau \in (0,R) : a(\tau) = 0\}$ is of measure 0. Therefore $\Delta_p w(x) = 0$ almost everywhere in B and $w \equiv 0$ on ∂B.

Multiplying first equation by w and integrating by parts we conclude the result. Assume that $\lambda > 0$. First observe that if (4.2) admits solution with an arbitrary positive λ then it also admits solution with $\lambda = 1$ on the possibly different ball. This follows by the rescalling argument: we substitute $w_t(x) := tw(x)$ to the equation for suitable t. Therefore, we can assume that $\lambda = 1$. Multiplying (2.2) with $\phi(\tau) = |\tau|^{q-2}\tau$ by $\tau^n u'(\tau)$ and integrating the resulting equation by parts, we arrive at the following

$$a(R)R^n|u'(R)|^p - \int_0^R a'(\tau)\tau^n|u'(\tau)|^p\,d\tau - \frac{1}{p}\int_0^R a(\tau)\tau^n(|u'(\tau)|^p)'\,d\tau \\ - \int_0^R a(\tau)\tau^{n-1}|u'(\tau)|^p\,d\tau = \frac{n}{q}\int_0^R |u(\tau)|^q\tau^{n-1}\,d\tau.$$

Applying integration by parts to the third term of the above equation we obtain the following radial variant of Derrick–Pohozhaev identity.

$$\int_0^R \left\{ -\left(1-\frac{1}{p}\right)a'(\tau)\tau - \left(1-\frac{n}{p}\right)a(\tau)\right\}|u'(\tau)|^p\tau^{n-1}\,d\tau \\ + a(R)R^n|u'(R)|^p\left(1-\frac{1}{p}\right) = \frac{n}{q}\int_0^R |u(\tau)|^q\tau^{n-1}\,d\tau.$$

This in turn implies inequality

$$\int_0^R \left\{ -\left(1-\frac{1}{p}\right)a'(\tau)\tau - \left(1-\frac{n}{p}\right)a(\tau)\right\}|u'(\tau)|^p\tau^{n-1}\,d\tau \\ \leq \frac{n}{q}\int_0^R |u(\tau)|^q\tau^{n-1}\,d\tau. \tag{4.3}$$

Similarly, multiplying (2.2) for $\phi(\tau) = |\tau|^{q-2}\tau$ by $\tau^{n-1}u(\tau)$ and integrating the resulting equations by parts we get

$$\int_0^R |u(\tau)|^q\tau^{n-1}\,d\tau = \int_0^R a'(\tau)\Phi_p(u'(\tau))u(\tau)\tau^{n-1}\,d\tau + \int_0^R a(\tau)|u'(\tau)|^p\tau^{n-1}\,d\tau.$$

Observe, that $\Phi_p(u')u = |u'|^{p-2}(u'u)$. As $w \geq 0$, part (a) of Corollary 2.3 implies that u is nonincreasing, in particular $u'u \leq 0$, henceforth $\Phi_p(u')u \leq 0$. Therefore, under assumption that $a' \geq 0$, the first term on the right hand side of the above equation is nonpositive. Hence

$$\int_0^R |u(\tau)|^q\tau^{n-1}\,d\tau \leq \int_0^R a(\tau)|u'(\tau)|^p\tau^{n-1}\,d\tau.$$

This, together with (4.3) implies, that

$$\int_0^R \left\{ -\left(1-\frac{1}{p}\right)a'(\tau)\tau - \left(1-\frac{n}{p}+\frac{n}{q}\right)a(\tau)\right\}|u'(\tau)|^p\tau^{n-1}\,d\tau \leq 0.$$

This inequality cannot hold for nontrivial u, as the expression

$$\left\{-\left(1-\frac{1}{p}\right)a'(\tau)\tau-\left(1-\frac{n}{p}+\frac{n}{q}\right)a(\tau)\right\}$$

is strictly positive.

(b) If $\lambda \le 0$ we proceed as in the proof of first part. If $\lambda > 0$ we may assume as before that $\lambda = 1$ and we adopt the technique described in the Evans book [35, Section 9.4.2] (where one considers the case $p = 2$) to the case of solutions to the PDE:

$$-\Delta_p w(x) = |w(x)|^{q-2}w, \quad w \equiv 0 \quad \text{on } \partial B. \tag{4.4}$$

Multiplying (4.4) by $x \cdot \nabla w(x)$ then integrating by parts leads to the following Derrick–Pohozhaev identity:

$$\left(\frac{n}{p}-1\right)\int_B |\nabla w|^p\,dx + \left(1-\frac{1}{p}\right)\int_{\partial B}|\nabla w|^p|x|\,dS = \frac{n}{q}\int_B |w|^q\,dx,$$

which implies an inequality:

$$\left(\frac{n}{p}-1\right)\int_B |\nabla w|^p\,dx \le \frac{n}{q}\int_B |w|^q\,dx. \tag{4.5}$$

Next, multiplying (4.4) by w, then integrating by parts one gets:

$$\int_B |w|^q\,dx = \int_B |\nabla w|^p\,dx.$$

Therefore (4.5) implies an inequality:

$$\left(\frac{n}{p}-\frac{n}{q}-1\right)\int_B |\nabla w|^p\,dx \le 0.$$

This is impossible for the nontrivial w under our assumptions. Easy details are left to the reader. □

5. Elements of sets $\mathcal{A}$ and $\mathcal{B}$

Members of set $\mathcal{A}$ can be easily recognized with help of the following practical observations.

Proposition 5.1. *The following statements hold true.*

- (a) *Every positive constant function belongs to* $\mathcal{A}$.
- (b) *Every positive nonincreasing function in the* $C^1((0,R))$ *class which is continuous on* $[0,R]$ *is a member of* $\mathcal{A}$.
- (c) *Every positive concave function* (*not necessary strictly concave*) *in the class* $W^{1,1}_{\mathrm{loc}}((0,R)) \cap L^\infty((0,R))$ *belongs to* $\mathcal{A}$.

Proof. Note that for nonnegative a the mapping $\beta \mapsto \kappa_\beta(a,\tau)$ is nondecreasing and $\beta(n,p) > 1$. As for concave function one has $\kappa_1(a,\tau) \geq 0$, therefore the last statement follows. The remaining statements are plain. $\square$

Proposition 5.2.

(a) *If $a \in \mathcal{A}$ and $g \in C([0,R]) \cap C^1((0,R))$ is nonnegative and $g' \leq 0$ almost everywhere in $(0,R)$ then $ga \in \mathcal{A}$;*

(b) *If $a \in \mathcal{A}$ (resp. $a \in \mathcal{B}$) and $F\colon [0,\infty) \to [0,\infty)$, $F \in C^1([0,\infty))$ is nondecreasing and concave (not necessarily strictly concave) then $F(a) \in \mathcal{A}$ (resp. $F(a) \in \mathcal{B}$).*

Proof. We give an argument only for part (b) for elements of $\mathcal{A}$, as the remaining cases follow easily. As $(F(a))' = F'(a)a'$, $F' \geq 0$ and $aF'(a) \leq F(a)$, we have

$$(F(a(\tau)))' = F'(a(\tau))a'(\tau) \leq \beta F'(a(\tau))\frac{a(\tau)}{\tau} \leq \beta\frac{F(a(\tau))}{\tau}.$$

Thus $F(a) \in \mathcal{A}$. $\square$

Example 5.3. Using the above properties it is easy to generate elements of set $\mathcal{A}$, e.g. every positive constant function belongs to $\mathcal{A}$, for every $\alpha \in [1,(n-1)p/(p-1)]$, $\beta \geq 0$ functions τ^α, $\tau^\alpha(R-\tau)^\beta$ belong to $\mathcal{A}$, as well as for $a(\tau) \in \mathcal{A}$ functions $e^{-\tau}a(\tau)$, $1-e^{-a(\tau)}$, $\log(1+a(\tau))$ are members of $\mathcal{A}$.

Example 5.4. In relation to Proposition 4.7 we list some elements of $\mathcal{B}$. Similarly as before functions $1-e^{-a(\tau)}$ and $\log(1+a(\tau))$ belong to $\mathcal{B}$ when $a \in \mathcal{B}$. This is because Proposition 5.2(b) holds also for elements of $\mathcal{B}$. An easy computation shows that the function $a(\tau) = \tau^\beta \in \mathcal{B}$ if and only if $0 \leq \beta < s$.

Notice, that on the contrary to properties of class $\mathcal{A}$ this time Proposition 5.2(a) may fail. To see this consider p, q, n such that $0 < s$. Then we chose $\beta \in (0,s)$, $a(\tau) = \tau^\beta \in \mathcal{B}$ and take $R > \beta$. For the function $h(\tau) = a(\tau)g(\tau)$ with $g(\tau) = e^{-\tau}$ we have $h'(\tau) < 0$ when $\tau > \beta$. Therefore $h \notin \mathcal{B}$.

Acknowledgements. This work was originated during the second author's visit at the Department of Mathematics of Syracuse University in April 2006 and continued during the first author's visit at Institute of Mathematics of Warsaw University in June 2007. The hospitality of both involved institutions is gratefully acknowledged. A. K. also thanks Professor Tadeusz Iwaniec for discussions and hospitality during her stay at Syracuse University.

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Manuscript received February 21, 2008

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