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NOT FINITELY BUT COUNTABLY HOPF-EQUIVALENT CLOPEN SETS IN A CANTOR MINIMAL SYSTEM

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Abstract. We investigate topological Hopf-equivalences in the Morse substitution system. In particular, we give an example of not finitely but countably Hopf-equivalent clopen sets in a Cantor minimal system.

1. Introduction

E. Hopf [11] introduced the notions of Hopf-equivalences for non-singular and bi-measurable transformations on non-atomic Lebesgue spaces, and completely characterized both the existence of an equivalent finite invariant measure and the conservativity of the transformation in terms of countable and finite Hopf-equivalences, respectively. Then, the countable Hopf-equivalence played fundamental roles in the classification of the transformations up to orbit equivalence; see for example [9], [12]. After that, topological versions of the Hopf-equivalences played fundamental and crucial roles even in the classifications of Cantor minimal systems up to topological orbit equivalences; see for example [4], [5], [7]. Motivated by the above importance of Hopf-equivalences, the author [21] studied the Hopf-equivalences in zero-dimensional, topological dynamical systems in connection with several recurrence properties of the systems.

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In the present paper, we continue the investigation into the Hopf-equivalences in the zero-dimensional systems. The main purpose of the present work is to give an example of clopen sets in a Cantor minimal system which are not finitely but countably Hopf-equivalent. Specifically, by following an algorithm developed by F. Durand, B. Host and C. Skau [1], we present a Bratteli–Vershik representation B [10] of a bilateral minimal subshift (X_σ, T_σ) arising from the Morse substitution σ (Proposition 4.2), and then completely classify the cylinder sets in the infinite path space X_B of B up to the finite and countable Hopf-equivalences, respectively (Theorems 4.4 and 4.5). By means of the Bratteli–Vershik representation B, we also compute the dimension group $K^0(X_\sigma, T_\sigma)$ modulo the coboundaries associated with (X_σ, T_σ) in an explicit and simple form (Proposition 4.6). Although A. H. Forrest [3] computed the abelian group $K^0(X_\sigma, T_\sigma)$ itself to be $\mathbb{Z} \oplus \mathbb{Z}[1/2]$, where $\mathbb{Z}[1/2]$ is the abelian group of dyadic rationals, he could not determine its order structure. Although H. Matui [15] also computed it together with its order structure and distinguished order unit, it may be impossible to determine which element of the resultant group is the equivalence class of the characteristic function of a given cylinder set in X_σ. Similar negative factor may be seen also in a computation by J. Kwiatkowski and M. Wata [13]. These defects do not exist in our result.

On the other hand, we show that [10, Proposition 5.1] is still valid under a weaker assumption: any point in a totally disconnected, compact metric space X is chain recurrent for a homeomorphism S on X, then the pair $(K^0(X, S), K^0(X, S)^+)$ is an ordered group (Proposition 3.4). This fact is a consequence of a characterization, in terms of integer-valued continuous functions, of an incompressibility under the finite Hopf-equivalence (Proposition 3.3). By [21], we already know the equivalence of the incompressibility and the condition that any point is chain recurrent. Another important result is complete characterizations of orbit structures of a Cantor minimal system in terms of the Hopf-equivalences (Proposition 3.8), though its proof heavily depends on the orbit equivalence theorems of T. Giordano, I. Putnam and C. Skau [4].

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2. Primitive substitutions and Vershik maps

The purpose of this section is to briefly review some facts on bilateral minimal subshifts arising from primitive substitutions in connection with Vershik maps. Throughout the present paper, we omit several definitions relevant to our discussion, for example, those for a properly ordered Bratteli diagram, a Vershik map, a dimension group, a primitive substitution and so on, because we already have excellent and concise expositions on those materials, for example [1], [4] and [18]. Also, we freely use the notation of [1], except when we take an opposite attitude. As in [4], [10] and so on, any Bratteli diagram is supposed to have a unique vertex which is only sources of edges; that is the top vertex. Let $B = (V, E, \geq)$ be a properly ordered Bratteli diagram. Let $n \in \mathbb{N}$ and $p = (p_1, \ldots, p_n)$ be a path in $E_1 \circ \ldots \circ E_n$. The range vertex $r(p_n)$ of the edge p_n is referred to as the *range vertex* of the cylinder set

$$[p] := \{x = (x(i))_{i \in \mathbb{N}} \in X_B : x(i) = p_i \text{ for all } i,\ 1 \leq i \leq n\}$$

in the infinite path space X_B of B. Let $r([p])$ denote $r(p_n)$. By the length of $[p]$, we mean the length n of p. Let λ_B denote the Vershik map associated with B. We denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{Z}^+$ the set of nonnegative integers.

2.1. Model theorems. We start with a model theorem of the Cantor minimal systems.

Theorem 2.1 ([10]). *Any Cantor minimal system (X, S) is conjugate to the Bratteli–Vershik system (X_B, λ_B) associated with a properly ordered Bratteli diagram B, and vice versa.*

The following model theorem is fundamental for our discussion. In fact, we can say more than the statement; see [1], [3] for details.

Theorem 2.2 ([1], [3]). *Given an aperiodic, primitive substitution σ on an alphabet A, we can construct a stationary, properly ordered Bratteli diagram $B = (V, E, \geq)$, in an explicit and algorithmic method, such that the Bratteli–Vershik system (X_B, λ_B) is conjugate to the bilateral minimal subshift (X_σ, T_σ) arising from the substitution σ.*

We shall review the construction, which depends on whether the substitution σ is proper, or not. Since the Morse substitution is not proper, it is enough for us to consider the case where σ is not proper. As the alphabet A is finite, we can find $l, r \in A$ and $k \in \mathbb{N}$ such that

(1) the word rl belongs to the language $\mathcal{L}(\sigma)$ of the substitution σ;
(2) the last letter of $\sigma^k(r)$ is r;
(3) the first letter of $\sigma^k(l)$ is l.

Then $x = \lim_{n\to\infty} \sigma^{kn}(\ldots rrrr \mid llll \ldots) \in A^{\mathbb{Z}}$ is a fixed point of the map $\sigma^k : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, where the vertical bar is, as in [1], the separation between the negative and the nonnegative coordinates. The bilateral minimal subshift (X_σ, T_σ) arising from σ coincides with the subshift of orbit-closure of x under the shift on $A^{\mathbb{Z}}$. Let $\mathcal{R}$ denote the set of return words to $r.l$ in x. Put $R = \{1, \ldots, \sharp\mathcal{R}\}$. Take a bijection $\phi : R \to \mathcal{R}$. For any $w \in \mathcal{R}$, $\sigma^k(w)$ can be a concatenation of words in $\mathcal{R}$. Then we can define a proper and primitive substitution τ on R so that $\sigma^k(\phi(i)) = \phi(\tau(i))$ for each $i \in R$. Then we construct $B = (V, E, \geq)$ as follows. Set $V_n = \{(n,i) : i \in R\}$ for $n \in \mathbb{N}$, and let V_0 be a singleton, say $\{v_0\}$. For each $i \in R$, we draw $|\phi(i)|$ edges from the top vertex v_0 to the vertex $(1,i)$, where $|\phi(i)|$ is the length of the word $\phi(i)$. For each $n \in \mathbb{N}$, we draw a single edge from (n,i) to $(n+1,j)$ with the numerical order $m-1$ if $\tau(j)_m = i$, where m varies from 1 to $|\tau(j)|$. This completes the construction.

If, by following Subsection 2.3 of [1], we construct an ordered Bratteli diagram from Kakutani–Rohlin partitions:

$$\mathcal{P}_n := \{{T_\sigma}^i[\sigma^{kn}(r).\sigma^{kn}(w)\sigma^{kn}(l)] : 0 \le i < |\sigma^{kn}(w)|,\ w \in \mathcal{R}\}, \quad n \ge 1,$$

then the resultant diagram is nothing but the ordered Bratteli diagram B, where for words u and v,

$$[u.v] = \{y \in X_\sigma : y_{[-|u|,|v|)} = uv\}.$$

This fact is not explicitly stated in [1] but is a consequence of results of [1]. It is verified with the aid of B. Mossé's bilateral recognizability [16].

2.2. Invariant measures of Vershik maps. The following lemma by F. Sugisaki tells us when the Vershik map associated with a properly ordered Bratteli diagram is uniquely ergodic.

LEMMA 2.3. *Suppose that $B = (V,E,\ge)$ is a properly ordered Bratteli diagram. For vertices $u,v \in V$, let $N(u,v)$ denote the number of paths starting from u and terminating at v. Then the following conditions are equivalent:*

(a) *the Vershik map λ_B is uniquely ergodic;*
(b) *for any $k \in \mathbb{N}$ and any $u \in V_k$, the limits*

$$m(u) := \lim_{l\to\infty} \min_{v \in V_{k+l}} \frac{N(u,v)}{N(v_0,v)} \quad \textit{and} \quad M(u) := \lim_{l\to\infty} \max_{v \in V_{k+l}} \frac{N(u,v)}{N(v_0,v)}$$

exist and coincide, where $V_0 = \{v_0\}$.

Furthermore, when these conditions hold, the unique λ_B-invariant probability measure assigns the value $m(r(C)) = M(r(C))$ to a given cylinder set $C \subset X_B$.

PROOF. (a) $\Rightarrow$ (b) Let μ denote the unique λ_B-invariant probability measure. Let $k \in \mathbb{N}$, $u \in V_k$ and $\varepsilon > 0$. Take a cylinder set $C \subset X_B$ with $r(C) = u$. By [17] (cf. [19, Theorem 6.19]), there is $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ and any $x \in X$,

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \chi_C({\lambda_B}^i x) - \mu(C) \right| < \varepsilon.$$

Take $l_0 \in \mathbb{N}$ so that $N(v_0,v) \ge n_0$ for any $l \ge l_0$ and any $v \in V_{k+l}$. This is possible because (V,E) is simple. For $l \ge l_0$ and $v \in V_{k+l}$, take $x_v \in X_B$ so that for every integer $1 \le i \le k+l$, the i-th edge $x_v(i)$ is minimal in E_i. Then, for any $l \geq l_0$ and any $v \in V_{k+l}$,

$$\left|\frac{N(u,v)}{N(v_0,v)} - \mu(C)\right| = \left|\frac{1}{N(v_0,v)} \sum_{i=0}^{N(v_0,v)-1} \chi_C({\lambda_B}^i x_v) - \mu(C)\right| < \varepsilon.$$

We hence have $m(u) = M(u) = \mu(C)$.

(b) $\Rightarrow$ (a) Take integers $n_0 = 0 < n_1 < n_2 < \dots$ so that for every $i \in \mathbb{Z}^+$, a path exists from any vertex in V_{n_i} to any vertex in $V_{n_{i+1}}$. Let μ be a λ_B-invariant probability measure. Let $i \in \mathbb{N}$ and $p \in E_1 \circ E_2 \circ \dots \circ E_{n_i}$. Put $u = r(p_{n_i})$. Let $j \in \mathbb{N}$ with $j > i$. Then

$$\mu([p]) = \sum_{v \in V_{n_j}} N(u,v)\mu([pq_v]) \quad \text{and} \quad 1 = \sum_{v \in V_{n_j}} N(v_0,v)\mu([pq_v]),$$

where q_v is a path in $E_{n_i+1} \circ \dots \circ E_{n_j}$ starting from $r(p_{n_i})$ and terminating at v. We hence obtain

$$\min_{v \in V_{n_j}} \frac{N(u,v)}{N(v_0,v)} \leq \mu([p]) \leq \max_{v \in V_{n_j}} \frac{N(u,v)}{N(v_0,v)},$$

and letting $j \to \infty$ shows that $m(u) \leq \mu([p]) \leq M(u)$. By the assumption, $\mu([p]) = m(u) = M(u)$. This completes the proof. $\square$

Remark 2.4. (a) The equivalence of (a) and (b) in Lemma 2.3 was originally proved by R. Gjerde [6]. However, instead of (b), he used another condition equivalent to (b) which is not so understandable as (b). Moreover, the above proof is much simpler.

(b) Recall that there is a bijective correspondence between the set of states on the dimension group $K^0(V,E)$ and the set of λ_B-invariant probability measures [10, Theorem 5.5]. It was shown by E. G. Effros [2] and R. Gjerde [6], respectively, that the dimension group $K_0(V,E)$ associated with a stationary, simple Bratteli diagram (V,E) has a unique state. Although E. G. Effros [2] showed how we can construct the unique state, he mentioned no reason why we must adopt the way of construction.

Under the assumption that $B = (V,E,\geq)$ is stationary, we shall compute $m(u)$ and $M(u)$. By definition of the stationarity, there is a constant $n \in \mathbb{N}$ such that $\sharp V_k = n$ for all $k \in \mathbb{N}$, and we may label the vertices in each V_k, say $(k,1), \dots, (k,n)$, so that the incidence matrices M_k are constant for all integers $k \geq 2$, where

$$(M_k)_{i,j} := \sharp\{e \in E_k : s(e) = (k-1,j),\ r(e) = (k,i)\} \quad \text{for } 1 \leq i,\ j \leq n.$$

Put $\alpha = M_1$, which is a column vector. Since (V,E) is simple, the matrix $M := M_2$ must be primitive. Hence, M has the so-called Perron–Frobenius eigenvalue, say θ, and corresponding left and right, positive eigenvectors β and

γ, respectively. We may assume that $\beta\gamma = 1$. By Perron–Frobenius Theorem; see for example [14], we have $\lim_{l\to\infty}(\theta^{-l}M^l)_{i,j} = \gamma_i\beta_j$ for any integers (i,j) with $1 \le i,j \le n$. Fix $k \in \mathbb{N}$ and $u := (k,i) \in V_k$. It follows that for any $l \in \mathbb{N}$ and any $v := (k+l,j) \in V_{k+l}$,

$$\frac{N(u,v)}{N(v_0,v)} = \frac{1}{\theta^{k-1}} \frac{(\theta^{-l}M^l)_{j,i}}{\sum_{h=1}^n (\theta^{-(k+l-1)}M^{k+l-1})_{j,h}\alpha_h}.$$

This implies that

$$m(u) = M(u) = \frac{\beta_i}{\theta^{k-1}\sum_{h=1}^n \beta_h\alpha_h}.$$

What we have shown is:

LEMMA 2.5. *Under the above setting, the Vershik map λ_B is uniquely ergodic, and the unique invariant probability measure assigns the value*

$$\frac{\beta_i}{\theta^{k-1}\sum_{h=1}^n \beta_h\alpha_h}$$

to a given cylinder set $C \subset X_B$, where $(k,i) = r(C)$.

2.3. Dimension groups. Given a properly ordered Bratteli diagram $B = (V,E,\ge)$, the dimension group $(K^0(X_B,\lambda_B), K^0(X_B,\lambda_B)^+)$ is order isomorphic to $(K_0(V,E), K_0(V,E)^+)$ by a map preserving the distinguished ordered units [10]. The dimension group is a complete invariant for the strong orbit equivalence class of (X_B,λ_B); see [4] for details. For $k \in \mathbb{Z}^+$, we define groups $G_k = \{(k,a) : a \in \mathbb{Z}^{\sharp V_k}\}$ with the addition $(k,a) + (k,b) = (k,a+b)$. For $k \in \mathbb{N}$, we define homomorphisms $f_k \colon G_{k-1} \to G_k, (k-1,a) \mapsto (k, M_k a)$, where M_k is the incidence matrix of B at the level k. Recall that the abelian group $K_0(V,E)$ is defined to be the inductive limit of the system of groups:

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} G_3 \xrightarrow{f_4} \cdots$$

For $k \in \mathbb{Z}^+$, let ψ_k denote the canonical morphism from G_k to $K_0(V,E)$. By definition, $K_0(V,E)^+ = \bigcup_{k=0}^\infty \psi_k(G_k^+)$, where $G_k^+ = \{(k,a) : a \in (\mathbb{Z}^+)^{\sharp V_k}\}$.

Suppose that B is stationary, so that M_k are constant for all $k \ge 2$. Let s denote the size of $M := M_2$. Following Sections 7.4 and 7.5 of [14], we set

$$\begin{aligned}
\mathcal{R}_M &= \bigcap_{k=1}^\infty M^k(\mathbb{Q}^s),\\
\Delta_M &= \{x \in \mathcal{R}_M : M^k x \in \mathbb{Z}^s \text{ for some } k \in \mathbb{Z}^+\},\\
\Delta_M^+ &= \{x \in \mathcal{R}_M : M^k x \in (\mathbb{Z}^+)^s \text{ for some } k \in \mathbb{Z}^+\}.
\end{aligned}$$

The abelian group Δ_M with the usual addition is a dimension group with a positive cone Δ_M^+. Notice that $\mathcal{R}_M = M^s(\mathbb{Q}^s)$, and the map $\mathcal{R}_M \to \mathcal{R}_M, x \mapsto Mx$ is both invertible and surjective. Then, the following lemma is readily verified.

Lemma 2.6. *Under the above setting,* $(K^0(X_B, \lambda_B), K^0(X_B, \lambda_B)^+)$ *is order isomorphic to* (Δ_M, Δ_M^+).

See also [14, Exercise 7.5.6] in connection with Lemma 2.6.

3. Hopf-equivalences

Definition 3.1. Let S be a homeomorphism on a totally disconnected, compact metric space X, and $C, D \subset X$ nonempty clopen sets.

(a) We say that C and D are *finitely Hopf-equivalent* if there are integers $\{n_i : 1 \le i \le k\}$ and decompositions

$$C = \bigcup_{i=1}^{k} C_i \quad \text{and} \quad D = \bigcup_{i=1}^{k} D_i$$

into nonempty clopen sets such that $S^{n_i}(C_i) = D_i$ for every integer $1 \le i \le k$.

(b) We say that C and D are *countably Hopf-equivalent* if there are integers $\{n_i : i \in \mathbb{Z}^+\}$ and decompositions

$$C = \bigcup_{i\in\mathbb{N}} C_i \cup \{x_0\} \quad \text{and} \quad D = \bigcup_{i\in\mathbb{N}} D_i \cup \{y_0\}$$

into nonempty clopen sets C_i, D_i and singletons $\{x_0\}$, $\{y_0\}$ such that

(i) $S^{n_0}(x_0) = y_0$, and $S^{n_i}(C_i) = D_i$ for every $i \in \mathbb{N}$,

(ii) the map $\alpha\colon C \to D$ defined by

$$\alpha(x) = \begin{cases} S^{n_i}(x) & \text{if } x \in C_i \text{ and } i \in \mathbb{N}, \\ y_0 & \text{if } x = x_0 \end{cases}$$

is a homeomorphism.

Remark 3.2. In [21], the notion of countable Hopf-equivalence is defined in a more general fashion: the clopen sets C and D are said to be countably Hopf-equivalent if there are partitions $\{C_i : i \in \mathbb{N}\}$ and $\{D_i : i \in \mathbb{N}\}$ into nonempty closed sets of C and D, respectively, and integers $\{n_i\}_{i\in\mathbb{N}}$ such that $S^{n_i}(C_i) = D_i$ for each $i \in \mathbb{N}$.

3.1. Chain recurrence, incompressibility and ordered group. Recall that a point x in a compact metric space X is said to be *chain recurrent* for a homeomorphism S on X if for any $\varepsilon > 0$, there are points $x_0 = x, x_1, \dots, x_{n-1}, x_n = x$ in X such that $d(Sx_i, x_{i+1}) < \varepsilon$ for every integer $0 \le i < n$, where d is a fixed metric on X.

PROPOSITION 3.3. *Suppose that S is a homeomorphism on a totally disconnected, compact metric space X. Then the following conditions are equivalent:*

(a) *any point in X is chain recurrent for S;*
(b) *there exists no clopen subset C of X such that $S(C) \subsetneq C$;*
(c) *there exists no clopen subset C of X which is finitely Hopf-equivalent to a proper subset of C;*
(d) *if $f \in C(X,\mathbb{Z})$ satisfies the property that $f(S(x)) \geq f(x)$ for any $x \in X$, then $f(S(x)) = f(x)$ for any $x \in X$;*
(e) *if C is a clopen subset of X whose characteristic function χ_C is a coboundary, then C is the empty set.*

PROOF. The equivalence of (a), (b) and (c) was proved by [21].

(b) $\Rightarrow$ (d) Let $f \in C(X,\mathbb{Z})$ be such that $f(S(x)) \geq f(x)$ for any $x \in X$. Set $f(X) = \{m_1 < m_2 < \ldots < m_k\}$. For $1 \leq i \leq k$, set $A_i = f^{-1}(m_i)$, which is clopen. Since $S(A_k) = \{x \in X : f(S^{-1}(x)) = m_k\}$ and $f(S(x)) \geq f(x)$ for any $x \in X$, we have $S(A_k) \subset A_k$ and so $S(A_k) = A_k$. Similar arguments show that $S(A_i) = A_i$ for all integers $1 \leq i \leq k$. Hence, f is S-invariant.

(d) $\Rightarrow$ (e) Suppose that $C \subset X$ is a clopen set such that $\chi_C = f \circ S - f$ for some $f \in C(X,\mathbb{Z})$. Condition (d) forces that f is S-invariant, and so $\chi_C = 0$.

(e) $\Rightarrow$ (b) Assume that (b) does not hold but (e) holds. There is a nonempty clopen set $C \subset X$ such that $S(C) \subsetneq C$. Then $\chi_{C\setminus S(C)}$ is a coboundary, a contradiction to (e). This completes the proof. $\square$

QUESTION. What dynamical property of a homeomorphism S on a totally disconnected, compact metric space X is equivalent to the nonexistence of clopen sets $C \subset X$ which are countably Hopf-equivalent to a proper subset of C?

In a measurable case, there is an answer due to E. Hopf [11]: a non-singular and bi-measurable transformation T on a non-atomic Lebesgue space has no measurable set C which is "countably Hopf-equivalent" to a proper subset of C if and only if T has an equivalent, finite invariant measure.

It was proved by [10, Proposition 5.1] that if S is an essentially minimal homeomorphism on a totally disconnected, compact metric space X, then the pair $(K^0(X,S), K^0(X,S)^+)$ is an ordered group with the distinguished order unit. This conclusion is still valid under a weaker assumption:

PROPOSITION 3.4. *If any point in a totally disconnected, compact metric space X is chain recurrent for a homeomorphism S on X, then the pair $(K^0(X,S)$, $K^0(X,S)^+)$ is an ordered group with the distinguished order unit.*

PROOF. It is sufficient to show that $K^0(X,S)^+ \cap (-K^0(X,S)^+) = \{0\}$. Suppose that the equivalence class of $f \in C(X,\mathbb{Z})$ belongs to $K^0(X,S)^+ \cap$

$(-K^0(X,S)^+)$. There exist $f_1, f_2 \in C(X,\mathbb{Z}^+)$ and $g_1, g_2 \in C(X,\mathbb{Z})$ such that

$$f = f_1 + g_1 \circ S - g_1 \quad \text{and} \quad f = -f_2 + g_2 \circ S - g_2.$$

Hence, $f_1 + f_2 = (g_2 - g_1) \circ S - (g_2 - g_1) \geq 0$.

Proposition 3.3 forces that $g_2 = g_1$, and so $f_1 = f_2 = 0$. This completes the proof. □

Question. The converse of Proposition 3.4 is also true?

3.2. Preliminaries for the next section. The following two lemmas will help us classify clopen sets in the Morse substitution system up to the Hopf-equivalences. For a Cantor minimal system (X,S), let $M(S)$ denote the S-invariant probability measures. E. Glasner and B. Weiss [7] obtained a criterion to determine whether given two clopen sets are countably Hopf-equivalent, or not:

Lemma 3.5 ([7]). *Suppose that (X,S) is a Cantor minimal system, and $C, D \subset X$ are nonempty clopen sets. Then, the clopen sets C and D are countably Hopf-equivalent if and only if $\mu(C) = \mu(D)$ for any $\mu \in M(S)$.*

The following lemma is a criterion to determine whether given clopen sets are finitely Hopf-equivalent, or not. Although it has been implicitly used in several literatures, we provide it with a proof.

Lemma 3.6. *Suppose that $B = (V, E, \geq)$ is a properly ordered Bratteli diagram, and $C, D \subset X_B$ are nonempty clopen sets. Then the following conditions are equivalent:*

(a) *C and D are finitely Hopf-equivalent;*
(b) *$\chi_C - \chi_D$ is a coboundary;*
(c) *there is $n \in \mathbb{N}$ such that for any $v \in V_n$,*

$$\sharp\{p \in P_n : r(p_n) = v, [p] \subset C\} = \sharp\{p \in P_n : r(p_n) = v, [p] \subset D\},$$

where $P_n = E_1 \circ \ldots \circ E_n$.

Proof. (a) ⇒ (b) Let $\{n_i\}_{i=1}^k$, $\{C_i\}_{i=1}^k$, $\{D_i\}_{i=1}^k$ be as in Definition 3.1(a). We have

$$\chi_C - \chi_D = \sum_{i=1}^{k} (\chi_{C_i} - \chi_{C_i} \circ {\lambda_B}^{-n_i}).$$

However, for any $f \in C(X_B, \mathbb{Z})$ and any $n \in \mathbb{Z}$,

$$f - f \circ {\lambda_B}^n = \begin{cases} \displaystyle\sum_{i=0}^{n-1} f \circ {\lambda_B}^i - \left(\sum_{i=0}^{n-1} f \circ {\lambda_B}^i\right) \circ \lambda_B & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \displaystyle\sum_{i=0}^{-n-1} f \circ {\lambda_B}^{-i} - \left(\sum_{i=0}^{-n-1} f \circ {\lambda_B}^{-i}\right) \circ \lambda_B & \text{if } n < 0. \end{cases}$$

Hence, $\chi_C - \chi_D$ is a coboundary.

(b) $\Rightarrow$ (c) There is $f \in C(X_B, \mathbb{Z})$ such that $\chi_C - \chi_D = f \circ \lambda_B - f$. Take $m \in \mathbb{N}$ so that the functions f, χ_C and χ_D are all constant on each cylinder set of length m. Since X_B has a unique, minimal infinite path $x_{\min}$, we can find $n \in \mathbb{N}$ with $n > m$ such that if a path $(p_{m+1}, \dots, p_n) \in E_{m+1} \circ \dots \circ E_n$ is such that every p_i is minimal in E_i, then $s(p_{m+1}) = r(x_{\min}(m))$. For $v \in V_n$, let p_v denote the path in $E_1 \circ \dots \circ E_n$ terminating at v such that for every integer $1 \leq i \leq n$, the i-th edge $(p_v)_i$ is minimal in E_i. For $v \in V_n$, set $n_v = \sharp\{p \in P_n : r(p_n) = v\}$, and fix a point $x_v \in [p_v] \subset [(x_{\min}(1), \dots, x_{\min}(m))]$. Since ${\lambda_B}^{n_v}(x_v) \in [(x_{\min}(1), \dots, x_{\min}(m))]$, we have

$$\sum_{i=0}^{n_v - 1} \{\chi_C \circ {\lambda_B}^i(x_v) - \chi_D \circ {\lambda_B}^i(x_v)\} = f({\lambda_B}^{n_v}(x_v)) - f(x_v) = 0.$$

This implies the conclusion.

(c) $\Rightarrow$ (a) For $n \in \mathbb{N}$ and $v \in V_n$, set

$$\begin{aligned} C_v &= \bigcup\{[p] \subset C : p \in P_n, r(p_n) = v\}, \\ D_v &= \bigcup\{[p] \subset D : p \in P_n, r(p_n) = v\}. \end{aligned}$$

By the assumption, C_v and D_v are finitely Hopf-equivalent, and so are C and D, because $C = \bigcup_{v \in V_n} C_v$ and $D = \bigcup_{v \in V_n} D_v$ are disjoint unions. This completes the proof. $\square$

REMARK 3.7. As a consequence of the equivalence of (a) and (c) in Lemma 3.6, it follows that for a properly ordered Bratteli diagram B, the finite Hopf-equivalences for the Vershik map λ_B and for the *finite coordinate change relation* R [8] on X_B are equivalent. The relation R is called the cofinal relation in [4]. For infinite paths $x, y \in X_B$, we define $(x, y) \in R$ if x and y have the same tail, i.e. $x(n) = y(n)$ for all sufficiently large $n \in \mathbb{N}$. If clopen sets $C, D \subset X_B$ are finitely Hopf-equivalent for $S := \lambda_B$ as in Definition 3.1(a), and if for each integer $1 \leq i \leq k$, the map $C_i \to D_i$, $x \mapsto S^{n_i}x$ in Definition 3.1(a) implements $(x, S^{n_i}x) \in R$ for any $x \in C_i$, then C and D are said to be finitely Hopf-equivalent for R. As a consequence of this equivalence, it also follows that the countable Hopf-equivalences for λ_B and R are equivalent, because we are allowed to choose both of the points $x_0 \in C$ and $y_0 \in D$ in Definition 3.1(b) from a single equivalence class of R, in view of the proof of Lemma 3.5 by E. Glasner and B. Weiss [7].

3.3. Complete characterizations of orbit structures. Lemmas 3.5 and 3.6 help us show that the Hopf-equivalences completely determine orbit structures of a Cantor minimal system:

PROPOSITION 3.8. *Suppose that (X,S) and (Y,T) are Cantor minimal systems. Then the following two statements hold.*

(a) *(X,S) and (Y,T) are strong orbit equivalent if and only if there is a homeomorphism $F\colon X \to Y$ which respects the finite Hopf-equivalence.*

(b) *(X,S) and (Y,T) are orbit equivalent if and only if there is a homeomorphism $F\colon X \to Y$ which respects the countable Hopf-equivalence.*

PROOF. (a) Let $F\colon X \to Y$ be an orbit map implementing the strong orbit equivalence of (X,S) and (Y,T). We have the associated orbit cocycles $m,n\colon X \to \mathbb{Z}$, i.e. $F \circ S(x) = T^{n(x)} \circ F(x)$ and $F \circ S^{m(x)}(x) = T \circ F(x)$ for all $x \in X$, such that a point $x_1 \in X$ is the possible discontinuity point of m and n. Suppose that clopen sets $C, D \subset X$ are finitely Hopf-equivalent. We shall show that $F(C)$ and $F(D)$ are finitely Hopf-equivalent. We may assume $C \cap D = \emptyset$. Since one of C and D does not contain x_1, the continuity of n except x_1 implies that $F(D)$ and $F(C)$ are finitely Hopf-equivalent. Similar arguments involving m show that if $F(C)$ and $F(D)$ are finitely Hopf-equivalent, then so are C and D.

Conversely, if a homeomorphism $F\colon X \to Y$ respects the finite Hopf-equivalence, then F induces the bijection $F_*\colon \partial_S C(X,\mathbb{Z}) \to \partial_T C(Y,\mathbb{Z}), f \mapsto f \circ F^{-1}$, where $\partial_S(f) = f \circ S - f$. Hence, F_* induces an order isomorphism from $K^0(X,S)$ to $K^0(Y,T)$ which preserves the distinguished order units; see also [7, Theorem 1.1]. Then the strong orbit equivalence theorem [4, Theorem 2.1] leads us to the conclusion.

(b) Let $F\colon X \to Y$ be an orbit map. Since F induces a bijective correspondence between $M(S)$ and $M(T)$, it follows that for clopen sets $C, D \subset X$, $\mu(C) = \mu(D)$ for any $\mu \in M(S)$ if and only if $\nu(F(C)) = \nu(F(D))$ for any $\nu \in M(T)$. Lemma 3.5 shows that F respects the countable Hopf-equivalence.

Conversely, if a homeomorphism $F\colon X \to Y$ respects the countable Hopf-equivalence, then the map $M(S) \to M(T), \mu \mapsto \mu \circ F^{-1}$ is both well-defined and bijective. Hence, the orbit equivalence theorem [4, Theorem 2.2] shows that (X,S) and (Y,T) are orbit equivalent. This completes the proof. $\square$

4. The Morse substitution

4.1. A Bratteli–Vershik model of the Morse substitution system. We again use definitions and notation of [1]. Let A be an alphabet of two letters, say $A = \{a,b\}$. Let σ denote the Morse substitution on A, that is, $\sigma(a) = ab$ and $\sigma(b) = ba$. Put $x = \lim_{n\to\infty} \sigma^{2n}(\dots bbbb \mid aaaa \dots)$. Since there is no nonempty word u such that uuu_1 occurs in x [18, Proposition 5.1.6], the subshift X_σ contains no periodic point. We shall construct a stationary, properly ordered Bratteli diagram B with (X_B, λ_B) conjugate to (X_σ, T_σ) as in Subsection 2.1.

LEMMA 4.1. *The return words to $b.a$ in x are*

$$w_1 := abb, \quad w_2 := ab, \quad w_3 := aabb, \quad \text{and} \quad w_4 := aab.$$

PROOF. Since the word ba occurs in each of $\sigma^2(a)$ and $\sigma^2(b)$, and since $\sigma^2(x) = x$, all the return words to $b.a$ occur in some word of the form $\sigma^2(cd)$ for $c, d \in A$. This leads us to the conclusion. □

Since

$$\begin{aligned}
\sigma^2(w_1) &= abbabaabbaab = w_1w_2w_3w_4,\\
\sigma^2(w_2) &= abbabaab = w_1w_2w_4,\\
\sigma^2(w_3) &= abbaabbabaabbaab = w_1w_3w_2w_3w_4,\\
\sigma^2(w_4) &= abbaabbabaab = w_1w_3w_2w_4,
\end{aligned}$$

we define a proper and primitive substitution $\tau\colon R \to R^+$, where $R := \{1, 2, 3, 4\}$, by

$$\tau(1) = 1234, \quad \tau(2) = 124, \quad \tau(3) = 13234, \quad \text{and} \quad \tau(4) = 1324.$$

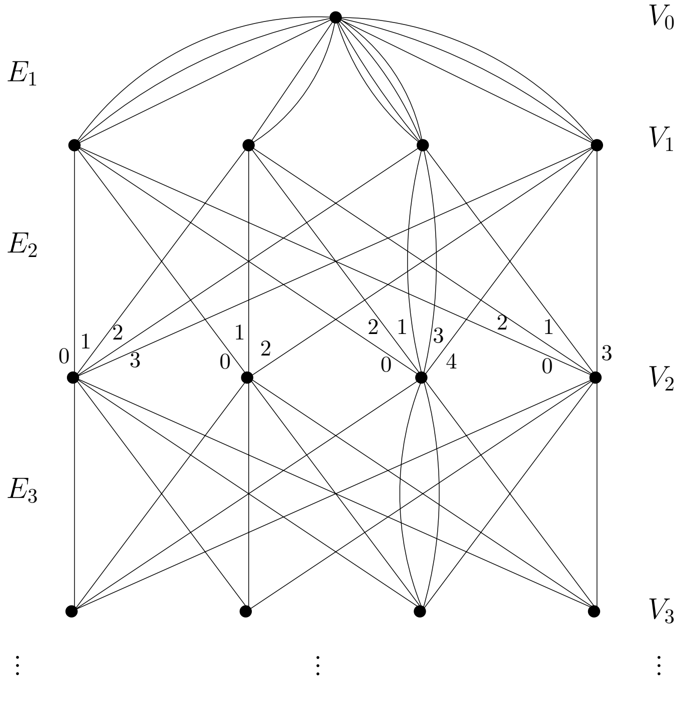

FIGURE 1. A stationary, properly ordered Bratteli diagram B

Consequently, we obtain the following.

PROPOSITION 4.2. *Let $B = (V, E, \geq)$ denote the stationary, properly ordered Bratteli diagram associated with Kakutani–Rohlin partitions of (X_σ, T_σ):*

$$\mathcal{P}_k = \{{T_\sigma}^j[u_k.w_{k,i}v_k] : 0 \leq j < |w_{k,i}|,\ i \in R\}, \quad k \in \mathbb{N},$$

where $u_k = \sigma^{2(k-1)}(b)$, $v_k = \sigma^{2(k-1)}(a)$ and $w_{k,i} = \sigma^{2(k-1)}(w_i)$. Then, the Bratteli–Vershik system (X_B, λ_B) is conjugate to (X_σ, T_σ).

PROOF. See Subsection 2.1. □

Figure 1 shows the diagram B, where the partial order $\geq$ is indicated by the numerical order by edges.

4.2. Hopf-equivalences in (X_B, λ_B). We continue to consider the stationary, properly ordered Bratteli diagram $B = (V, E, \geq)$. For each $k \in \mathbb{N}$, we shall label the vertices in V_k $(k,1)$, $(k,2)$, $(k,3)$ and $(k,4)$ from the leftmost one. The incidence matrices $\{M_k\}_{k\in\mathbb{N}}$ of B are

$$M_1 = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad M_k = M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (k \geq 2),$$

where $M_{i,j} := \sharp\{e \in E_2 : s(e) = (1, j),\ r(e) = (2, i)\}$ for all $i, j \in R$. The Perron–Frobenius eigenvalue of M is 4, and a corresponding left, positive eigenvector is $\beta := [1, 1, 1, 1]$. By Lemma 2.5, we obtain:

LEMMA 4.3. *Let μ be the unique λ_B-invariant probability measure. Then, for a cylinder set $C \subset X_B$, we have $\mu(C) = 1/(3 \cdot 2^{2k})$, where k is the length of C.*

We are now in a position to classify the cylinder sets in X_B up to the finite and countable Hopf-equivalences, respectively.

THEOREM 4.4. *Suppose that $C, D \subset X_B$ are cylinder sets. We may write $r(C) = (m, i)$ and $r(D) = (n, j)$ for some $(m, i), (n, j) \in \mathbb{N} \times R$. Then the following two statements hold.*

(a) *C and D are countably Hopf-equivalent if and only if $m = n$.*
(b) *C and D are finitely Hopf-equivalent if and only if $m = n$ and one of the following exclusive conditions holds:*
  (i) *$i \neq 3$ and $j \neq 3$,*
  (ii) *$i = j = 3$.*

PROOF. (a) It is obvious from Lemmas 3.5 and 4.3.

(b) The eigenvalues of M are 0, 1, and 4 whose multiplicities are 2, 1, and 1, respectively. Corresponding right eigenvectors are respectively

$$\alpha_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \alpha_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \beta := \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \gamma := \begin{bmatrix} 3 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

These vectors constitute a basis of $\mathbb{R}^4$. Let $\{u_1, u_2, u_3, u_4\}$ denote the standard basis of $\mathbb{R}^4$. By Lemma 3.6, C and D are finitely Hopf-equivalent if and only if $m = n$ and there is $k \in \mathbb{N}$ such that $M^k(u_i - u_j) = 0$. Since for each $k \in \mathbb{N}$,

$$\begin{aligned} M^k u_1 &= \frac{1}{3}\beta + \frac{4^k}{12}\gamma = \frac{1}{3}(\beta + 4^{k-1}\gamma), \\ M^k u_2 &= \frac{1}{3}\beta + \frac{4^k}{12}\gamma = \frac{1}{3}(\beta + 4^{k-1}\gamma), \\ M^k u_3 &= -\frac{2}{3}\beta + \frac{4^k}{12}\gamma = \frac{1}{3}(-2\beta + 4^{k-1}\gamma), \\ M^k u_4 &= \frac{1}{3}\beta + \frac{4^k}{12}\gamma = \frac{1}{3}(\beta + 4^{k-1}\gamma), \end{aligned}$$

we obtain the conclusion. □

The following is an immediate consequnece of Theorem 4.4.

THEOREM 4.5. *Suppose that $C, D \subset X_B$ are cylinder sets. We may write $r(C) = (m, i)$ and $r(D) = (n, j)$ for some $(m, i), (n, j) \in \mathbb{N} \times R$. Then the following conditions are equivalent:*

(a) *C and D are not finitely but countably Hopf-equivalent;*
(b) *$m = n$, and moreover, one of i and j does not equal* 3 *but so does another.*

4.3. The dimension group $K^0(X_\sigma, T_\sigma)$. In view of Lemma 2.6, we shall compute (Δ_M, Δ_M^+). It is easy to show that $\mathcal{R}_M = \{s\beta + t\gamma : s, t \in \mathbb{Q}\}$. Assume that $s\beta + t\gamma \in \Delta_M$ for some $s, t \in \mathbb{Q}$. It is necessary that there is $k \in \mathbb{N}$ such that

$$3 \cdot 2^{2k} t \in \mathbb{Z}, \tag{4.1}$$

$$s + 2^{2k+1} t \in \mathbb{Z}, \tag{4.2}$$

$$-s + 2^{2k+2} t \in \mathbb{Z}. \tag{4.3}$$

By (4.2) $\times$ 2 $-$ (4.3), we obtain $s \in (1/3)\mathbb{Z}$. Since (4.2) implies that

$$2^{2k+1}(t - s) + (2^{2k+1} + 1)s \in \mathbb{Z},$$

and since $2^{2k+1}+1$ is a multiple of 3, we obtain $t-s \in \mathbb{Z}[1/2]$, where $\mathbb{Z}[1/2] = \{m2^{-n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$. What we have shown is the implication that for $s, t \in \mathbb{Q}$,

$$s\beta + t\gamma \in \Delta_M \Rightarrow s \in \frac{1}{3}\mathbb{Z},\ t-s \in \mathbb{Z}[1/2].$$

It is straightforward to prove that the converse implication also holds. We obtain

$$\Delta_M = \left\{ s\beta + t\gamma : s \in \frac{1}{3}\mathbb{Z},\ t-s \in \mathbb{Z}[1/2] \right\}.$$

The condition that $t = 0 \Rightarrow s = 0$ is necessary for a pair $(s,t) \in \mathbb{Q}^2$ with both $s \in (1/3)\mathbb{Z}$ and $t - s \in \mathbb{Z}[1/2]$ to satisfy the condition that

$$M^k(s\beta + t\gamma) = \begin{bmatrix} 3 \cdot 2^{2k} t \\ s + 2^{2k+1} t \\ -s + 2^{2k+2} t \\ 3 \cdot 2^{2k} t \end{bmatrix} \in (\mathbb{Z}^+)^4 \quad \text{for some } k \in \mathbb{N}.$$

It is clear that $M^k(s\beta + t\gamma) \in (\mathbb{Z}^+)^4$ for some $k \in \mathbb{N}$ whenever $t > 0$. We hence obtain $\Delta_M^+ \subset \{s\beta + t\gamma \in \Delta_M : t > 0\} \cup \{0\}$. It is easy to show that the converse inclusion relation also holds. We obtain

$$\Delta_M^+ = \{s\beta + t\gamma \in \Delta_M : t > 0\} \cup \{0\}.$$

The order isomorphism $f\colon K_0(V,E) \to \Delta_M$, which was defined in Lemma 2.6, maps the distinguished order unit of $K_0(V,E)$ to γ. It also follows that for all $(k,i) \in \mathbb{N} \times R$,

$$f(k,u_i) = \begin{cases} \dfrac{1}{3}\left(\beta + \dfrac{1}{4^k}\gamma\right) & \text{if } i \neq 3, \\ \dfrac{1}{3}\left(-2\beta + \dfrac{1}{4^k}\gamma\right) & \text{if } i = 3. \end{cases}$$

What we have shown is:

Proposition 4.6. *The dimension group $(K^0(X_\sigma, T_\sigma), K^0(X_\sigma, T_\sigma)^+)$ is order isomorphic to (Δ_M, Δ_M^+) by a map which maps the distinguished order unit of $K^0(X_\sigma, T_\sigma)$ to γ, where*

$$\begin{aligned} \Delta_M &= \left\{ s\beta + t\gamma : s \in \frac{1}{3}\mathbb{Z},\ t-s \in \mathbb{Z}[1/2] \right\}, \\ \Delta_M^+ &= \{s\beta + t\gamma \in \Delta_M : t > 0\} \cup \{0\}, \\ \beta &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 3 \end{bmatrix}. \end{aligned}$$

Furthermore, for $k \in \mathbb{N}$, $i \in R$ and $0 \leq j < |w_{k,i}|$, the element of Δ_M whose representative is the characterisric function of the cylinder set ${T_\sigma}^j B_{k,i}$ is $3^{-1}(\beta + \gamma/4^k)$ if $i \neq 3$; otherwise, $3^{-1}(-2\beta + \gamma/4^k)$.

We shall compute the infinitesimal subgroup $\mathrm{Inf}(\Delta_M)$ of Δ_M. By [4, Definition 1.10],

$$\mathrm{Inf}(\Delta_M) = \{\alpha \in \Delta_M : -n\gamma \le m\alpha \le n\gamma \text{ for any integers } m > n > 0\},$$

where $a \le b$ means $b - a \in \Delta_M^+$. Suppose $\alpha := s\beta + t\gamma \in \mathrm{Inf}(\Delta_M)$. If $t > 0$, then $n\gamma - m\alpha \notin \Delta_M^+$ for some integers $m > n > 0$. Hence, t is necessarily zero, and so $\mathrm{Inf}(\Delta_M) \subset \{s\beta : s \in (1/3)\mathbb{Z} \cap \mathbb{Z}[1/2]\} = \{s\beta : s \in \mathbb{Z}\}$. It is easy to show that the converse inclusion relation also holds, and so

$$\mathrm{Inf}(\Delta_M^+) = \{s\beta : s \in \mathbb{Z}\}.$$

The quotient $(\Delta_M/\mathrm{Inf}(\Delta_M), \Delta_M^+/\mathrm{Inf}(\Delta_M))$ with the order unit $[\gamma]$ is order isomorphic to $((1/3)\mathbb{Z}[1/2], (1/3)\mathbb{Z}[1/2] \cap \mathbb{R}^+)$ with the order unit 1 by a map which maps $[\gamma]$ to 1, where $[\gamma]$ denotes the equivalence class of γ, and $\mathbb{R}^+$ is the set of nonnegative real numbers. By [4, Theorem 2.2], the Morse substitution system (X_σ, T_σ) is orbit equivalent to the odometer with stationary base $(3, 2, 2, \dots)$, which was shown in [20].

We shall determine the order automorphisms δ on (Δ_M, Δ_M^+) which preserve the distinguished order unit γ. Thus, $\delta(\gamma) = \gamma$. Since the restriction of δ onto $\mathrm{Inf}(\Delta_M)$ is an automorphism of $\mathrm{Inf}(\Delta_M)$, one of $\delta(\beta) = \beta$ and $\delta(\beta) = -\beta$ must hold. Hence, the order automorphisms are $\{\mathrm{id}_{\Delta_M}, \delta\}$, where $\delta\colon \Delta_M \to \Delta_M$, $s\beta + t\gamma \mapsto -s\beta + t\gamma$.

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