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GLOBAL REGULAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN AN AXIALLY SYMMETRIC DOMAIN

Wojciech M. Zajączkowski

Abstract. We prove the existence of global regular solutions to the Navier–Stokes equations in an axially symmetric domain in $\mathbb{R}^3$ and with boundary slip conditions. We assume that initial angular component of velocity and angular component of the external force and angular derivatives of the cylindrical components of initial velocity and of the external force are sufficiently small in corresponding norms. Then there exists a solution such that velocity belongs to $W^{2,1}_{5/2}(\Omega^T)$ and gradient of pressure to $L_{5/2}(\Omega^T)$, and we do not have restrictions on T.

1. Introduction

We examine the following problem

$$
\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\overline{n} = 0 && \text{on } S^T = S\times(0,T),\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\
&v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

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where $\Omega \subset \mathbb{R}^3$ is a bounded axially symmetric domain with the boundary S.

By $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ we denote the velocity of the fluid, $p \in \mathbb{R}$ the pressure, $f = (f_1, f_2, f_3) \in \mathbb{R}^3$ the external force, $\overline{n}$ is the unit outward vector normal to S, $\overline{\tau}_\alpha$, $\alpha = 1, 2$, are tangent to S, γ is the constant slip coefficient. Moreover, the dot denotes the scalar product in $\mathbb{R}^3$.

By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI,$$

where ν is the constant viscosity coefficient, $\mathbb{D}(v)$ the dilatation tensor of the form

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3},$$

and I is the unit matrix.

Let (x_1, x_2, x_3) be a local Cartesian system such that the x_3 axis is the axis of symmetry of Ω. Let (r, φ, z) be the cylindrical coordinates such that $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$.

Let $\overline{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\overline{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, $\overline{e}_z = (0,0,1)$ be vectors connected with cylindrical coordinates r, φ, z, respectively. Let u be any vector. Then cylindrical coordinates of u are denoted by $u_r = u \cdot \overline{e}_r$, $u_\varphi = u \cdot \overline{e}_\varphi$, $u_z = u \cdot \overline{e}_z$.

Let $R > 0$, $a > 0$ be given numbers. We assume that Ω is axially symmetric and is located in the rectangle $r \in [0, R]$, $z \in [-a, a]$, S is described by the relation $\psi(r,z) = 0$ which meets the x_3 axis in two points only: $z = -a$ and $z = a$.

Now we formulate the main result of this paper. Let

$$g = f_{r,\varphi}\overline{e}_r + f_{\varphi,\varphi}\overline{e}_\varphi + f_{z,\varphi}\overline{e}_z, \quad h = v_{r,\varphi}\overline{e}_r + v_{\varphi,\varphi}\overline{e}_\varphi + v_{z,\varphi}\overline{e}_z,$$

and

$$\begin{aligned} X_1(T) = {} & \|g\|_{L_2(0,T;L_{6/5}(\Omega))} + \|g\|_{L_{2,-(1+\varepsilon_*)}(\Omega^T)} + \|f_\varphi\|_{L_{2,1-\mu}(\Omega^T)} \\ & + \|h(0)\|_{H^1_{-(1+\varepsilon_*)}(\Omega)} + \|v_\varphi(0)\|_{H^1_{1-\mu}(\Omega)}, \end{aligned}$$

$\mu \in (0,1)$, $\varepsilon_* \in (0,1)$ and ε_* can be chosen arbitrary small. The above and below introduced spaces are defined in Section 2.

Let $F = \operatorname{rot} f$, $F' = F_r\overline{e}_r + F_z\overline{e}_z$, $\alpha = \operatorname{rot} v$, $\alpha' = \alpha_r\overline{e}_r + \alpha_z\overline{e}_z$,

$$\begin{aligned} X_2(T) = {} & \|F'\|_{L_2(0,T;L_{6/5}(\Omega))} + \|F_r\|_{L_2(0,T;L_{6/5,-\mu}(\Omega))} + \|F'\|_{L_{2,1-\mu}(\Omega^T)} \\ & + \|\alpha'(0)\|_{L_2(\Omega)} + \|\alpha_r(0)\|_{L_{2,-\mu}(\Omega)} + \|\alpha'(0)\|_{W^1_{2,1-\mu}(\Omega)}, \\ Y_1(T) = {} & \|F_\varphi\|_{L_2(0,T;L_{6/5,-1}(\Omega))} + \|\alpha_\varphi(0)\|_{L_{2,-1}(\Omega)}, \\ d_2(T) = {} & (1+T)(\|f\|_{L_1(0,T;L_2(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \equiv (1+T)d_1, \\ Y_2(T) = {} & \|f\|_{L_{5/2}(\Omega^T)} + \|v(0)\|_{W^{6/5}_{5/2}(\Omega)}, \end{aligned}$$

$$K_* = \left\| \frac{1}{r}\left(k - \frac{\gamma}{2\nu}\right)\right\|_{W^1_\infty(S)},$$

where k is the curvature of the curve $S \cap P$, where P is the plane passing through the axis of symmetry of Ω.

Theorem 1.1 (Existence). *Let $X_1(T) < \infty$, $Y_1(T) < \infty$, $d_2(T) < \infty$, $K_* < \infty$. Let*

$$A = 2\sigma[\varphi_2^2(X_1)Y_1^2 + c_1K_*(K_*+1)(d_2^2 + X_1^2)] + cY_2,$$

where $\sigma > 2$, φ_2 is an increasing positive function and c_1 is the constant from (3.56)*. Let $X_3 = X_1 + X_2$ be so small that*

$$2\varphi_1^2(T, X_1, A)X_3^2 \le \left(1 - \frac{1}{\sigma}\right)A,$$

where φ_1 is an increasing positive function. Then there exists a solution to problem (1.1) *such that $v \in W^{2,1}_{5/2}(\Omega^T)$, $\nabla p \in L_{5/2}(\Omega^T)$ and*

$$\|v\|_{W^{2,1}_{5/2}(\Omega^T)} + |\nabla p|_{5/2,\Omega^T} \le A,$$

The existence of global weak solutions to the Navier–Stokes equations for the Cauchy problem and the Dirichlet–Cauchy problem was proved long time ago (J. Leray [8] (1933) and E. Hopf [3] (1951)) (see [7]).

A similar result can be also proved for problem (1.1). However, up to now, we do not know how to increase regularity of the weak solutions by assuming sufficiently regular data. Therefore many mathematicians tried to prove the existence of global regular solutions to the Navier–Stokes equations by imposing some geometrical restrictions on solutions of the considered initial-boundary value problems.

We recall the results:

(1) two-dimensional problems [5];
(2) axially symmetric solutions — $v_{r,\varphi} = v_{z,\varphi} = p_{,\varphi} = 0$, v_φ (see notation below)
  - in axially symmetric domains [6],
  - in all space [14];
(3) helically symmetric solutions [9];
(4) existence in thin domains $\Omega = \Omega' \times [0,\varepsilon]$, $\Omega' \subset \mathbb{R}^2$, ε-small [11], [12].

In view of the above results we considered the existence of global regular solutions to the Navier–Stokes equations which are close either to two-dimensional or to axially symmetric solutions. In [17] we proved the existence of global regular solutions to problem (1.1) by assuming that Ω is an axially symmetric cylinder and

$$f_{r,\varphi}, f_{\varphi,\varphi}, f_{z,\varphi}, v_r(0)_{,\varphi}, v_\varphi(0)_{,\varphi}, v_z(0)_{,\varphi} \text{ and } v_\varphi(0) \tag{1.2}$$

are sufficiently small in some norms.

Under (1.2) and for an axially symmetric domain we proved global regular solutions to (1.1) in [20]. The existence of solutions in [20], [17] was proved by the method of successive approximation. We have to underline that the main step in proofs of existence in [6], [14], [17], [20] is the proof of estimate and existence of solutions to problem for $\chi = (\operatorname{rot} v)_\varphi$ with appropriate boundary conditions which are such that $\chi|_S$ depends on $v|_S$ only (not on $v_{,x}|_S$).

The appropriate boundary conditions follow from the slip boundary conditions $(1.1)_{3,4}$. This is the main reason why problems with slip boundary conditions are only considered. We are not able to prove the existence of global regular solutions for non-slip boundary condition: $v|_S = 0$.

In [17] $\chi|_S = 0$ because Ω is axially symmetric cylinder. Moreover, in [20] to show a global estimate we had to assume that $k - \gamma/2\nu$ must be small in some norms. In this paper we were able to omit the restriction. However, we use some estimates from [20] the proof of existence in this paper is much more simpler and elegant because the Leray–Schauder fixed point theorem was applied. We have to underline that the proof of existence in this paper is essentially different from the proof from [20] however some points are the same. We should mention that the condition: v_φ is small in (1.2) is not natural for axially symmetric solutions. In [15], [18], [19], [26] we tried to understand and relax it. But the existence of global regular axially symmetric solutions to the Navier–Stokes equations with large swirl (large v_φ near the axis of symmetry) is still an open problem.

In [13], [21], [22] we proved the existence of solutions which are close to $2d$ solutions. The solutions are such that Ω is a cylinder with x_3 axis and

$$v_{,x_3},\ p_{,x_3} \text{ are small in some norms.}$$

In [22] we proved the existence with inflow and outflow but in [21] with impermeable boundary.

In [13], [21], [22] the existence is proved by the Leray-Schauder fixed point theorem. In [21], [22] the existence is proved in Besov spaces but in [4], [13], the proofs from [21], [22] were simplified by using Sobolev spaces only.

We have to underline that the main step in proofs of global existence in [21], [22], [13], [4] was the estimate for the component $\chi = (\operatorname{rot} v)_3$ of vorticity, so the slip boundary conditions must be also assumed.

In [25] stability of arbitrary linear combination of $2d$ and axially symmetric solutions to the Navier–Stokes equations in a cylinder is proved.

The paper is organized in the following way. In Section 2 we introduce the notation, recall the results on existence of solutions to the heat equation and to the Stokes system in weighted Sobolev spaces and define problems for $\alpha = \operatorname{rot} v, h, q$ (see (2.7)).

We replace problem (1.1) by a system of problems (2.8)–(2.14) because the Leary–Schauder fixed point theorem can not be applied directly to (1.1). Problems (2.8)–(2.14) generate a mapping Φ which fixed points are solutions to problem (1.1).

The smallness restriction on $k-\gamma/2\nu$ assumed in [20] is relaxed in this paper by applying Lemma 2.6.

In Section 3 we show all inequalities which are necessary to obtain an estimate for a fixed point of mapping Φ. However, to obtain the estimate we need that quantity $X_2(T)$ (see Theorem 1.1) must be sufficiently small.

We have to underline that the crucial inequality (3.1) implies that weighted Sobolev spaces must be used.

In Section 4 we show that Φ is compact but in Section 5 that it is uniformly continuous.

Hence the assumptions of the Leray-Schauder fixed point theorem are satisfied so the existence of solutions to (1.1) follows.

2. Notation and auxiliary results

By c we denote the generic constant. By $c(\sigma)$ we denote the generic function which is always positive and increasing.

To simplify considerations we introduce

$$\begin{aligned}
|u|_{p,Q} &= \|u\|_{L_p(Q)}, && Q\in\{\Omega,S,\Omega^T,S^T\},\ p\in[1,\infty],\\
\|u\|_{s,Q} &= \|u\|_{H^s(Q)}, && Q\in\{\Omega,S\},\ s\in\mathbb{R}_+\cup\{0\},\\
\|u\|_{s,Q} &= \|u\|_{H^{s,s/2}(Q)}, && Q\in\{\Omega^T,S^T\},\ s\in\mathbb{R}_+\cup\{0\}\\
& && \text{and } \|u\|_{0,Q}=|u|_{2,Q},\\
|u|_{p,q,\Omega^T} &= \bigg[\int_0^T dt\bigg(\int_\Omega |u(x,t)|^p\,dx\bigg)^{q/p}\bigg]^{1/q}, && p,q\in[1,\infty],\\
|u|_{p,q,\mu,\Omega^T} &= \bigg(\int_0^T dt\|u(t)\|^q_{L_{p,\mu}(\Omega)}\bigg)^{1/q}, && p,q\in[1,\infty],\ \mu\in\mathbb{R}.
\end{aligned}$$

Let us introduce the energy norm

$$\|v\|_{V_2^s(\Omega^T)}=\operatorname*{ess\,sup}_{t\le T}\|v(t)\|_{H^s(\Omega)}+\bigg(\int_0^t\|\nabla v(t)\|^2_{H^s(\Omega)}\,dt\bigg)^{1/2},\quad 0\le s\in\mathbb{N}\cup\{0\}.$$

Now we introduce weighted spaces

$$\|u\|_{L_{p,\mu}(Q)}=\bigg(\int_Q |u|^p r^{p\mu}dQ\bigg)^{1/p},\quad p\in[1,\infty],\ \mu\in\mathbb{R},\ Q\in\{\Omega,S,\Omega^T,S^T\},$$

where dQ is the measure connected with the set Q, with the notation

$$|u|_{p,\mu,Q}=\|u\|_{L_{p,\mu}(Q)}.$$

Let us define $H^s_\mu(Q)$ for $Q\in\{\Omega,S\}$, $s\in\mathbb{Z}_+\cup\{0\}$, $\mu\in\mathbb{R}$ by

$$\|u\|_{H^s_\mu(Q)}=\bigg(\sum_{|\alpha|\le s}\int_Q|D^\alpha_x u|^2r^{2(\mu-s+|\alpha|)}\,dQ\bigg)^{1/2}<\infty,$$

where $L_{2,\mu}(\Omega)=H^0_\mu(\Omega)$ and $H^{s,s/2}_\mu(Q)$ for $Q\in\{\Omega^T,S^T\}$, $s\in\mathbb{Z}_+\cup\{0\}$, $\mu\in\mathbb{R}$ by

$$\|u\|_{H^{s,s/2}_\mu(Q)}=\bigg(\sum_{|\alpha|+2i\le s}\int_Q|D^\alpha_x\partial^i_t u|^2r^{2(\mu-s+|\alpha|+2i)}\,dQ\bigg)^{1/2}<\infty.$$

To simplify notation we introduce

$$\|u\|_{s,\mu,Q}=\|u\|_{H^s_\mu(Q)}\quad\text{for } Q\in\{\Omega,S\}$$

and

$$\|u\|_{s,\mu,Q}=\|u\|_{H^{s,s/2}_\mu(Q)}\quad\text{for } Q\in\{\Omega^T,S^T\}.$$

Similarly we introduce spaces $V^s_{p,\mu}(Q)$ by

$$\|u\|_{V^s_{p,\mu}(Q)}=\bigg(\sum_{|\alpha|\le s}\int_Q|D^\alpha_x u|^pr^{p(\mu-s+|\alpha|)}\,dQ\bigg)^{1/p}\quad\text{for } Q\in\{\Omega,S\}$$

and

$$\|u\|_{V^{s,s/2}_{p,\mu}(Q)}=\bigg(\sum_{|\alpha|+2i\le s}\int_Q|D^\alpha_x\partial^i_t u|^pr^{p(\mu-s+|\alpha|+2i)}\,dQ\bigg)^{1/p}$$

for $Q\in\{\Omega^T,S^T\}$, $p\in[1,\infty]$, $s\in\mathbb{Z}_+\cup\{0\}$, $\mu\in\mathbb{R}$.

Finally, we define

$$W^{2,1}_{2,\mu}(\Omega^T)=\bigg\{u:\|u\|_{W^{2,1}_{2,\mu}(\Omega^T)}=\bigg(\int_{\Omega^T}(u^2_{,xx}+u^2_{,t}+u^2)r^{2\mu}\,dx\,dt\bigg)^{1/2}<\infty\bigg\},$$

and use the notation

$$\|u\|_{W^{2,1}_{2,\mu}(\Omega^T)}=|||u|||_{2,2,\mu,\Omega^T}.$$

Moreover,

$$W^2_{2,\mu}(\Omega)=\bigg\{u:\|u\|_{W^2_{2,\mu}(\Omega)}=\bigg(\int_\Omega(u^2_{,xx}+u^2)r^{2\mu}\,dx\bigg)^{1/2}<\infty\bigg\}$$

and

$$\|u\|_{W^2_{2,\mu}(\Omega)}=|||u|||_{2,2,\mu,\Omega}.$$

Now we recall inequalities and imbedding theorems used in this paper.

From [10] we have the imbedding

$$\|u\|_{V^s_{q,\beta+s-l+n/p-n/q}(\Omega)}\le c\|u\|_{V^l_{p,\beta}(\Omega)},\quad \Omega\subset\mathbb{R}^n,\tag{2.1}$$

and $s-l+n/p-n/q\le 0$.

Let us consider the problem

$$u_{,t} - \Delta u = f \quad \text{in } \Omega^T, \qquad u = g \quad \text{on } S^T,\ u|_{t=0} = u_0. \tag{2.2}$$

From [16] we have:

Lemma 2.1. *Let*

$$f \in L_{2,\mu}(\Omega^T), \quad g \in W_{2,\mu}^{3/2,3/4}(S^T),\ u_0 \in W_{2,\mu}^1(\Omega),\ \mu \in \mathbb{R}_+. \tag{2.3}$$

Then there exists a solution to problem (2.2) *such that* $u \in W_{2,\mu}^{2,1}(\Omega^T)$ *and*

$$|||u|||_{2,2,\mu,\Omega^T} \le c(|f|_{2,\mu,\Omega^T} + |||g|||_{3/2,2,\mu,S^T} + |||u_0|||_{1,2,\mu,\Omega}).$$

Similarly, let us consider the problem

$$u_{,t} - \Delta u = f \quad \text{in } \Omega^T, \quad \frac{\partial u}{\partial n} = g \quad \text{on } S^T,\ u|_{t=0} = u_0. \tag{2.4}$$

From [16] we have:

Lemma 2.2. *Let* (2.3) *for* f *and* u_0 *hold and let* $g \in W_{2,\mu}^{1/2,1/4}(S^T)$. *Then there exists a solution to problem* (2.4) *such that* $u \in W_{2,\mu}^{2,1}(\Omega^T)$ *and*

$$|||u|||_{2,2,\mu,\Omega^T} \le c(|f|_{2,\mu,\Omega^T} + |||g|||_{1/2,2,\mu,S^T} + |||u_0|||_{1,2,\mu,\Omega}).$$

From Section 2 in [2] we have the Hardy inequality

$$|x^{-\beta}u|_{p,\mathbb{R}_+} \le \frac{1}{|\beta - 1/p|}|x^{-\beta+1}u_{,x}|_{p,\mathbb{R}_+}$$

which in our case takes the form

$$|u|_{2,\mu-1,\Omega} \le \frac{1}{|\mu|}|u_{,x}|_{2,\mu,\Omega}, \quad \mu \ne 0.$$

From Lemma 3.3 [20] we have

Lemma 2.3. *Let* $\eta = \overline{e}_0 \times \overline{x}$, $\overline{e}_0 = (0,0,1)$, $\overline{x} = (x_1,x_2,x_3)$, $\eta = \overline{x}_{,\varphi}$, Ω *have the axis of symmetry* $\overline{e}_0$. *Let* v *be a solution to* (1.1) *and let*

$$\left|\int_\Omega v(0)\cdot\eta\,dx\right| < \infty, \qquad \left|\int_{\Omega^t} f\cdot\eta\,dx\,dt'\right| < \infty.$$

Then

$$\int_\Omega v\cdot\eta\,dx + \gamma\int_{S^t} v\cdot\eta\,dx\,dt' = \int_\Omega v(0)\cdot\eta\,dx + \int_{\Omega^t} f\cdot\eta\,dx\,dt'.$$

Lemma 3.4 [20] reads

LEMMA 2.4. *Let*

$$E_\Omega(v)=\int_\Omega (v_{i,x_j}+v_{j,x_i})^2\,dx$$

and let $|\int_\Omega v\cdot\eta\,dx|<\infty$, $E_\Omega(v)<\infty$. *Then*

$$\|v\|_{1,\Omega}^2\le c\bigg(E_\Omega(v)+\bigg|\int_\Omega v\cdot\eta\,dx\bigg|^2\bigg).$$

Finally Lemma 3.5 [20] implies

LEMMA 2.5. *Assume that* $v(0)\in L_2(\Omega)$, $f\in L_{2,1}(\Omega^T)$, $|\int_{\Omega^t} v\cdot\eta\,dx\,dt'|<\infty$. *Then*

$$|v(t)|_{2,\Omega}\le |f|_{2,1,\Omega^t}+|v(0)|_{2,\Omega}\equiv d_1,\quad t\le T, \tag{2.5}$$

and

$$\|v\|_{V_2^0(\Omega^t)}\le c(1+t)(|f|_{2,1,\Omega^t}+|v(0)|_{2,\Omega})\equiv d_2(t),\quad t\le T. \tag{2.6}$$

To show the existence of a global regular solution to problem (1.1) we need the following quantities (see [20])

$$\begin{aligned}
h&=v_{r,\varphi}\bar e_r+v_{\varphi,\varphi}\bar e_\varphi+v_{z,\varphi}\bar e_z,\quad q=p_{,\varphi},\\
\alpha&=\operatorname{rot}v,\quad \chi=\alpha_\varphi=v_{r,z}-v_{z,r},\quad w=v_\varphi,\quad F=\operatorname{rot}f,\\
g&=f_{r,\varphi}\bar e_r+f_{\varphi,\varphi}\bar e_\varphi+f_{z,\varphi}\bar e_z.
\end{aligned} \tag{2.7}$$

From Section 1.1 [20] we obtain problems for h, q, α, w. Let v be given then (h,q) is a solution of the problem

$$\begin{aligned}
&h_{,t}-\operatorname{div}\mathbb{T}(h,q)=-v\cdot\nabla h-h\cdot\nabla v+g\equiv G &&\text{in }\Omega^T,\\
&\operatorname{div}h=0 &&\text{in }\Omega^T,\\
&h\cdot\bar n=0,\quad \nu\bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha+\gamma h\cdot\bar\tau_\alpha=0,\quad \alpha=1,2, &&\text{on }S^T,\\
&h|_{t=0}=h(0) &&\text{in }\Omega.
\end{aligned} \tag{2.8}$$

For given v, h, q we have

$$\begin{aligned}
&w_{,t}+v\cdot\nabla w+\frac{v_r}{r}w-\nu\Delta w+\nu\frac{w}{r^2}=\frac1r q+\frac{2\nu}{r^2}h_r+f_\varphi &&\text{in }\Omega^T,\\
&\nu\bar n\cdot\nabla w=-\gamma w+\nu\frac{a_1}{r}w &&\text{on }S^T,\\
&w|_{t=0}=w(0) &&\text{in }\Omega,
\end{aligned} \tag{2.9}$$

where the boundary S is described by the equation $\psi(r,z)=0$ and $a_1=\psi_{,r}/\sqrt{\psi_{,r}^2+\psi_{,z}^2}$, $a_2=\psi_{,z}/\sqrt{\psi_{,r}^2+\psi_{,z}^2}$. The cylindrical components of vorticity assume the form

$$\alpha_r=\frac1r(v_{z,\varphi}-rv_{\varphi,z}),\quad \alpha_\varphi=v_{r,z}-v_{z,r}\equiv\chi,\quad \alpha_z=\frac1r[(rv_\varphi)_{,r}-v_{r,\varphi}]. \tag{2.10}$$

Then $\alpha' = (\alpha_r, \alpha_z)$ is a solution to the problem

$$
(2.11)\quad
\begin{aligned}
&\alpha_{r,t} + v\cdot\nabla\alpha_r - (\alpha_r v_{r,r} + \alpha_z v_{r,z}) - \frac{\chi}{r}h_r \\
&\qquad + \frac{2\nu}{r^2}(h_{r,z} - h_{z,r}) + \frac{\nu\alpha_r}{r^2} - \nu\Delta\alpha_r = F_r && \text{in } \Omega^T,\\
&\alpha_{z,t} + v\cdot\nabla\alpha_z - (\alpha_r v_{z,r} + \alpha_z v_{z,z}) - \frac{\chi}{r}h_z - \nu\Delta\alpha_z = F_z && \text{in } \Omega^T,\\
&\overline{\tau}_2\cdot\alpha' = -\frac{2a_1}{r}w + \frac{\gamma}{\nu}w && \text{on } S^T,\\
&(\overline{n}\cdot\alpha')_{,n} = \beta_1 h_r + \beta_2 h_z + \beta_3 w_{,r} + \beta_4 w_{,z} + \beta_5 w && \text{on } S^T,\\
&\alpha'|_{t=0} = \alpha'(0) && \text{in } \Omega,
\end{aligned}
$$

where $\overline{\tau}_2 = a_2\overline{e}_r - a_1\overline{e}_z$, $\overline{n} = a_1\overline{e}_r + a_2\overline{e}_z$, $\overline{\tau}_2\cdot\alpha' = a_2\alpha_r - a_1\alpha_z$, $\overline{n}\cdot\alpha' = a_1\alpha_r + a_2\alpha_z$, $\partial_n = \overline{n}\cdot\nabla$, β_i, $i = 1,\dots,5$, depend on a_1, a_2 (see Lemma 3.2 from [20]).

Next, χ is a solution to the problem

$$
(2.12)\quad
\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi + (v_{r,r} + v_{z,z})\chi \\
&\qquad - \nu\bigg[\bigg(r\Big(\frac{\chi}{r}\Big)_{,r}\bigg)_{,r} + \frac{1}{r^2}\chi_{,\varphi\varphi} + \chi_{,zz} + 2\Big(\frac{\chi}{r}\Big)_{,r}\bigg] \\
&\qquad = \frac{2\nu}{r}\Big(-h_{\varphi,z} + \frac{1}{r}h_{z,\varphi}\Big) \\
&\qquad\quad - \frac{1}{r}\Big(w_{,z}h_r - w_{,r}h_z + \frac{w}{r}h_z\Big) + \frac{2}{r}ww_{,z} + F_\varphi && \text{in } \Omega^T,\\
&\qquad \chi = 2\Big(k - \frac{\gamma}{2\nu}\Big)v\cdot\overline{\tau}_2 && \text{on } S^T,\\
&\qquad \chi|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned}
$$

where k is the curvature of the curve $S' = \{r, z : \psi(r,z) = 0\}$ which generates S by rotating it around the x_3 axis.

Finally, v and p are calculated from the elliptic problems

$$
(2.13)\quad
\begin{aligned}
\operatorname{rot} v &= \alpha && \text{in } \Omega,\\
\operatorname{div} v &= 0 && \text{in } \Omega,\\
v\cdot\overline{n} &= 0 && \text{on } S
\end{aligned}
$$

and

$$
(2.14)\quad
\begin{aligned}
\Delta p &= -\nabla v\cdot\nabla v + \operatorname{div} f && \text{in } \Omega,\\
\frac{\partial p}{\partial n} &= f\cdot\overline{n} + \nu\overline{n}\cdot\Delta v - \overline{n}\cdot v\cdot\nabla v && \text{on } S.
\end{aligned}
$$

To obtain an estimate for χ we need

Lemma 2.6. *Assume that* $A_i = \alpha_{i1}v_{r,r} + \alpha_{i2}v_{r,z} + \alpha_{i3}v_{z,r} + \alpha_{i4}v_{z,z}$ *where* α_{ij} $i,j = 1,\dots,4$, *depend on* a_1, a_2. *Assume that*

$$\det\{-\alpha_{i1}a_2 - \alpha_{i3}a_1 + \alpha_{i4}a_2, -\alpha_{i1}a_1 + \alpha_{i2}a_2 + \alpha_{i4}a_1, -\alpha_{i2}a_1, \alpha_{i3}a_2\}_{i=1,\dots,4} \neq 0.$$

Assume that the function B depends on a_1, a_2 and their derivatives and depends linearly on components of $v_{,x}$, $\nabla'(h_\varphi/r)$, $\nabla'(v_r/r)$, where $\nabla' = (\partial_r, \partial_z)$. Then

$$\overline{n} \cdot \nabla\chi|_S = \sum_{i=1}^{4} b_i \partial_s A_i + B, \quad \partial_s = \overline{\tau}_2 \cdot \nabla, \tag{2.15}$$

where b_i, $i = 1, \dots, 4$, depend on a_1 and a_2.

Proof. From the form of χ we have

$$\overline{n} \cdot \nabla\chi|_S = a_1\chi_{,r} + a_2\chi_{,z} = a_1(v_{r,rz} - v_{z,rr}) + a_2(v_{r,zz} - v_{z,rz}). \tag{2.16}$$

We want to express $\overline{n} \cdot \nabla\chi|_S$ in terms of $\partial_s A_i$, $i = 1, \dots, 4$. Performing calculations in $\partial_s A_i$, $i = 1, \dots, 4$, we obtain the identities

$$\begin{aligned} &\partial_s\alpha_{i1}v_{r,r} + \partial_s\alpha_{i2}v_{r,z} + \partial_s\alpha_{i3}v_{z,r} + \partial_s\alpha_{i4}v_{z,z} \\ &\quad + \alpha_{i1}(a_2v_{r,rr} - a_1v_{r,rz}) + \alpha_{i2}(a_2v_{r,rz} - a_1v_{r,zz}) \\ &\quad\quad + \alpha_{i3}(a_2v_{z,rr} - a_1v_{z,rz}) + \alpha_{i4}(a_2v_{z,rz} - a_1v_{z,zz}) = \partial_s A_i, \end{aligned} \tag{2.17}$$

for $i = 1, \dots, 4$. Since $\overline{n} \cdot \nabla\chi|_S$ depends on four different second derivatives we have to eliminate the remaining two derivatives. For this purpose we use the continuity equation

$$v_{r,r} + v_{z,z} = -\frac{1}{r}(h_\varphi + v_r) \tag{2.18}$$

Differentiating (2.18) with respect to r and z we get

$$\begin{aligned} v_{r,rr} + v_{z,rz} &= -\left(\frac{h_\varphi}{r} + \frac{v_r}{r}\right)_{,r} \equiv d_1, \\ v_{r,rz} + v_{z,zz} &= -\frac{1}{r}(h_{\varphi,z} + v_{r,z}) \equiv d_2. \end{aligned} \tag{2.19}$$

Calculating $v_{r,rr}$ and $v_{z,zz}$ from (2.19) and inserting them to (2.17) we obtain

$$\begin{aligned} &(-\alpha_{i1}a_2 - \alpha_{i3}a_1 + \alpha_{i4}a_2)v_{z,rz} + (-\alpha_{i1}a_1 + \alpha_{i2}a_2 + \alpha_{i4}a_1)v_{r,rz} \\ &\quad - \alpha_{i2}a_1v_{r,zz} + \alpha_{i3}a_2v_{z,rr} \\ &= \partial_s A_i - (\partial_s\alpha_{i1}v_{r,r} + \partial_s\alpha_{i2}v_{r,z} + \partial_s\alpha_{i3}v_{z,r} + \partial_s\alpha_{i4}v_{z,z}) \\ &\quad - \alpha_{i1}a_2d_1 + \alpha_{i4}a_1d_2, \end{aligned} \tag{2.20}$$

for $i = 1, \dots, 4$. In view of the assumptions of the lemma we can calculate the second derivatives of v from (2.20) and insert them to (2.16). In this way we obtain (2.15). □

From [24] we recall

Lemma 2.7. *Let* $g \in L_2(0,T;L_{6/5}(\Omega))$, $h(0) \in L_2(\Omega)$, $v \in L_2(0,T;W_3^1(\Omega))$. *Then solutions to* (2.8) *satisfy*

$$\|h\|_{V_2^0(\Omega^t)} \le c\exp(c\|v_{,x}\|_{3,2,\Omega^t}^2)[|g|_{6/5,2,\Omega^t} + |h(0)|_{2,\Omega}], \quad \textit{for } t \le T.$$

Let $\delta \in (0,1)$ and let

$$\mathfrak{N}_\delta(\Omega^T) = L_\infty(0,T;L_{4,-\delta}(\Omega)) \cap L_\infty(0,T;W_{2,-\delta}^1(\Omega)) \cap L_2(0,T;W_3^1(\Omega)). \tag{2.21}$$

Lemma 2.8 (see [24]). *Let* $\delta \in (0,1)$, $v \in \mathfrak{N}_\delta(\Omega^T)$, $g \in L_2(0,T;L_{6/5}(\Omega)) \cap L_{2,-(1+\varepsilon_*)}(\Omega^T)$, $h(0) \in H^1(\Omega) \cap H^1_{-(1+\varepsilon_*)}(\Omega)$, $\varepsilon_* \in (0,\delta)$. *Then solutions of* (2.8) *satisfy*

$$\begin{aligned} \|h\|_{2,-(1+\varepsilon_*),\Omega^t} &+ \|q\|_{L_2(0,t;H^1_{-(1+\varepsilon_*)}(\Omega))} \\ &\le \varphi(\|v\|_{\mathfrak{N}_\delta(\Omega^t)})(|g|_{6/5,2,\Omega^t} + |h(0)|_{2,\Omega}) \\ &\quad + c(|g|_{2,-(1+\varepsilon_*),\Omega^t} + \|h(0)\|_{1,-(1+\varepsilon_*),\Omega}), \end{aligned} \tag{2.22}$$

$t \le T$, *where* φ *is an increasing positive function.*

3. Estimates

In this section we show a global a priori estimate for solutions to problem (1.1). First for given w and h we obtain an estimate for χ.

Lemma 3.1. *Assume that* $h \in L_2(0,T;H^2_{-1}(\Omega))$, $w \in L_\infty(0,T;H^1_0(\Omega))$, $v_{\varphi,z} \in L_2(0,T;L_{4,-3/4-\varepsilon}(\Omega))$, $\varepsilon > 0$ *is a small number which will be chosen later,* $F_\varphi \in L_2(0,T;L_{6/5,-1}(\Omega))$. *Let*

$$K = \left|k - \frac{\gamma}{2\nu}\right|_{\infty,-1,S} + \sup_i \left\|\frac{1}{r} b_i\left(k - \frac{\gamma}{2\nu}\right)\right\|_{W^1_\infty(S)} \le cK_*, \quad i \in \{1,\dots,4\},$$

where $b_1,\dots,b_4$ *are introduced in* (2.15). *Let* $\{\varphi_j(x,t)\}$ *be a partition of unity near* S^t, $t \le T$. *Then*

$$\begin{aligned} |\chi(t)|_{2,-1,\Omega}^2 &+ \nu\int_0^t \left\|\nabla\frac{\chi(t')}{r}\right\|_{0,\Omega}^2 dt' \le c\exp(c|h|_{3,2,-1,\Omega^t}^2) \\ &\cdot\bigg[K(K+1)\sum_j \int_{S^t\cap\operatorname{supp}\varphi_j}\left(|v_{,x}|^2 + |v|^2 + \left|\nabla\frac{h}{r}\right|^2\right)dx\,dt' \\ &+ (1+\sup_t \|w\|_{1,0,\Omega}^2)\int_0^t \|h(t')\|_{2,-1,\Omega}^2\,dt' \\ &+ \frac{R^{2\varepsilon}}{\varepsilon^2}\sup_t\|w\|_{1,0,\Omega}^2\int_0^t |v_{\varphi,z}(t')|_{4,-3/4-\varepsilon,\Omega}^2\,dt' \\ &+ |F_\varphi|_{6/5,2,-1,\Omega^t}^2 + |\chi(0)|_{2,-1,\Omega}^2\bigg], \end{aligned} \tag{3.1}$$

for $t \le T$, where the constants c do not depend on t.

PROOF. First we introduce the set $\Omega_* = \{x \in \Omega : 0 < \varepsilon_* < r\}$ and add the artificial boundary condition

$$\chi|_{r=\varepsilon_*} = 0. \tag{3.2}$$

Multiplying $(2.12)_1$ by χ/r^2 and integrating over Ω_* we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\chi|^2_{2,-1,\Omega_*} &+ \int_{\Omega_*}[v\cdot\nabla\chi + (v_{r,r}+v_{z,z})\chi]\frac{\chi}{r^2}\,dx \\ &- \nu\int_{\Omega_*}\bigg[\bigg(r\bigg(\frac{\chi}{r}\bigg)_{,r}\bigg)_{,r} + \frac{1}{r^2}\chi_{,\varphi\varphi} + \chi_{,zz} + 2\bigg(\frac{\chi}{r}\bigg)_{,r}\bigg]\frac{\chi}{r^2}\,dx \\ = &-2\nu\int_{\Omega_*}\frac{1}{r^2}\bigg(-h_{\varphi,z} + \frac{1}{r}h_{z,\varphi}\bigg)\frac{\chi}{r^2}\,dx \\ &- \int_{\Omega_*}\frac{1}{r}\bigg(w_{,z}h_r - w_{,r}h_z + \frac{w}{r}h_z\bigg)\frac{\chi}{r^2}\,dx \\ &+ 2\int_{\Omega_*}\frac{1}{r}ww_{,z}\frac{\chi}{r^2}\,dx + \int_{\Omega_*}F_\varphi\frac{\chi}{r^2}\,dx. \end{aligned} \tag{3.3}$$

The second term on the l.h.s. of (3.3) equals

$$\frac{1}{2}\int_{\partial\Omega_*} v\cdot\overline{n}\frac{\chi^2}{r^2}\,d\partial\Omega_* + \int_{\Omega_*}\bigg(v_{r,r}+v_{z,z}+\frac{v_r}{r}\bigg)\frac{\chi^2}{r^2}\,dx \equiv I_1,$$

where the first term in I_1 equals

$$\frac{1}{2}\int_{S_*} v\cdot\overline{n}\frac{\chi^2}{r^2}dS + \frac{1}{2}\int_{-a}^{a} v\cdot\overline{n}\frac{\chi^2}{r^2}\bigg|_{r=\varepsilon_*}dz = 0,$$

where $S_* = \{x \in S : 0 < \varepsilon_* < r\}$ and we used $(1.1)_3$ and (3.2).

The second term in I_1 assumes the form

$$-\int_{\Omega_*}\frac{h_\varphi}{r}\frac{\chi^2}{r^2}\,dx$$

which can be estimated by

$$\varepsilon_1\bigg|\frac{\chi}{r}\bigg|^2_{6,\Omega_*} + c(1/\varepsilon_1)|h_\varphi|^2_{3,-1,\Omega_*}\bigg|\frac{\chi}{r}\bigg|^2_{2,\Omega_*}.$$

The last term on the l.h.s. of (3.3) takes the form

$$\begin{aligned} -\nu\int_{\Omega_*}\bigg[\bigg(r\bigg(\frac{\chi}{r}\bigg)_{,r}\bigg)_{,r}\frac{\chi}{r} + \bigg(r\bigg(\frac{\chi}{r}\bigg)_{,z}\bigg)_{,z}\frac{\chi}{r}\bigg]\,dr\,dz\,d\varphi \\ + \nu\int_{\Omega_*}\bigg|\frac{1}{r^2}\chi_{,\varphi}\bigg|^2 dx - \nu\int_{\Omega_*}2\bigg(\frac{\chi}{r}\bigg)_{,r}\frac{\chi}{r}\,dr\,dz\,d\varphi \equiv I_2. \end{aligned}$$

Integrating by parts the first integral in I_2 equals

$$\nu\int_{\Omega_*}\left|\nabla'\frac{\chi}{r}\right|^2dx-\nu\int_{S_*}\overline{n}\cdot\nabla\frac{\chi}{r}\chi\,dS_*+\nu\int_{-a}^{a}\int_0^{2\pi}d\varphi\,dz\left(\frac{\chi}{r}\right)_{,r}\chi\bigg|_{r=\varepsilon_*},$$

where $\nabla'=(\partial_r,\partial_z)$.

Finally, the last integral in I_2 reads

$$\begin{aligned}-\nu\int_{\Omega_*}\partial_r\left(\frac{\chi^2}{r^2}\right)dr\,dz\,d\varphi\\ =-\nu\int_{-a_*}^{a_*}\int_0^{2\pi}\frac{\chi^2}{r^2}\bigg|_{r\in S'_*}dz\,d\varphi+\nu\int_{-a_*}^{a_*}\int_0^{2\pi}\frac{\chi^2}{r^2}\bigg|_{r=\varepsilon_*}dz\,d\varphi,\end{aligned}$$

where $S'=\{(r,z)\in[0,R]\times[-a,a]:\psi(r,z)=0 \text{ and } r=\psi_1(z) \text{ and } z=\mp\psi_2(r)\}$, $S'_*=\{(r,x)\in S':\varepsilon_*\le r\}$ and $a_*=\psi_2(\varepsilon_*)<a$. Since $\overline{n}=(a_1,a_2)$ is the unit normal vector to S', $\overline{\tau}=(-a_2,a_1)$ is the unit tangent vector to S'. Assuming that S' is described in the parametric form $(r(\tau),z(\tau))$, we have that $(-a_2,a_1)=(dr/d\tau,dz/d\tau)$, so $dz=a_1d\tau$. Denoting $dS=d\tau d\varphi$ we express the first integral on the r.h.s. by

$$-\nu\int_{S_*}\frac{\chi^2}{r^2}a_1\,dS_*.$$

The first term on the r.h.s. of (3.3) is estimated by

$$\left|2\nu\int_{\Omega_*}\left(\frac{h_\varphi}{r^3}\left(\frac{\chi'}{r}\right)_{,z}-\frac{h_z}{r^3}\frac{1}{r}\left(\frac{\chi'}{r}\right)_{,\varphi}\right)dx\right|\le\varepsilon_2\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega_*}+c(1/\varepsilon_2)|h|^2_{2,-3,\Omega_*},$$

where the last norm is estimated by $\|h\|^2_{2,-1,\Omega_*}$.

By the Hölder and Young inequalitites we estimate the second term on the r.h.s. of (3.3) by

$$\varepsilon_3\left|\frac{\chi}{r}\right|^2_{6,\Omega_*}+c(1/\varepsilon_3)\|w\|^2_{1,0,\Omega_*}|h|^2_{3,-2,\Omega_*},$$

where the imbedding

$$|h|_{3,-2,\Omega_*}\le c\|h\|_{2,-1,\Omega_*}$$

will be used.

We estimate the third term on the r.h.s. of (3.3) by

$$\varepsilon_4\left|\frac{\chi}{r^{2-\varepsilon'}}\right|^2_{2,\Omega_*}+\frac{c}{\varepsilon_4}\|w\|^2_{1,0,\Omega_*}|v_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*},$$

where $\varepsilon'>0$ is a small number which will be chosen later.

The last three cases are more explicitly described in the proof of Lemma 4.1 from [20].

Finally, the last term on the r.h.s. of (3.3) is estimated by

$$\varepsilon_5\left|\frac{\chi}{r}\right|^2_{6,\Omega_*}+c(1/\varepsilon_5)\left|\frac{F_\varphi}{r}\right|^2_{6/5,\Omega_*}.$$

Emploing the above estimates in (3.3) and using that $\varepsilon_i\le\varepsilon/8$, $i=1,3,5$, $\varepsilon_2,\varepsilon_4$ are sufficiently small and (3.2) we obtain

$$\begin{aligned}(3.4)\quad \frac{d}{dt}|\chi|^2_{2,-1,\Omega_*}+\nu\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega_*}&\le\varepsilon\left|\frac{\chi}{r}\right|^2_{6,\Omega_*}+c\int_{S_*}\frac{\chi^2}{r^2}\,dS_*+c\left|\int_{\Omega_*}\frac{h_\varphi}{r}\frac{\chi^2}{r^2}\,dx\right|\\&\quad+2\nu\int_{S_*}\overline{n}\cdot\nabla\frac{\chi}{r}\chi\,dS_*+c(1/\varepsilon)\bigg[(1+\|w\|^2_{1,0,\Omega_*})\|h\|^2_{2,-1,\Omega_*}\\&\quad+\frac{R^{2\varepsilon'}}{\varepsilon'^2}\|w\|^2_{1,0,\Omega_*}|v_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*}+|F_\varphi|^2_{6/5,-1,\Omega_*}\bigg],\end{aligned}$$

where we used the Hardy inequality

$$\left|\frac{\chi}{r}\right|^2_{2,-1+\varepsilon',\Omega}\le\frac{1}{|\varepsilon'|^2}\int_\Omega\left|\nabla\frac{\chi}{r}\right|^2r^{2\varepsilon'}\,dx\le\frac{R^{2\varepsilon'}}{|\varepsilon'|^2}\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega}.$$

In view of the Poincare inequality

$$\left|\frac{\chi}{r}\right|_{2,\Omega_*}\le c\left|\frac{\chi}{r}\right|_{2,S_*}+c\left|\nabla\frac{\chi}{r}\right|_{2,\Omega_*}$$

and for sufficiently small ε we obtain from (3.4) the inequality

$$\begin{aligned}(3.5)\qquad \frac{d}{dt}|\chi|^2_{2-1,\Omega_*}+\nu\left\|\frac{\chi}{r}\right\|^2_{1,\Omega_*}&\le c|\chi|^2_{2,-1,S_*}+c\left|\int_{\Omega_*}\frac{h_\varphi}{r}\frac{\chi^2}{r^2}\,dx\right|\\&\quad+c\nu\int_{S_*}\overline{n}\cdot\nabla\frac{\chi}{r}\chi\,dS_*+c\bigg[(1+\|w\|^2_{1,0,\Omega_*})\|h\|^2_{2,-1,\Omega_*}\\&\quad+\frac{R^{2\varepsilon'}}{|\varepsilon'|^2}\|w\|^2_{1,0,\Omega_*}|v_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*}+|F_\varphi|^2_{6/5,-1,\Omega_*}\bigg].\end{aligned}$$

From the boundary condition $(2.12)_2$ the first term on the r.h.s. of (3.5) is estimated by

$$c\left|\left(k-\frac{\gamma}{2\nu}\right)v\cdot\overline{\tau}_2\right|^2_{2,-1,S_*}\le c\sup_S\left(\frac{|k-\gamma/(2\nu)|^2a_1^2}{r^2}\right)|v|^2_{2,S}\le cK^2|v|^2_{2,S}.$$

Applying the Hölder and Young inequalities we estimate the second term on the r.h.s. of (3.5) by

$$\varepsilon_6\left|\frac{\chi}{r}\right|^2_{6,\Omega_*}+c(1/\varepsilon_6)|h_\varphi|^2_{3,-1,\Omega_*}|\chi|^2_{2,-1,\Omega_*}.$$

Using the above estimates in (3.5) and assuming that ε_6 is sufficiently small yields

$$\frac{d}{dt}|\chi|^2_{2,-1,\Omega_*}+\nu\left\|\frac{\chi}{r}\right\|^2_{1,\Omega_*}\le cK^2|v|^2_{2,S}+c|h_\varphi|^2_{3,-1,\Omega_*}|\chi|^2_{2,-1,\Omega_*}+c\int_S\overline{n}\cdot\nabla\frac{\chi}{r}\chi\,dS$$
$$+c\bigg[(1+\|w\|^2_{1,0,\Omega_*})\|h\|^2_{2,-1,\Omega_*}+\frac{R^{2\varepsilon'}}{|\varepsilon'|^2}\|w\|^2_{1,0,\Omega_*}|v_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*}+|F_\varphi|^2_{6/5,-1,\Omega_*}\bigg].$$

Integrating the above inequality with respect to time, replacing ε' by ε and passing with ε_* to 0 we obtain, for $t\le T$,

$$|\chi(t)|^2_{2,-1,\Omega}+\nu\int_0^t\left\|\frac{\chi(t')}{r}\right\|^2_{1,\Omega}dt' \tag{3.6}$$
$$\le c\exp(c|h|^2_{3,2,-1,\Omega^t})\bigg[\bigg|\int_{S^t}\overline{n}\cdot\nabla\frac{\chi}{r}\chi dS\,dt'\bigg|+K^2|v|^2_{2,S^t}$$
$$+(1+\sup_t\|w\|^2_{1,0,\Omega})\int_0^t\|h(t')\|^2_{2,-1,\Omega}\,dt'$$
$$+\frac{R^{2\varepsilon}}{\varepsilon^2}\sup_t\|w\|^2_{1,0,\Omega}\int_0^t|v_{\varphi,z}(t')|^2_{4,-3/4-\varepsilon,\Omega}\,dt'$$
$$+|F_\varphi|^2_{6/5,2,-1,\Omega^t}+|\chi(0)|^2_{2,-1,\Omega}\bigg].$$

To examine the first term on the r.h.s. of (3.6) we introduce a partition of unity $\varphi_j(x,t)$ in a neighbourhood of S^T. Since $\sum_j\varphi_j(x,t)=1$ we have

$$I_1\equiv\sum_j\int_{S^t}\varphi_j(x,t')\overline{n}\cdot\nabla\frac{\chi}{r}\chi\,dS\,dt'$$
$$=\sum_j\int_{S^t}\varphi_j(x,t')\frac{1}{r}\overline{n}\cdot\nabla\chi\chi\,dS\,dt'-\sum_j\int_{S^t}\varphi_j(x,t')a_1\frac{\chi^2}{r^2}dS\,dt'\equiv I_2+I_3.$$

In view of Lemma 2.4,

$$I_2=\sum_j\int_{S^t}\varphi_j(x,t')\frac{1}{r}\sum_{i=1}^4(b_i\partial_sA_i+B)\bigg(k-\frac{\gamma}{2\nu}\bigg)v\cdot\overline{\tau}_2\,dS\,dt'.$$

Integrating by parts yields

$$I_2=-\sum_j\int_{S^t}\sum_{i=1}^4A_i\partial_s\bigg[\varphi_j\frac{1}{r}b_i\bigg(k-\frac{\gamma}{2\nu}\bigg)v\cdot\overline{\tau}_2\bigg]\,dS\,dt'$$
$$+\sum_j\int_{S^t}\varphi_j\frac{1}{r}B\bigg(k-\frac{\gamma}{2\nu}\bigg)v\cdot\overline{\tau}_2\,dS\,dt'\equiv I_4+I_5,$$

where

$$|I_4| \le \sum_j \int_{S^t\cap\operatorname{supp}\varphi_j} \varphi_j \frac{1}{r}\bigg|k-\frac{\gamma}{2\nu}\bigg| v_{,x}^2\,dS\,dt' + \sum_j \int_{S^t\cap\operatorname{supp}\varphi_j} \bigg|\sum_{i=1}^4 \partial_s\bigg(\varphi_j\frac{b_i}{r}\bigg(k-\frac{\gamma}{2\nu}\bigg)\bigg)\bigg| |v_{,x}|\,|v|\,dS\,dt' \le cK\sum_j \int_{S^t\cap\operatorname{supp}\varphi_j} v_{,x}^2\,dS\,dt' + cK\sum_j \int_{S^t\cap\operatorname{supp}\varphi_j} |v_{,x}||v|\,dS\,dt',$$

and

$$|I_5| \le cK\sum_j \int_{S^t\cap\operatorname{supp}\varphi_j} |v|\bigg(|v_{,x}|+|v|+\bigg|\nabla\frac{h}{r}\bigg|\bigg)\,dS\,dt'.$$

Finally,

$$|I_3| \le cK^2\sum_j \int_{S^t\cap\operatorname{supp}\varphi_j} |v|^2\,dS\,dt'.$$

Summarizing, we obtain

$$|I_1| \le cK(K+1)\sum_j \int_{S^t\cap\operatorname{supp}\varphi_j} \bigg(v_{,x}^2+v^2+\bigg|\nabla\frac{h}{r}\bigg|^2\bigg)\,dS\,dt'.$$

Using this estimate in (3.6) we obtain (3.1). □

To obtain a global estimate necessary for the proof of global existence we have to estimate all norms from the r.h.s. of (3.1). First we shall examine the second factor from the third term on the r.h.s. of (3.1). For this purpose we use $(2.10)_1$ in the form

$$v_{\varphi,z} = -\alpha_r + \frac{1}{r}h_z. \tag{3.7}$$

By (2.1) we have

$$\begin{aligned} |v_{\varphi,z}|_{4,-3/4-\varepsilon,\Omega} &\le |\alpha_r|_{4,-3/4-\varepsilon,\Omega} + |h|_{4,-7/4-\varepsilon,\Omega} \\ &\le c(\|\alpha_r\|_{1,-1/2-\varepsilon,\Omega} + \|h\|_{1,-3/2-\varepsilon,\Omega}). \end{aligned} \tag{3.8}$$

To estimate the first norm on the r.h.s. of (3.8) we need energy type estimates for $\alpha' = (\alpha_r,\alpha_z)$.

Lemma 3.2. *Assume that*

$$v\in L_4(\Omega^T)\cap L_3(0,T;W_3^1(\Omega))\cap L_\infty(0,T;H^1(\Omega)),$$
$$h\in H^{2,1}_{1-\mu}(\Omega^T),\quad w\in H^{2,1}_{1-\mu}(\Omega^T),\quad F'\in L_{2,1-\mu}(\Omega^T),$$
$$\alpha'(0)\in W^1_{2,1-\mu}(\Omega),\ \alpha'\in L_\infty(0,T;L_{2,1-\mu}(\Omega)),\ \alpha_r\in L_{2,-(1+\mu)}(\Omega^T),\ a_1\in C^2,$$

$$\sum_{i=0}^{2} r^i\left|\nabla^i\frac{a_1}{r}\right| \le c, \qquad \sum_{s=0}^{1} r^{2+s}|\nabla^s\beta_j| \le c, \quad j=1,2,$$
$$\sum_{s=0}^{1} r^s|\nabla^s\beta_k| \le c, \quad k=3,4, \qquad \sum_{s=0}^{1} r^{1+s}|\nabla^s\beta_5| \le c.$$

Then solutions of problem (2.11) *satisfy, for $t \le T$,*

$$\begin{aligned}(3.9)\quad |||\alpha'|||_{2,2,1-\mu,\Omega^t} \le{}& \varphi(|v|_{4,\Omega^t}, |\nabla v|_{3,\Omega^t})|\alpha'|_{2,1-\mu,\Omega^t}\\ &+ c|\nabla v|_{2,\infty,\Omega^t}\|h\|_{2,-\mu,\Omega^t} + c(\|h\|_{2,-\mu,\Omega^t} + \|w\|_{2,1-\mu,\Omega^t})\\ &+ c|\alpha_r|_{2,-(1+\mu),\Omega^t} + c|F'|_{2,1-\mu,\Omega^t} + c|||\alpha'(0)|||_{1,2,1-\mu,\Omega},\end{aligned}$$

where φ is an increasing positive function.

Proof. Applying [16] to problem (2.11) yields (see also Lemmas 2.1 and 2.2)

$$\begin{aligned}(3.10)\quad |||\alpha'|||_{2,2,1-\mu,\Omega^t} \le c\bigg(&|v\cdot\nabla\alpha'|_{2,1-\mu,\Omega^t} + |\alpha_r v_{r,r} + \alpha_z v_{r,z}|_{2,1-\mu,\Omega^t}\\ &+ |\alpha_r v_{z,r} + \alpha_z v_{z,z}|_{2,1-\mu,\Omega^t} + \left|\frac{\chi}{r}h'\right|_{2,1-\mu,\Omega^t}\\ &+ \left|\frac{1}{r^2}(h_{r,z} - h_{z,r})\right|_{2,1-\mu,\Omega^t} + \left|\frac{\alpha_r}{r^2}\right|_{2,1-\mu,\Omega^t}\\ &+ |F'|_{2,1-\mu,\Omega^t} + \left|\left|\left|\frac{2a_1}{r}w - \frac{\gamma}{\nu}w\right|\right|\right|_{3/2,2,1-\mu,S^t}\\ &+ |||\beta_1 h_r + \beta_2 h_z + \beta_3 w_{,r} + \beta_4 w_{,z} + \beta_5 w|||_{1/2,2,1-\mu,S^t}\\ &+ |||\alpha'(0)|||_{1,2,1-\mu,\Omega}\bigg),\end{aligned}$$

where $h' = (h_r, h_z)$.

Now we estimate the terms from the r.h.s. of (3.10). The first term is estimated by

$$|\nabla\alpha'|_{4,1-\mu,\Omega^t}|v|_{4,\Omega^t} \le \varepsilon_1|||\alpha'|||_{2,2,1-\mu,\Omega^t} + \varphi(1/\varepsilon_1, |v|_{4,\Omega^t})|\alpha'|_{2,1-\mu,\Omega^t},$$

where φ is an increasing positive function.

The second and the third by

$$\begin{aligned}c|\alpha'\cdot\nabla v|_{2,1-\mu,\Omega^t} &\le c|\alpha'|_{6,1-\mu,\Omega^t}|\nabla v|_{3,\Omega^t}\\ &\le \varepsilon_2|||\alpha'|||_{2,2,1-\mu,\Omega^t} + \varphi(1/\varepsilon_2, |\nabla v|_{3,\Omega^t})|\alpha'|_{2,1-\mu,\Omega^t},\end{aligned}$$

where φ as above.

The fourth term by

$$\begin{aligned}\left(\int_0^t |\chi|^2_{2,\Omega}|h|^2_{\infty,-\mu,\Omega}\,dt'\right)^{1/2} &\le c\left(\int_0^t |\chi|^2_{2,\Omega}\|h\|^2_{2,-\mu,\Omega}\,dt'\right)^{1/2}\\ &\le c|\nabla v|_{2,\infty,\Omega^t}\|h\|_{L_2(0,t;H^2_{-\mu}(\Omega))}.\end{aligned}$$

The fifth term by $c\|h\|_{L_2(0,t;H^2_{-\mu}(\Omega))}$. The eighth by $c\|w\|_{2,1-\mu,\Omega^t}$. Finally, the ninth by $c(\|h\|_{2,-\mu,\Omega^t}+\|w\|_{2,1-\mu,\Omega^t})$. To show the last estimate see the proof of Lemma 4.3 from [20]. From the above estimates we obtain (3.9). □

Next we have

Lemma 3.3. *Assume that*

$$v\in L_2(0,T;W^1_3(\Omega))\cap L_\infty(0,T;H^1(\Omega))\cap L_2(0,T;L_\infty(\Omega)),$$
$$h\in H^{2,1}_{-1}(\Omega^T),\quad w\in H^{2,1}_{1-\mu}(\Omega^T),\quad F'\in L_2(0,T;L_{6/5}(\Omega)),\quad \alpha'(0)\in L_2(\Omega).$$

Assume that

$$\sum_{i=0}^2 r^i\left|\nabla^i\frac{a_1}{r}\right|\le c,\qquad \sum_{i=0}^1 r^{3-\mu+i}|\nabla^i\beta_j|\le c,\quad j=1,2,$$
$$\sum_{i=0}^1 r^i|\nabla^i\beta_j|\le c,\quad j=3,4,\qquad \sum_{i=0}^1 r^{1+i}|\nabla^i\beta_5|\le c.$$

$S_1=\{x\in S:a_1>0\}$ *and* $S_2=\{x\in S:\ a_2>0\}$. *Assume that any point of* Ω *can be reached by a curve from some point of* S_1. *Assume that*

$$\sum_{i=0}^2 r^{2-i}\left|\nabla^i\frac{a_1}{a_2}\right|\le c\quad \text{on } S_2.$$

Then solutions to problem (2.11) *satisfy the estimate*

$$\begin{aligned}(3.11)\quad &|\alpha'(t)|^2_{2,\Omega}+\nu\int_0^t\|\alpha'(t')\|^2_{1,\Omega}\,dt'+\nu|\alpha_r|^2_{2,-1,\Omega^t}\\ &\le c\exp(c|v_{,x}|^2_{3,2,\Omega^t})[\varphi(|v_{,x}|_{2,\infty,\Omega^t},|v|_{\infty,2,\Omega^t},|v_{,x}|_{3,2,\Omega^t})\\ &\quad\cdot(\|h\|^2_{2,-1,\Omega^t}+\|w\|^2_{2,1-\mu,\Omega^t})+|F'|^2_{6/5,2,\Omega^t}+|\alpha(0)|^2_{2,\Omega}],\end{aligned}$$

for $t\le T$, *where* φ *is an increasing positive function*

Proof. To show (3.11) we introduce functions $\widetilde\alpha'=(\widetilde\alpha_r,\widetilde\alpha_z)$ as solutions of the problem

$$(3.12)\quad \begin{aligned}&\widetilde\alpha_{r,t}-\nu\Delta\widetilde\alpha_r=0 &&\text{in } \Omega^T,\\ &\widetilde\alpha_{z,t}-\nu\Delta\widetilde\alpha_z=0 &&\text{in } \Omega^T,\\ &a_2\widetilde\alpha_r-a_1\widetilde\alpha_z=-\frac{2a_1}{r}w+\frac{\gamma}{\nu}w=g_1 &&\text{on } S^T,\\ &(a_1\widetilde\alpha_r+a_2\widetilde\alpha_z)_{,n}=\beta_1h_r+\beta_2h_z+\beta_3w_{,r}+\beta_4w_{,z}+\beta_5w\equiv g_2 &&\text{on } S^T,\\ &\widetilde\alpha_r|_{t=0}=0,\quad \widetilde\alpha_z|_{t=0}=0 &&\text{in } \Omega.\end{aligned}$$

Defining the functions

$$(3.13)\qquad \overline\alpha_r=\alpha_r-\widetilde\alpha_r,\quad \overline\alpha_z=\alpha_z-\widetilde\alpha_z,\quad \overline\alpha'=(\overline\alpha_r,\overline\alpha_z),\quad \widetilde\alpha'=(\widetilde\alpha_r,\widetilde\alpha_z),$$

we see that they are solutions to the problem

$$(3.14)\quad \begin{aligned} &\overline{\alpha}_{r,t} + v\cdot\nabla\overline{\alpha}_r - (\overline{\alpha}_r v_{r,r} + \overline{\alpha}_z v_{r,z}) - \frac{\chi}{r}h_r + \frac{2\nu}{r^2}(h_{r,z} - h_{z,r})\\ &\quad + \nu\frac{\overline{\alpha}_r}{r^2} - \nu\Delta\overline{\alpha}_r = F_r - v\cdot\nabla\widetilde{\alpha}_r + (\widetilde{\alpha}_r v_{r,r} + \widetilde{\alpha}_z v_{r,z}) - \nu\frac{\widetilde{\alpha}_r}{r^2} &&\text{in } \Omega^T,\\ &\overline{\alpha}_{z,t} + v\cdot\nabla\overline{\alpha}_z - (\overline{\alpha}_r v_{z,r} + \overline{\alpha}_z v_{z,z}) - \frac{\chi}{r}h_z - \nu\Delta\overline{\alpha}_z\\ &\quad = F_z - v\cdot\nabla\widetilde{\alpha}_z + (\widetilde{\alpha}_r v_{z,r} + \widetilde{\alpha}_z v_{z,z}) &&\text{in } \Omega^T,\\ &a_2\overline{\alpha}_r - a_1\overline{\alpha}_z = 0 &&\text{on } S^T,\\ &(a_1\overline{\alpha}_r + a_2\overline{\alpha}_z)_{,n} = 0 &&\text{on } S^T,\\ &\overline{\alpha}'|_{t=0} = \overline{\alpha}'(0) &&\text{in } \Omega. \end{aligned}$$

For solutions of (3.12) we have (for more details see the proof of Lemma 4.4 in [20])

$$(3.15)\qquad |||\widetilde{\alpha}'|||_{2,2,1-\mu,\Omega^t} \le c(\|h\|_{2,-1,\Omega^t} + \|w\|_{2,1-\mu,\Omega^t}).$$

Now we obtain an energy estimate for solutions to (3.14). Multiplying $(3.14)_1$ by $\overline{\alpha}_r$, $(3.14)_2$ by $\overline{\alpha}_z$, integrating the results over Ω and adding yields

$$(3.16)\quad \begin{aligned} &\frac{1}{2}\frac{d}{dt}|\overline{\alpha}'|^2_{2,\Omega} - \int_\Omega [\overline{\alpha}_r^2 v_{r,r} + \overline{\alpha}_r\overline{\alpha}_z(v_{r,z} + v_{z,r}) + \overline{\alpha}_z^2 v_{z,z}]\,dx\\ &\quad - \int_\Omega \frac{\chi}{r}(h_r\overline{\alpha}_r + h_z\overline{\alpha}_z)\,dx + 2\nu\int_\Omega \frac{1}{r^2}(h_{r,z} - h_{z,r})\overline{\alpha}_r\,dx\\ &\quad + |\overline{\alpha}_r|^2_{2,-1,\Omega} - \nu\int_\Omega (\Delta\overline{\alpha}_r\overline{\alpha}_r + \Delta\overline{\alpha}_z\overline{\alpha}_z)\,dx\\ &= \int_\Omega (F_r\overline{\alpha}_r + F_z\overline{\alpha}_z)\,dx\\ &\quad - \int_\Omega (v\cdot\nabla\widetilde{\alpha}_r\overline{\alpha}_r + v\cdot\nabla\widetilde{\alpha}_z\overline{\alpha}_z)\,dx - \nu\int_\Omega \frac{\widetilde{\alpha}_r}{r^2}\overline{\alpha}_r\,dx\\ &\quad + \int_\Omega [(\widetilde{\alpha}_r v_{r,r} + \widetilde{\alpha}_z v_{r,z})\overline{\alpha}_r + (\widetilde{\alpha}_r v_{z,r} + \widetilde{\alpha}_z v_{z,z})\overline{\alpha}_z]\,dx. \end{aligned}$$

The term with laplacians equals $\nu|\nabla\overline{\alpha}'|^2_{2,\Omega}$ (see the proof of Lemma 4.4 [20]).

The second term on the l.h.s. of (3.16) is estimated by

$$\varepsilon|\overline{\alpha}'|^2_{6,\Omega} + c(1/\varepsilon)|v_{,x}|^2_{3,\Omega}|\overline{\alpha}'|^2_{2,\Omega}.$$

The third by

$$\varepsilon(|\overline{\alpha}_r|^2_{2,-1,\Omega} + |\overline{\alpha}_z|^2_{2,\Omega}) + c(1/\varepsilon)\int_\Omega \frac{\chi^2}{r^2}(h_r^2 + h_z^2)\,dx.$$

The fourth by

$$\varepsilon|\overline{\alpha}_r|^2_{2,-1,\Omega} + c(1/\varepsilon)\|h\|^2_{2,-1,\Omega}.$$

The first term on the r.h.s. of (3.16) is bounded by

$$\varepsilon|\overline{\alpha}'|^2_{6,\Omega} + c(1/\varepsilon)|F'|^2_{6/5,\Omega},$$

the second by

$$\varepsilon|\nabla\overline{\alpha}'|^2_{2,\Omega} + c(1/\varepsilon)|v|^2_{\infty,\Omega}|\widetilde{\alpha}'|^2_{2,\Omega},$$

the third by

$$\varepsilon|\overline{\alpha}_r|^2_{2,-1,\Omega} + c(1/\varepsilon)|\widetilde{\alpha}_r|^2_{2,-1,\Omega},$$

and finally, the last by

$$\varepsilon|\overline{\alpha}'|_{6,\Omega} + c(1/\varepsilon)|v_{,x}|^2_{3,\Omega}|\widetilde{\alpha}'|^2_{2,\Omega}.$$

Summarizing the above results we obtain

$$\begin{aligned}(3.17)\quad \frac{1}{2}\frac{d}{dt}|\overline{\alpha}'|^2_{2,\Omega} &+ \nu|\nabla\overline{\alpha}'|^2_{2,\Omega} + \nu|\overline{\alpha}_r|^2_{2,-1,\Omega} \le \varepsilon(|\overline{\alpha}_z|^2_{6,\Omega} + |\overline{\alpha}_z|^2_{2,\Omega})\\ &+ c(1/\varepsilon)|v_{,x}|^2_{3,\Omega}|\overline{\alpha}'|^2_{2,\Omega} + c(1/\varepsilon)|\chi|^2_{2,\Omega}\|h\|^2_{2,-1,\Omega} + c\|h\|^2_{2,-1,\Omega}\\ &+ c|F'|^2_{6/5,\Omega} + c|v|^2_{\infty,\Omega}|\widetilde{\alpha}'|^2_{2,\Omega} + c|\widetilde{\alpha}_r|^2_{2,-1,\Omega} + c(1/\varepsilon)|v_{,x}|^2_{3,\Omega}|\widetilde{\alpha}'|^2_{2,\Omega}.\end{aligned}$$

In view of (4.25) from [20] we have

$$(3.18)\qquad |\overline{\alpha}_z|_{2,\Omega} \le c(|\nabla\overline{\alpha}_z|_{2,\Omega} + \|\overline{\alpha}_r\|_{1,\Omega}).$$

In virtue of (3.18) inequality (3.17) takes the form

$$\begin{aligned}(3.19)\qquad \frac{d}{dt}|\overline{\alpha}'|^2_{2,\Omega} &+ \nu\|\overline{\alpha}'\|^2_{1,\Omega} + \nu|\overline{\alpha}_r|^2_{2,-1,\Omega} \le c|v_{,x}|^2_{3,\Omega}|\overline{\alpha}'|^2_{2,\Omega}\\ &+ c|v_{,x}|^2_{2,\Omega}\|h\|^2_{2,-1,\Omega} + c\|h\|^2_{2,-1,\Omega} + c|F'|^2_{6/5,\Omega}\\ &+ c|v|^2_{\infty,\Omega}|\widetilde{\alpha}'|^2_{2,\Omega} + c|\widetilde{\alpha}_r|^2_{2,-1,\Omega} + c|v_{,x}|^2_{3,\Omega}|\widetilde{\alpha}'|^2_{2,\Omega}.\end{aligned}$$

By the energy method we obtain for solutions to problem (3.12) the inequality (see (4.32) in [20])

$$(3.20)\qquad |\widetilde{\alpha}'|^2_{2,\Omega} + |\nabla\widetilde{\alpha}'|^2_{2,\Omega^t} + |||\widetilde{\alpha}'|||^2_{2,2,1-\mu,\Omega^t} \le c(\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t}).$$

By (4.35) in [20] (see also (4.32) and (4.34) in [20]) we have

$$(3.21)\qquad \|\widetilde{\alpha}_r\|_{2,1-\mu,\Omega^t} \le c(\|w\|_{2,1-\mu,\Omega^t} + \|h\|_{2,-1,\Omega^t}) \equiv cI,$$

so

$$(3.22)\qquad \|\widetilde{\alpha}_r\|_{1,-\mu,\Omega^t} \le cI.$$

Integrating (3.19) with respect to time and using (3.20) and (3.22) yields

$$
\begin{aligned}
(3.23)\qquad |\overline{\alpha}'(t)|^2_{2,\Omega} &+ \nu\int_0^t \|\overline{\alpha}'(t')\|^2_{1,\Omega}\,dt' + \nu|\overline{\alpha}_r|^2_{2,-1,\Omega^t}\\
&\le c\exp(c|v_{,x}|^2_{3,2,\Omega^t})[|v_{,x}|^2_{2,\infty,\Omega^t}\|h\|^2_{2,-1,\Omega^t}\\
&\quad + (1+|v|^2_{\infty,2,\Omega^t} + |v_{,x}|^2_{3,2,\Omega^t})(\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t})\\
&\quad + |F'|^2_{6/5,2,\Omega^t} + |\alpha'(0)|^2_{2,\Omega}].
\end{aligned}
$$

Using (3.15) and (3.22) in (3.23) we obtain (3.11). □

Finally, we have

Lemma 3.4. *Assume that*

$$
\begin{aligned}
v \in L_\infty(0,T;W^1_2(\Omega)) &\cap L_2(0,T;W^1_\infty(\Omega)) \cap L_\infty(0,T;W^1_{p'}(\Omega))\\
&\cap L_2(0,T;W^1_3(\Omega)) \cap L_4(\Omega^T) \cap L_3(0,T;W^1_3(\Omega)) \equiv \mathfrak{N}_1(\Omega^T),
\end{aligned}
$$

for $p'<3$. Assume that

$$
\begin{gathered}
h \in H^{2,1}_{-1}(\Omega^T),\quad w \in H^{2,1}_{1-\mu}(\Omega^T),\quad F' \in L_2(0,T;L_{6/5}(\Omega)),\\
F_r \in L_2(0,T;L_{6/5,-\mu}(\Omega)),\quad \alpha'(0)\in L_2(\Omega),\quad \alpha_r(0)\in L_{2,-\mu}(\Omega),\\
F' \in L_{2,1-\mu}(\Omega^T),\quad \alpha'(0)\in W^1_{2,1-\mu}(\Omega),\quad \mu\in(0,1).
\end{gathered}
$$

Then, for $t\le T$,

$$
\begin{aligned}
(3.24)\quad \nu\int_0^t \|\alpha_r(t')\|^2_{1,-\mu,\Omega} \le \varphi(\|v\|_{\mathfrak{N}_1(\Omega^T)})[\|h\|^2_{2,-1,\Omega^t} &+ \|w\|^2_{2,1-\mu,\Omega^t}\\
+ |F'|^2_{6/5,2,\Omega^t} &+ |F_r|^2_{6/5,2,-\mu,\Omega^t} + |\alpha'(0)|^2_{2,\Omega}\\
+ |\alpha_r(0)|^2_{2,-\mu,\Omega} &+ |F'|^2_{2,1-\mu,\Omega^t} + |||\alpha'(0)|||^2_{1,2,1-\mu,\Omega}],
\end{aligned}
$$

where φ is an increasing positive function.

Proof. In view of (3.20), (3.22) we have to find an estimate for $\|\overline{\alpha}_r\|_{1,-\mu,\Omega^t}$. Multiplying $(3.14)_1$ by $\overline{\alpha}_r r^{-2\mu}$ and integrating over Ω implies

$$
\begin{aligned}
(3.25)\quad \frac12\frac{d}{dt}|\overline{\alpha}_r|^2_{2,-\mu,\Omega} &+ \int_\Omega v\cdot\nabla\overline{\alpha}_r\overline{\alpha}_r r^{-2\mu}\,dx - \int_\Omega(\overline{\alpha}_r v_{r,r} + \overline{\alpha}_z v_{r,z})\overline{\alpha}_r r^{-2\mu}\,dx\\
&- \int_\Omega \frac{\chi}{r}h_r\overline{\alpha}_r r^{-2\mu}\,dx + 2\nu\int_\Omega\frac{1}{r^2}(h_{r,z}-h_{z,r})\overline{\alpha}_r r^{-2\mu}\,dx\\
&+ \nu|\overline{\alpha}_r|^2_{2,-(1+\mu),\Omega} - \nu\int_\Omega \Delta\overline{\alpha}_r\overline{\alpha}_r r^{-2\mu}\,dx\\
&= \int_\Omega F_r\overline{\alpha}_r r^{-2\mu}\,dx - \int_\Omega v\cdot\nabla\widetilde{\alpha}_r\overline{\alpha}_r r^{-2\mu}\,dx\\
&+ \int_\Omega(\widetilde{\alpha}_r v_{r,r} + \widetilde{\alpha}_z v_{r,z})\overline{\alpha}_r r^{-2\mu}\,dx - \nu\int_\Omega\frac{\widetilde{\alpha}_r}{r^2}\overline{\alpha}_r r^{-2\mu}\,dx.
\end{aligned}
$$

Now we examine the particular terms in (3.25). The second term on the l.h.s. equals

$$\frac{1}{2}\int_\Omega v\cdot\nabla\overline{\alpha}_r^2 r^{-2\mu}\,dx = \frac{1}{2}\int_\Omega v\cdot\nabla(\overline{\alpha}_r^2 r^{-2\mu})\,dx + \mu\int_\Omega \overline{\alpha}_r^2 r^{-2\mu-1} v\cdot\nabla r\,dx,$$

where the first term vanishes and the second is estimated by

$$\varepsilon|\overline{\alpha}_r|^2_{2,-(1+\mu),\Omega} + c(1/\varepsilon)|v|^2_{\infty,\Omega}|\overline{\alpha}_r|^2_{2,-\mu,\Omega}.$$

The third term on the l.h.s. of (3.25) is bounded by

$$\varepsilon|\overline{\alpha}_r|^2_{6,-\mu,\Omega} + c(1/\varepsilon)|v_{,x}|^2_{p',\Omega}|\overline{\alpha}'|^2_{p,-\mu,\Omega},$$

where $1/p+1/p'=5/6$, $p'<3$, $p>2$, the fourth by

$$\varepsilon|\overline{\alpha}_r|^2_{2,-(1+\mu),\Omega} + c(1/\varepsilon)|\chi|^2_{2,\Omega}|h|^2_{\infty,-\mu,\Omega},$$

the fifth by

$$\varepsilon|\overline{\alpha}_r|^2_{2,-(1+\mu),\Omega} + c(1/\varepsilon)|h_{,x}|^2_{2,-(1+\mu),\Omega}.$$

Integrating by parts the term with laplacian takes the form

$$\nu\int_\Omega \overline{\alpha}_{r,x}^2 r^{-2\mu}\,dx - \nu\int_\Omega \operatorname{div}(\nabla\overline{\alpha}_r\overline{\alpha}_r r^{-2\mu})\,dx \\ - 2\mu\nu\int_\Omega \nabla\overline{\alpha}_r\overline{\alpha}_r r^{-2\mu-1}\nabla r\,dx \equiv \nu|\nabla\overline{\alpha}_r|^2_{2,-\mu,\Omega} + I_1 + I_2,$$

where (see the proof of Lemma 4.5 in [20])

$$|I_1| \le \varepsilon_1|||\overline{\alpha}'|||_{2,2,1-\mu,\Omega} + c(1/\varepsilon_1)\|\overline{\alpha}'\|_{1,\Omega}$$

and

$$|I_2| \le \nu\bigg[\frac{\varepsilon}{2}|\overline{\alpha}_{r,x}|^2_{2,-\mu,\Omega} + \frac{2\mu^2}{\varepsilon}|\overline{\alpha}_r|^2_{2,-(1+\mu),\Omega}\bigg].$$

The first term on the r.h.s. of (3.25) is estimated by

$$\varepsilon|\overline{\alpha}_r|^2_{6,-\mu,\Omega} + c(1/\varepsilon)|F_r|^2_{6/5,-\mu,\Omega},$$

the second by

$$\varepsilon|\nabla\widetilde{\alpha}_r|^2_{2,-\mu,\Omega} + c(1/\varepsilon)|v|^2_{\infty,\Omega}|\overline{\alpha}_r|^2_{2,-\mu,\Omega},$$

the third by

$$\varepsilon|\overline{\alpha}_r|^2_{6,-\mu,\Omega} + c(1/\varepsilon)|v_{,x}|^2_{p',\Omega}|\widetilde{\alpha}'|^2_{p,-\mu,\Omega}, \quad \frac{1}{p}+\frac{1}{p'}=\frac{5}{6}, \quad p'<3,\ p>2,$$

and finally, the last by

$$\varepsilon|\overline{\alpha}_r|^2_{2,-(1+\mu),\Omega} + c(1/\varepsilon)|\widetilde{\alpha}_r|^2_{2,-(1+\mu),\Omega}.$$

In view of the above estimates and for sufficiently small ε we obtain from (3.25) the inequality

$$\begin{aligned}(3.26)\quad \frac{d}{dt}|\overline{\alpha}_r|^2_{2,-\mu,\Omega} &+ \nu\|\overline{\alpha}_r\|^2_{1,-\mu,\Omega} \le c|v|^2_{\infty,\Omega}|\overline{\alpha}_r|^2_{2,-\mu,\Omega}\\ &+ c|v_{,x}|^2_{p',\Omega}|\overline{\alpha}'|^2_{p,-\mu,\Omega} + c|v_{,x}|^2_{2,\Omega}\|h\|^2_{2,-\mu,\Omega} + c\|h\|^2_{2,-\mu,\Omega}\\ &+ \varepsilon_1|||\overline{\alpha}'|||^2_{2,2,1-\mu,\Omega} + c(1/\varepsilon_1)\|\overline{\alpha}'\|^2_{1,\Omega} + c|F_r|^2_{6/5,-\mu,\Omega}\\ &+ c|\nabla\widetilde{\alpha}_r|^2_{2,-\mu,\Omega} + c|v_{,x}|^2_{p',\Omega}|\widetilde{\alpha}'|^2_{p,-\mu,\Omega} + c|\widetilde{\alpha}_r|^2_{2,-(1+\mu),\Omega},\end{aligned}$$

where $1/p + 1/p' = 5/6$, $p' < 3$, $p > 2$.

Integrating (3.26) with respect to time yields

$$\begin{aligned}(3.27)\quad |\overline{\alpha}_r|^2_{2,-\mu,\Omega} &+ \nu\int_0^t \|\overline{\alpha}_r\|^2_{1,-\mu,\Omega}\,dt'\\ &\le c\exp(c|v|^2_{\infty,2,\Omega^t})\cdot\Big[|v_{,x}|^2_{p',\infty,\Omega^t}(|\overline{\alpha}'|^2_{p,2,-\mu,\Omega^t} + |\widetilde{\alpha}'|^2_{p,2,-\mu,\Omega^t})\\ &\quad + |v_{,x}|^2_{2,\infty,\Omega^t}\|h\|^2_{2,-\mu,\Omega^t} + \|h\|^2_{2,-\mu,\Omega^t}\\ &\quad + \varepsilon_1\int_0^t |||\overline{\alpha}'(t')|||^2_{2,2,1-\mu,\Omega}\,dt' + c(1/\varepsilon_1)\int_0^t \|\overline{\alpha}'(t')\|^2_{1,\Omega}\,dt'\\ &\quad + |F_r|^2_{6/5,2,-\mu,\Omega^t} + |\widetilde{\alpha}_r|^2_{2,-(1+\mu),\Omega^t} + |\alpha_r(0)|^2_{2,-\mu,\Omega}\Big].\end{aligned}$$

Using (3.20)–(3.22) in (3.27) implies

$$\begin{aligned}(3.28)\quad |\overline{\alpha}_r|^2_{2,-\mu,\Omega} &+ \nu\int_0^t \|\alpha_r\|^2_{1,-\mu,\Omega}\,dt' \le \varphi(|v|_{\infty,2,\Omega^t}, |v_{,x}|_{p',\infty,\Omega^t}, |v_{,x}|_{2,\infty,\Omega^t})\\ &\cdot\Big[|\overline{\alpha}'|^2_{p,2,-\mu,\Omega^t} + \|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t}\\ &\quad + \varepsilon_1\int_0^t |||\overline{\alpha}'(t')|||^2_{2,2,1-\mu,\Omega}\,dt' + c(1/\varepsilon_1)\int_0^t \|\overline{\alpha}'(t')\|^2_{1,\Omega}\,dt'\\ &\quad + |F_r|^2_{6/5,2,-\mu,\Omega^t} + |\alpha_r(0)|^2_{2,-\mu,\Omega}\Big],\end{aligned}$$

where φ is an increasing positive function.

Employing (3.23) in (3.28) gives

$$\begin{aligned}(3.29)\quad \nu\int_0^t \|\alpha_r(t')\|^2_{1,-\mu,\Omega}\,dt' &\le \varphi(|v|_{\infty,2,\Omega^t}, |v_{,x}|_{p',\infty,\Omega^t}, |v_{,x}|_{2,\infty,\Omega^t}, |v_{,x}|_{3,2,\Omega^t})\\ &\cdot\Big[\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t} + |F'|^2_{6/5,2,\Omega^t} + |F_r|^2_{6/5,2,-\mu,\Omega^t}\\ &\quad + |\alpha'(0)|^2_{2,\Omega} + |\alpha_r(0)|^2_{2,-\mu,\Omega} + \varepsilon\int_0^t |||\overline{\alpha}'(t')|||^2_{2,2,1-\mu,\Omega}\,dt'\Big].\end{aligned}$$

Exploiting (3.9) with (3.11) in (3.29) and assuming that ε is sufficiently small we get (3.24). □

In virtue of (3.24) we obtain from (3.8) the inequality

$$
\begin{aligned}
(3.30)\qquad & |v_{\varphi,z}|^2_{4,2,-3/4-\varepsilon,\Omega^t} \\
&\le c\biggl(\int_0^t \|\alpha_r(t')\|^2_{1,-1/2-\varepsilon,\Omega}\,dt' + \|h\|^2_{2,-1,\Omega^t}\biggr)\\
&\le \varphi(\|v\|_{\mathfrak{N}'_1(\Omega^t)})\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t}\\
&\quad + |F'|^2_{6/5,2,\Omega^t} + |F_r|^2_{6/5,2,-\mu,\Omega^t} + |\alpha'(0)|^2_{2,\Omega} + |\alpha_r(0)|^2_{2,-\mu,\Omega}\\
&\quad + |F'|^2_{2,1-\mu,\Omega^t} + |||\alpha'(0)|||^2_{1,2,1-\mu,\Omega}],
\end{aligned}
$$

where $1/2+\varepsilon<\mu$, $\mu\in(0,1)$ and

$$
(3.31)\quad \|v\|_{\mathfrak{N}'_1(\Omega^T)} = |v|_{\infty,2,\Omega^T} + |v_{,x}|_{p',\infty,\Omega^T} + |v_{,x}|_{2,\infty,\Omega^T} + |v_{,x}|_{3,2,\Omega^T},\quad p'<3.
$$

We need an estimate for $\|w\|_{2,1-\mu,\Omega^t}$. However, it must be shown in a different way than in Lemma 5.1 from [20].

Lemma 3.5. *Assume that*

$$
\begin{gathered}
v\in L_\infty(0,T;L_4(\Omega)),\quad w\in L_{2,-(1+\mu)}(\Omega^T),\quad h\in L_{2,-(1+\mu)}(\Omega^T),\\
q\in L_{2,-\mu}(\Omega^T),\quad f_\varphi\in L_{2,1-\mu}(\Omega^T),\quad w(0)\in H^1_{1-\mu}(\Omega),\quad \mu\in(0,1),\\
|\nabla(a_1/r)|\le c/r^2,\quad |a_1/r|\le c/r^{1-\varepsilon},\quad \varepsilon>0.
\end{gathered}
$$

Then, for $t\le T$,

$$
\begin{aligned}
(3.32)\quad \|w\|_{2,1-\mu,\Omega^t} &\le \varphi(|v|_{4,\infty,\Omega^t})|w|_{2,1-\mu,\Omega^t} + c(|w|_{2,-(1+\mu),\Omega^t}\\
&\quad + |h|_{2,-(1+\mu),\Omega^t} + |q|_{2,-\mu,\Omega^t} + |f_\varphi|_{2,1-\mu,\Omega^t} + \|w(0)\|_{1,1-\mu,\Omega}),
\end{aligned}
$$

where φ is an increasing positive function.

Proof. Applying [16] (see also Lemmas 2.1, 2.2) to problem (2.9) yields

$$
\begin{aligned}
(3.33)\quad \|w\|_{2,1-\mu,\Omega^t} \le c\biggl(&|v\cdot\nabla w|_{2,1-\mu,\Omega^t} + \biggl|\frac{v_r w}{r}\biggr|_{2,1-\mu,\Omega^t} + |w|_{2,-(1+\mu),\Omega^t}\\
&+ |q|_{2,-\mu,\Omega^t} + |h|_{2,-(1+\mu),\Omega^t} + |f_\varphi|_{2,1-\mu,\Omega^t}\\
&+ \|w\|_{1/2,1-\mu,S^t} + \biggl\|\frac{a_1}{r}w\biggr\|_{1/2,1-\mu,S^t} + \|w(0)\|_{1,1-\mu,\Omega}\biggr).
\end{aligned}
$$

Let us examine the particular terms from the r.h.s. of (3.33). The first term is estimated by

$$
\sup_t |v|_{2p,\Omega}\|\nabla w\|_{L_2(0,t;L_{2p',1-\mu}(\Omega))}\equiv I_1,
$$

where $1/p+1/p'=1$.

Using the interpolation inequality

$$
\|\nabla w\|_{L_2(0,T;L_{2p',1-\mu}(\Omega))}\le \varepsilon\|w\|_{2,1-\mu,\Omega^T} + c(1/\varepsilon)|w|_{2,1-\mu,\Omega^T}
$$

which holds for $p' < 3$, so we can choose $p = p' = 2$. Hence we obtain

$$I_1 \le \varepsilon_1 \|w\|_{2,1-\mu,\Omega^t} + \varphi(1/\varepsilon_1, |v|_{4,\infty,\Omega^t})|w|_{2,1-\mu,\Omega^t},$$

where φ is an increasing positive function. The second term on the r.h.s. of (3.33) is treated in the way

$$\left|v_r \frac{w}{r}\right|_{2,1-\mu,\Omega^t} \le \sup_t |v_r|_{2p,\Omega} \left\|\frac{w}{r}\right\|_{L_2(0,t;L_{2p',1-\mu}(\Omega))} \equiv I_2,$$

where $1/p + 1/p' = 1$.

Using the interpolation inequality

$$\begin{aligned}\left\|\frac{w}{r}\right\|_{L_2(0,T;L_{2p',1-\mu}(\Omega))} &= \|w\|_{L_2(0,T;L_{2p',-\mu}(\Omega))} \\ &\le \varepsilon \|w\|_{2,1-\mu,\Omega^T} + c(1/\varepsilon)|w|_{2,1-\mu,\Omega^T},\end{aligned}$$

which holds for $p' < 3$, so we can choose $p = p' = 2$. Hence

$$I_2 \le \varepsilon_2 \|w\|_{2,1-\mu,\Omega^t} + \varphi(1/\varepsilon_2, |v|_{4,\infty,\Omega^t})|w|_{2,1-\mu,\Omega^t}.$$

The boundary terms on the r.h.s. of (3.33) are estimated in the way

$$\|w\|_{1/2,1-\mu,S^t} \le \varepsilon_3 \|w\|_{2,1-\mu,\Omega^t} + c(1/\varepsilon_3)|w|_{2,1-\mu,\Omega^t},$$

and

$$\begin{aligned}\left\|\frac{a_1}{r} w\right\|_{1/2,1-\mu,S^t} &\le c\left\|\frac{a_1}{r} w\right\|_{1,1-\mu,\Omega^t} \le c\Bigg(\left|\nabla\left(\frac{a_1}{r}\right) w\right|_{2,1-\mu,\Omega^t} \\ &\quad + \left|\frac{a_1}{r}\nabla w\right|_{2,1-\mu,\Omega^t} + \left|\frac{a_1}{r}\langle w\rangle_{1/2,2,(0,t),t}\right|_{2,1-\mu,\Omega}\Bigg) \equiv I_3,\end{aligned}$$

where

$$\langle w\rangle_{\alpha,p,(0,T),t} = \left(\int_0^T \int_0^T \frac{|w(t) - w(t')|^p}{|t-t'|^{1+\alpha p}}\, dt\, dt'\right)^{1/p}.$$

Assuming that

$$\left|\nabla\left(\frac{a_1}{r}\right)\right| \le \frac{c}{r^2}$$

the first term in I_3 is estimated by $c\|w\|_{2,-(1+\mu),\Omega^t}$. Assuming,

$$\left|\frac{a_1}{r}\right| \le \frac{c}{r^{1-\varepsilon}}, \quad \varepsilon > 0 \tag{3.34}$$

small positive number, the second term in I_3 is bounded by

$$|\nabla w|_{2,-(\mu-\varepsilon),\Omega^t} \le \varepsilon_4 \|w\|_{2,1-\mu,\Omega^t} + c(1/\varepsilon_4)|w|_{2,1-\mu,\Omega^t},$$

because

$$\frac{5}{2} - \frac{5}{2} + 1 + 1 - \mu - (-\mu + \varepsilon) = 2 - \varepsilon < 2. \tag{3.35}$$

In view of (3.34) the last term in I_3 is estimated by

$$|\langle w\rangle_{1/2,2,(0,t),t}|_{2,-(\mu-\varepsilon),\Omega} \le \varepsilon_5\|w\|_{2,1-\mu,\Omega^t} + c(1/\varepsilon_5)|w|_{2,1-\mu,\Omega^t}$$

because (3.35) also holds.

Employing the above estimates in (3.33) implies (3.32) for sufficiently small $\varepsilon_1 - \varepsilon_5$. □

We need to estimate $|w|_{2,-(1+\mu),\Omega^t}$ which appears in (3.32).

LEMMA 3.6. *Assume that*

$$v \in L_2(0,T;L_\infty(\Omega)), \quad q \in L_{2,-\mu}(\Omega^T), \quad h \in L_{2,-(1+\mu)}(\Omega^T),$$
$$f_\varphi \in L_1(0,T;L_2(\Omega)), \quad w(0) \in L_{2,-\mu}(\Omega), \quad \mu \in (0,1), \quad |a_1/r| \le c.$$

Then

$$\begin{aligned}(3.36)\quad & |w(t)|^2_{2,-\mu,\Omega} + \nu|\nabla w|^2_{2,-\mu,\Omega^t} + \nu|w|^2_{2,-(1+\mu),\Omega^t} + \gamma|w|^2_{2,-\mu,S^t} \\ &\le c\exp(\|v\|^2_{\infty,2,\Omega^t} + t)[|q|^2_{2,-\mu,\Omega^t} + |h|^2_{2,-(1+\mu),\Omega^t} + |f_\varphi|^2_{2,1,\Omega^t} + |w(0)|^2_{2,-\mu,\Omega}].\end{aligned}$$

PROOF. Multiplying $(2.9)_1$ by $wr^{-2\mu}$, integrating over Ω_* (see the proof of Lemma 3.1) and assuming the artificial boundary condition

$$w|_{r=\varepsilon_*} = 0 \tag{3.37}$$

we obtain

$$\begin{aligned}(3.38)\quad & \frac{1}{2}\frac{d}{dt}|w|^2_{2,-\mu,\Omega_*} - \nu\int_{\Omega_*}\Delta w w r^{-2\mu}\,dx + \nu|w|^2_{2,-(1+\mu),\Omega_*} \\ &= -\int_{\Omega_*}\left(v\cdot\nabla w + \frac{v_r}{r}w\right)wr^{-2\mu}\,dx + \int_{\Omega_*}\frac{1}{r}qwr^{-2\mu}\,dx \\ &\quad + 2\nu\int_{\Omega_*}\frac{1}{r^2}h_r wr^{-2\mu}\,dx + \int_{\Omega_*}f_\varphi wr^{-2\mu}\,dx.\end{aligned}$$

The second term on the l.h.s. of (3.38) equals

$$\begin{aligned}&-\nu\int_{S_*}\frac{a_1}{r}w^2r^{-2\mu}dS + \nu\int_{-a}^{a}dz\int_0^{2\pi}d\varphi\frac{\partial w}{\partial r}wr^{-2\mu+1}\bigg|_{r=r_*} \\ &+\nu\int_{\Omega_*}|\nabla w|^2r^{-2\mu}\,dx + \gamma\int_{S_*}w^2r^{-2\mu}dS - 2\mu\nu\int_\Omega\nabla w w r^{-2\mu-1}\nabla r\,dx \equiv I_1,\end{aligned}$$

where $S_* = \partial\Omega\cap\partial\Omega_*$.

In view of (3.37) the second term in I_1 vanishes. Assuming $|a_1/r| \le c$ the first term in I_1 is estimated by

$$\varepsilon_1|w_{,x}|^2_{2,-\mu,\Omega} + c(1/\varepsilon_1)|w|^2_{2,-\mu,\Omega}.$$

Applying the Hölder and Young inequalities we estimate the last term in I_1 by

$$\nu\frac{\varepsilon_0}{2}|\nabla w|^2_{2,-\mu,\Omega}+\nu\frac{2\mu^2}{\varepsilon_0}|w|^2_{2,-(1+\mu),\Omega}.$$

The first term on the r.h.s. of (3.38) takes the form

$$\begin{aligned}-\int_{\Omega_*}\Big(v\cdot\nabla ww+\frac{v_r}{r}w^2\Big)r^{-2\mu}\,dx&=-\int_{\Omega_*}\Big(\frac{1}{2}v\cdot\nabla w^2+\frac{v_r}{r}w^2\Big)r^{-2\mu}\,dx\\&=-\int_{\Omega_*}\Big[\frac{1}{2}v\cdot\nabla(w^2r^{-2\mu})+(1+\mu)\frac{v_r}{r}w^2r^{-2\mu}\Big]\,dx\\&=-(1+\mu)\int_{\Omega_*}\frac{v_r}{r}w^2r^{-2\mu}\,dx\equiv I_2,\end{aligned}$$

where the boundary condition $(1.1)_3$ and (3.37) were exploited. Hence

$$|I_2|\le\varepsilon_2|w|^2_{2,-(1+\mu),\Omega_*}+c(1/\varepsilon_2)|v|^2_{\infty,\Omega_*}|w|^2_{2,-\mu,\Omega_*}.$$

The last three terms on the r.h.s. of (3.38) are estimated by

$$\varepsilon_3|w|^2_{2,-(1+\mu),\Omega_*}+c(1/\varepsilon_3)(|q|^2_{2,-\mu,\Omega_*}+|h|^2_{2,-(1+\mu),\Omega_*}+|f_\varphi|^2_{2,1,\Omega_*}).$$

Using the above estimates in (3.38) and assuming that $\varepsilon_1-\varepsilon_3$ are sufficiently small and $\varepsilon_0\in(2\mu^2,2)$ we obtain

$$\begin{aligned}&\frac{d}{dt}|w|^2_{2,-\mu,\Omega_*}+\nu|\nabla w|^2_{2,-\mu,\Omega_*}+\nu|w|^2_{2,-(1+\mu),\Omega_*}+\gamma|w|^2_{2,-\mu,S_*}\\&\le c(|v|^2_{\infty,\Omega_*}+1)|w|^2_{2,-\mu,\Omega_*}+c(|q|^2_{2,-\mu,\Omega_*}+|h|^2_{2,-(1+\mu),\Omega_*}+|f_\varphi|^2_{2,1,\Omega_*}).\end{aligned}\tag{3.39}$$

Integrating (3.39) with respect to time and passing with ε_* to 0 we obtain (3.36). □

Remark 3.7. Let us assume that $\psi=z+r^{\alpha+2}-a$ near the point $(r,z)=(0,a)$ (similar assumption can be set near $(r,z)=(0,-a)$, i.e. $\psi=2+r^{\alpha+2}+a$). Then $|\psi_{,r}|\le cr^{\alpha+1}$, so $|a_1|\le cr^{\alpha+1}$ and the first integral in I_1 (see the proof of Lemma 3.6) can be considered in the way

$$\widetilde{I}=\int_{S\cap\{x:r<r_0\}}r^\alpha w^2r^{-2\mu}dS+\int_{S\cap\{x:r\ge r_0\}}r^\alpha w^2r^{-2\mu}dS\equiv\widetilde{I}_1+\widetilde{I}_2.$$

Let $\alpha<2\mu$. Then

$$\begin{aligned}|\widetilde{I}_1|&\le cr_0^\alpha|w|^2_{2,-\mu,S}\le cr_0^\alpha(|w_{,x}|^2_{2,-\mu,\Omega}+|w|^2_{2,-\mu,\Omega})\equiv\widetilde{I}_1'\\|\overline{I}_2|&\le cr_0^{\alpha-2\mu}|w|^2_{2,S}\le cr_0^{\alpha-2\mu}\|w\|^2_{1,\Omega}\equiv\widetilde{I}_2'\end{aligned}$$

Hence for sufficiently small r_0 the term $\widetilde{I}_1'$ can be absorbed by the l.h.s. of (3.39), but $\widetilde{I}_2'$ can be bounded by the estimate of the weak solution.

For $\alpha \ge 2\mu$, we have $|\widetilde{I}| \le c\|w\|_{1,\Omega}^2$, so it is estimated in terms of the weak solutions. Then instead of (3.36) we obtain

$$|w(t)|_{2,-\mu,\Omega}^2 + \nu\|w\|_{1,-\mu,\Omega^t}^2 + \gamma|w|_{2,-\mu,S^t}^2 \le c\exp(c|v|_{\infty,2,\Omega^t}^2) \cdot \bigg[\int_0^t \|w(t)\|_{1,\Omega}^2\,dt + |q|_{2,-\mu,\Omega^t}^2 + |h|_{2,-(1+\mu),\Omega^t}^2 + |f_\varphi|_{2,1,\Omega^t}^2 + |w(0)|_{2,-\mu,\Omega}^2\bigg].$$

□

From (3.32) and (3.36) we obtain

$$\|w\|_{2,1-\mu,\Omega^t} \le \varphi(t,|v|_{4,\infty,\Omega^t},|v|_{\infty,2,\Omega^t}) \cdot [|q|_{2,-\mu,\Omega^t} + |h|_{2,-(1+\mu),\Omega^t} + |f_\varphi|_{2,1-\mu,\Omega^t} + \|w(0)\|_{1,1-\mu,\Omega}]. \tag{3.40}$$

Let

$$\|v\|_{\mathfrak{N}_2(\Omega^T)} = \|v\|_{\mathfrak{N}_\delta(\Omega^T)} + |v|_{4,\infty,\Omega^T} + |v|_{\infty,2,\Omega^T},$$

where the first norm on the r.h.s. is defined by (2.21) and let

$$X_1(T) = |g|_{6/5,2,\Omega^T} + |g|_{2,-(1+\varepsilon_*),\Omega^T} + |f_\varphi|_{2,1-\mu,\Omega^T} + \|h(0)\|_{1,-(1+\varepsilon_*),\Omega} + \|w(0)\|_{1,1-\mu,\Omega}. \tag{3.41}$$

Then in view of (2.22) we obtain from (3.40) the inequality

$$\|w\|_{2,1-\mu,\Omega^t} \le \varphi(t,\|v\|_{\mathfrak{N}_2(\Omega^t)})X_1(t), \quad t \le T. \tag{3.42}$$

Let

$$X_2(T) = |F'|_{6/5,2,\Omega^T} + |F_r|_{6/5,2,-\mu,\Omega^T} + |F'|_{2,1-\mu,\Omega^T} + |\alpha'(0)|_{2,\Omega} + |\alpha_r(0)|_{2,-\mu,\Omega} + |||\alpha'(0)|||_{1,2,1-\mu,\Omega} \tag{3.43}$$

and $\mathfrak{N}(\Omega^T) = \mathfrak{N}_1(\Omega^T) \cap \mathfrak{N}_2(\Omega^T)$. Using (3.42) and (2.33) in (3.30) yields

$$|v_{\varphi,z}|_{4,2,-3/4-\varepsilon,\Omega^t} \le \varphi(t,\|v\|_{\mathfrak{N}(\Omega^t)})X_3(t), \tag{3.44}$$

where $X_3(t) = X_1(t) + X_2(t)$. In view of (3.44), (2.22) we obtain from (3.1) the inequality

$$\left\|\frac{\chi}{r}\right\|_{V_2^0(\Omega^t)} \le \varphi(X_1)\sqrt{K(K+1)}\bigg(\sum_j \|v\|_{L_2(0,t;H^1(S_j))} + X_1\bigg) + \varphi(t,\|v\|_{\mathfrak{N}(\Omega^t)})(1+\sup_t\|w\|_{1,0,\Omega})X_3 + \varphi(X_1)Y_1, \tag{3.45}$$

where $S_j = S \cap \operatorname{supp}\varphi_j$, $\{\varphi_j\}$ is the partition of unity and

$$Y_1(t) = |F_\varphi|_{6/5,2,-1,\Omega^t} + |\chi(0)|_{2,-1,\Omega}, \tag{3.46}$$

and we used that $\mathfrak{N}_1(\Omega^T) \subset \mathfrak{N}_1'(\Omega^T)$, where $\mathfrak{N}_1(\Omega^T)$ is defined in assumptions of Lemma 3.4 and $\mathfrak{N}_1'(\Omega^T)$ by (3.31).

Now we have to find an estimate for $\sup_t \|w(t)\|_{1,0,\Omega}$. In view of 5.2.22 from [17] and (5.15) from [20] we obtain

$$\begin{aligned}(3.47)\quad \|w(t)\|_{1,0,\Omega}^2 \le c\exp(c|v|^2_{\infty,2,\Omega^t})[(1+|v|^4_{\infty,4,\Omega^t})\sup_t |w(t)|^2_{2,1,\Omega} \\ + |w|^2_{2,\Omega^t} + |q|^2_{2,\Omega^t} + |h|^2_{2,-1,\Omega^t} + |f_\varphi|^2_{2,1,\Omega^t} + e^{-t}\|w(0)\|^2_{1,0,\Omega}],\end{aligned}$$

where

$$(3.48)\quad |w(t)|_{2,1,\Omega} \le |w(0)|_{2,1,\Omega} + c\int_0^t (|q(t')|_{2,\Omega} + |h(t')|_{2,-1,\Omega} + |f_\varphi(t')|_{2,1,\Omega})\,dt'.$$

Hence

$$\|w(t)\|_{1,0,\Omega} \le \varphi(t, \|v\|_{\mathfrak{N}_2(\Omega^t)})(X_1 + |w|_{2,\Omega^t}).$$

Using (3.36) gives

$$|w|_{2,\Omega^t} \le \varphi(t, \|v\|_{\mathfrak{N}_2(\Omega^t)})X_1.$$

Hence

$$(3.49)\qquad \|w(t)\|_{1,0,\Omega} \le \varphi(t, \|v\|_{\mathfrak{N}_2(\Omega^t)})X_1.$$

Using (3.49) in (3.45) yields

$$\begin{aligned}\left\|\frac{\chi}{r}\right\|_{V_2^0(\Omega^t)} \le \varphi(X_1)\sqrt{K(K+1)}\Big(\sum_j \|v\|_{L_2(0,t;H^1(S_j))} + X_1\Big) \\ + \varphi(t, \|v\|_{\mathfrak{N}(\Omega^t)}, X_1)X_3 + \varphi(X_1)Y_1.\end{aligned}$$

From (3.11) we get

$$\|\alpha'\|_{V_2^0(\Omega^t)} \le \varphi(t, X_1, \|v\|_{\mathfrak{N}(\Omega^t)})X_2.$$

Hence, from problem (2.13) we have

$$\begin{aligned}(3.50)\quad |v|_{10,\Omega^t} + |\nabla v|_{10/3,\Omega^t} \le \varphi_1(t, X_1, \|v\|_{\mathfrak{N}(\Omega^t)})X_3 \\ + \varphi_2(X_1)Y_1 + \sqrt{K(K+1)}\Big(\sum_j \|v\|_{L_2(0,t;H^1(S_j))} + X_1\Big).\end{aligned}$$

Finally, we obtain an a priori estimate for solutions to problem (1.1). First we recall that

$$\begin{aligned}\mathfrak{N}(\Omega^T) = \mathfrak{N}_1(\Omega^T)\cap\mathfrak{N}_2(\Omega^T) = L_\infty(0,T;W^1_{2,-\delta}(\Omega))\cap L_2(0,T;W^1_\infty(\Omega)) \\ \cap L_\infty(0,T;W^1_{p'}(\Omega))\cap L_3(0,T;W^1_3(\Omega)),\end{aligned}$$

where $p' \in (2,3)$ and $\delta \in (0,1)$.

LEMMA 3.8. *Let $T < \infty$ be given, let $X_1(T) < \infty$ (see (3.41)), $X_2(T) < \infty$ (see (3,43)), $Y_1(T) < \infty$ (see (3.46)). Let*

$$|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega} < \infty.$$

Let solutions of (1.1) satisfy estimates for the weak solutions (see (2.5), (2.6)). Then there exists the constant

$$A = 2\sigma[\varphi_2^2(X_1)Y_1^2 + c_1(K+1)(d_2^2 + X_1^2) + c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega})],$$

where $\sigma > 2$, function φ_2 appears in (3.50) and c_1 is the constant introduced in (3.54), such that for X_3 so small that (3.58) holds we have the estimate

$$\|v\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} \le A. \tag{3.51}$$

PROOF. Let us consider problem (1.1). In view of (3.50) we have

$$\begin{aligned} \|v\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} &\le \varphi_1^2(T, X_1, \|v\|_{2,5/2,\Omega^T})X_3^2 + \varphi_2^2(X_1)Y_1^2 \\ &+ K(K+1)\bigg(\sum_j \|v\|^2_{L_2(0,T;H^1(S_j))} + X_1^2\bigg) + c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega}), \end{aligned} \tag{3.52}$$

where we used that $\|v\|_{\mathfrak{N}(\Omega^T)} \le c\|v\|_{2,5/2,\Omega^T}$.

To examine the third term on the r.h.s. of (3.52) we apply the interpolation inequality

$$|u|_{2,S} \le c|\nabla u|_{2,\Omega}^{1/2}|u|_{2,\Omega}^{1/2} + c|u|_{2,\Omega},$$

which after integration with respect to time takes the form

$$|u|_{2,S^T} \le c|\nabla u|_{2,\Omega^T}^{1/2}|u|_{2,\Omega^T}^{1/2} + c|u|_{2,\Omega^T}. \tag{3.53}$$

Using (3.53) we obtain

$$\begin{aligned} \|v\|_{L_2(0,T;H^1(S_j))} &= \|v\|_{L_2(0,T;L_2(S_j))} + \|\nabla v\|_{L_2(0,T;L_2(S_j))} \\ &\le c|\nabla^2 v|_{2,\Omega_j^T}^{1/2}|\nabla v|_{2,\Omega_j^T}^{1/2} + c|\nabla v|_{2,\Omega_j^T} + c|\nabla v|_{2,\Omega_j^T}^{1/2}|v|_{2,\Omega_j^T}^{1/2} + c|v|_{2,\Omega_j^T} \\ &\le c_1 d_2^{1/2}|\nabla^2 v|_{2,\Omega_j^T}^{1/2} + c_1 d_2, \end{aligned} \tag{3.54}$$

where $\Omega_j^T = \Omega^T \cap \operatorname{supp}\varphi_j$ and (2.6) was employed.

By the Hölder inequality we have

$$|\nabla^2 v|_{2,\Omega_j^T} \le |\Omega_j^T|^{1/10}|\nabla^2 v|_{5/2,\Omega_j^T}. \tag{3.55}$$

Utilizing (3.54) and (3.55) in (3.52) yields

$$\begin{aligned} \|v\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} &\le \varphi_1^2(T, X_1, \|v\|_{2,5/2,\Omega^T})X_3^2 + \varphi_2^2(X_1)Y_1^2 \\ &+ c_1K(K+1)[\sup_j |\Omega_j^T|^{1/10}|\nabla^2 v|_{5/2,\Omega^T} d_2 + d_2^2 + X_1^2] \\ &+ c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega}). \end{aligned} \tag{3.56}$$

Assuming that $c_1 \sup_j |\Omega_j^T|^{1/10} K(K+1)d_2 \le 1/2$ we obtain from (3.56) the inequality

$$\begin{aligned}(3.57)\quad \|v\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} \le 2\varphi_1^2(T, X_1, \|v\|_{2,5/2,\Omega^T})X_3^2 + 2\varphi_2^2(X_1)Y_1^2 \\ + 2c_1K(K+1)(d_2^2 + X_1^2) + 2c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega}).\end{aligned}$$

Let us introduce the quantity

$$A = 2\sigma[\varphi_2^2(X_1)Y_1^2 + c_1K(K+1)(d_2^2 + X_1^2) + c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega})]$$

and let $\|v\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} \le A$. Then inequality (3.57) implies

$$(3.58)\qquad 2\varphi_1^2(T, X_1, A)X_3^2 \le \left(1 - \frac{1}{\sigma}\right)A,$$

which can be satisfied for sufficiently small X_3.

In view of (3.58) estimate (3.51) holds. □

4. Existence

We prove the existence of solutions to problem (1.1) by the Leray–Schauder fixed point theorem. Therefore we consider the problem

$$(4.1)\qquad \begin{aligned} & v_{,t} - \operatorname{div} \mathbb{T}(v,p) = -\lambda\widetilde{v}(v')\cdot\nabla\widetilde{v}(v') + f && \text{in } \Omega^T,\\ & \operatorname{div} v = 0 && \text{in } \Omega^T,\\ & v\cdot\overline{n} = 0, \quad \overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\ & v|_{t=0} = v(0) && \text{in } \Omega,\end{aligned}$$

where $\lambda \in [0,1]$, $v' \in \mathfrak{N}(\Omega^T)$, $\widetilde{v} \in \mathfrak{M}_0(\Omega^T) = \{v : |v|_{10,\Omega^T} + |\nabla v|_{10/3,\Omega^T} < \infty\}$.

In view of Section 3 we have the mapping $\Phi_1 \colon \mathfrak{N}(\Omega^T) \to \mathfrak{M}_0(\Omega^T)$ so

$$\mathfrak{N}(\Omega^T) \ni v' \to \Phi_1(v') = \widetilde{v} \in \mathfrak{M}_0(\Omega^T).$$

For any $\lambda \in [0,1]$ and $\widetilde{v} \in \mathfrak{M}_0(\Omega^T)$ problem (4.1) generates the mapping

$$\Phi_2 \colon \mathfrak{M}_0(\Omega^T) \ni \widetilde{v} \to \Phi_2(\widetilde{v}) = v \in W^{2,1}_{5/2}(\Omega^T).$$

Therefore we define the mapping $v = \Phi(v',\lambda) = \Phi_2(\Phi_1(v'),\lambda)$ such that for any $\lambda \in [0,1]$ we have

$$(4.2)\qquad \Phi \colon \mathfrak{N}(\Omega^T) \to W^{2,1}_{5/2}(\Omega^T) \subset \mathfrak{N}(\Omega^T),$$

where the last imbedding is compact.

LEMMA 4.1. *The mapping* Φ *is compact, continuous on* $\mathfrak{N}(\Omega^T)\times[0,1]$ *and index* $\Phi|_{\lambda=0}=1$.

PROOF. Compactness follows from imbedding (4.2). Continuity on $\mathfrak{N}(\Omega^T)\times[0,1]$ can be proved in the same way as in Lemma 5.2 from [24]. For $\lambda=0$ there exists a unique solution to problem (4.1) (see [1]).

In view of Lemma 4.1 and (3.51), where a fixed point of mapping Φ is estimated we can apply the Leray–Schauder fixed point theorem. Hence Theorem 1.1 from Section 1 is proved. □

5. Continuity

In this Section we show that mappings Φ_1 and Φ_2 are continuous. First we show that Φ_2 is continuous.

LEMMA 5.1. *Let the assumptions of Lemma* 3.8 *hold. Then* Φ_2 *is continuous.*

PROOF. To show continuity of Φ_2 we consider the problems

$$\begin{aligned}
&v_{i,t}-\operatorname{div}\mathbb{T}(v_i,p_i)=-\lambda\widetilde{v}_i\nabla\widetilde{v}_i+f,\\
&\operatorname{div}v_i=0,\\
&v_i\cdot\overline{n}|_S=0,\quad (\nu\overline{n}\cdot\mathbb{D}(v_i)\cdot\tau_\alpha+\gamma v_i\cdot\overline{\tau}_\alpha)|_S=0,\quad \alpha=1,2,\\
&v_i|_{t=0}=v(0),
\end{aligned}$$

where $\nabla p_i\in L_{5/2}(\Omega^T)$, $\widetilde{v}_i\in\mathfrak{M}_0(\Omega^T)$, $v_i\in W^{2,1}_{5/2}(\Omega^T)$ and $i=1,2$.

Taking the differences $V=v_1-v_2$, $P=p_1-p_2$, $\widetilde{V}=\widetilde{v}_1-\widetilde{v}_2$, we see that they are solutions to the problem

$$\begin{aligned}
&V_{,t}-\operatorname{div}\mathbb{T}(V,P)=-\lambda[\widetilde{V}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{V}],\\
&\operatorname{div}V=0,\\
&V\cdot\overline{n}|_S=0,\quad (\nu\overline{n}\cdot\mathbb{D}(V)\cdot\overline{\tau}_\alpha+\gamma V\cdot\overline{\tau}_\alpha)|_S=0,\quad \alpha=1,2,\\
&V|_{t=0}=0.
\end{aligned}\tag{5.1}$$

For solutions to problem (5.1) we have

$$\begin{aligned}
\|V\|_{2,5/2,\Omega^T}&\le cA(|\widetilde{V}|_{10,\Omega^T}+|\nabla\widetilde{V}|_{10/3,\Omega^T})+c\sum_{\alpha=1}^2\|V\cdot\tau_\alpha\|_{3/5,5/2,S^T}\\
&\le cA\|\widetilde{V}\|_{\mathfrak{M}_0(\Omega^T)}+c\sum_{\alpha=1}^2\|V\cdot\overline{\tau}_\alpha\|_{3/5,5/2,S^T}.
\end{aligned}\tag{5.2}$$

To estimate the last integral on the r.h.s. of (5.2) we use the energy estimate. Multiplying $(5.1)_1$ by V, integrating over Ω and using the Korn inequality we obtain

$$\frac{1}{2}\frac{d}{dt}|V|_{2,\Omega}^2+\frac{\nu}{2}\|V\|_{1,\Omega}^2+\gamma|V\cdot\overline{\tau}_\alpha|_{2,S}^2\le c(|\widetilde{V}|_{3,\Omega}^2|\nabla\widetilde{v}_1|_{2,\Omega}^2+|\widetilde{v}_2|_{3,\Omega}^2|\nabla\widetilde{V}|_{2,\Omega}^2).$$

Integrating with respect to time yields

$$(5.3)\quad |V(t)|_{2,\Omega}^2+\nu\int_0^t\|V(t')\|_{1,\Omega}^2\,dt'+\gamma\int_0^t|V\cdot\bar\tau_\alpha(t')|_{2,S}^2\,dt' \\ \le c(|\widetilde V|_{3,5,\Omega^T}^2|\nabla\widetilde v_1|_{2,\frac{10}{3},\Omega^T}^2+|\widetilde v_2|_{3,5,\Omega^T}^2|\nabla\widetilde V|_{2,\frac{10}{3},\Omega^T}^2)\le cA^2\|\widetilde V\|_{\mathfrak{M}_0(\Omega^T)}^2.$$

We have the inequality

$$(5.4)\quad \|V\|_{W_{5/2}^{3/5,3/10}(S^T)}\le c\|V\|_{W_{5/2}^{1,1/2}(\Omega^T)}\le\varepsilon\|V\|_{W_{5/2}^{2,1}(\Omega^T)}+c(1/\varepsilon)\|V\|_{L_{5/2}(\Omega^T)}.$$

From (5.2)–(5.4) we obtain $\|V\|_{W_{5/2}^{2,1}(\Omega^T)}\le c(A)\|\widetilde V\|_{\mathfrak{M}_0(\Omega^T)}$. □

Lemma 5.2. *Let the assumptions of Lemma 3.8 hold. Then Φ_1 is continuous.*

Proof. Let $v_1',v_2'\in\mathfrak{N}(\Omega^T)$. Then (2.7) takes the form

$$\begin{aligned}
&h_{s,t}-\operatorname{div}\mathbb{D}(h_s)+\nabla q_s=-v_s'\cdot\nabla h_s-h_s\cdot\nabla v_s'+g && \text{in } \Omega^T,\\
&\operatorname{div}h_s=0 && \text{in } \Omega^T,\\
&\bar n\cdot h_s=0,\quad \bar n\cdot\mathbb{D}(h_s)\cdot\bar\tau_\alpha+\gamma h_s\cdot\bar\tau_\alpha=0 && \text{on } S^T,\\
&h_s|_{t=0}=h(0) && \text{in } \Omega,
\end{aligned}$$

where $s=1,2$. Let v_s',h_s,q_s, $s=1,2$, be given. Then w_s is a solution to the problem

$$\begin{aligned}
&w_{s,t}+v_s'\cdot\nabla w_s+\frac{v_{sr}'}{r}w_s-\nu\Delta w_s+\nu\frac{w_s}{r^2}=\frac1r q_s+\frac{2\nu}{r^2}h_s+f_\varphi && \text{in } \Omega^T,\\
&\nu\bar n\cdot\nabla w_s=-\gamma w_s+\nu\frac{a_1}{r}w_s && \text{on } S^T,\\
&w_s|_{t=0}=w(0) && \text{in } \Omega.
\end{aligned}$$

Next, $\alpha_s'=(\alpha_{sr},\alpha_{sz})$, $s=1,2$, is a solution to the problem

$$\begin{aligned}
&\alpha_{sr,t}+v_s'\cdot\nabla\alpha_{sr}-(\alpha_{sr}v_{sr,r}'+\alpha_{sz}v_{sz,z}')-\frac{\chi}{r}h_{sr}\\
&\qquad+\frac{2\nu}{r^2}(h_{sr,z}-h_{sz,r})+\frac{\nu\alpha_{sr}}{r^2}-\nu\Delta\alpha_{sr}=F_r && \text{in } \Omega^T,\\
&\alpha_{sz,t}+v_s'\cdot\nabla\alpha_{sz}-(\alpha_{sr}v_{sz,r}'+\alpha_{sz}v_{sz,z}')\\
&\qquad-\frac{\chi_s}{r}h_{sz}-\nu\Delta\alpha_{sz}=F_2 && \text{in } \Omega^T,\\
&\bar\tau_z\cdot\alpha_s'=-\frac{2a_1}{r}w_s+\frac{\gamma}{\nu}w_s && \text{on } S^T,\\
&(\bar n\cdot\alpha_s')_{,n}=\beta_1h_{sr}+\beta_2h_{sz}+\beta_3w_{s,r}+\beta_4w_{s,z}+\beta_5w_s && \text{on } S^T,\\
&\alpha_s'|_{t=0}=\alpha'(0) && \text{in } \Omega,
\end{aligned}$$

where $s=1,2$. Next, χ_s satisfies

$$\begin{aligned}
&\chi_{s,t}+v'_s\cdot\nabla\chi_s+(v'_{sr,r}+v'_{sz,z})\chi_s-\nu\bigg[\bigg(r\bigg(\frac{\chi_s}{r}\bigg)_{,r}\bigg)_{,r}\\
&\quad+\frac{1}{r^2}\chi_{s,\varphi\varphi}+\chi_{s,zz}+2\bigg(\frac{\chi_s}{r}\bigg)_{,r}\bigg]=\frac{2\nu}{r}\bigg(-h_{s\varphi,z}+\frac{1}{r}h_{sz,\varphi}\bigg)\\
&\quad-\frac{1}{r}\bigg(w_{s,z}h_{sr}-w_{s,r}h_{sz}+\frac{w_s}{r}h_{sz}\bigg)+\frac{2}{r}w_sw_{s,z}+F_\varphi \qquad \text{in } \Omega^T,\\
&\chi_s=2\bigg(k-\frac{\gamma}{2\nu}\bigg)v'_s\cdot\bar\tau_2 \qquad \text{on } S^T,\\
&\chi_s|_{t=0}=\chi(0) \qquad \text{in } \Omega,
\end{aligned}$$

where $s=1,2$. Finally, $\widetilde v_s$ is a solution to the elliptic problem

$$\begin{aligned}
\operatorname{rot}\widetilde v_s&=\alpha_s && \text{in } \Omega,\\
\operatorname{div}\widetilde v_s&=0 && \text{in } \Omega,\\
\widetilde v_s\cdot\bar n&=0 && \text{on } S,
\end{aligned}$$

$s=1,2$. Let us introduce the quantities

$$\begin{aligned}
V'&=v'_1-v'_2, \quad W=w_1-w_2, \quad H=h_1-h_2, \quad Q=q_1-q_2,\\
A&=\alpha_1-\alpha_2, \quad N=\chi_1-\chi_2, \quad \widetilde V=\widetilde v_1-\widetilde v_2.
\end{aligned}$$

First we have the problem for H and Q

$$\begin{aligned}
&H_{,t}-\nu\operatorname{div}\mathbb{D}(H)+\nabla Q=-v'_1\cdot\nabla H\\
&\qquad -V'\cdot\nabla h_2-h_1\cdot\nabla V'-H\cdot\nabla v'_2 && \text{in } \Omega^T,\\
&\operatorname{div}H=0 && \text{in } \Omega^T,\\
&\bar n\cdot H=0, \quad \nu\bar n\cdot\mathbb{D}(H)\cdot\bar\tau_\alpha+\gamma H\cdot\bar\tau_\alpha=0, \quad \alpha=1,2, && \text{on } S^T,\\
&H|_{t=0}=0 && \text{in } \Omega.
\end{aligned}\tag{5.5}$$

Then the problem for W takes the form

$$\begin{aligned}
&W_{,t}+v'_1\cdot\nabla W+V'\cdot\nabla w_2+\frac{v'_{1r}}{r}W+\frac{V'_r}{r}w_2\\
&\qquad-\nu\Delta W+\nu\frac{W}{r^2}=\frac{1}{r}Q+\frac{2\nu}{r^2}H_r && \text{in } \Omega^T,\\
&\bar n\cdot\nabla W=-\frac{\gamma}{\nu}W+\frac{a_1}{r}W && \text{on } S^T,\\
&W|_{t=0}=0 && \text{in } \Omega.
\end{aligned}\tag{5.6}$$

Continuing, $A' = (A_r, A_z)$ is a solution to the problem

$$
(5.7)\quad
\begin{aligned}
&A_{r,t} + v_1' \cdot \nabla A_r + V' \cdot \nabla\alpha_{2r}\\
&\qquad - (\alpha_{1r}V'_{r,r} + A_r v'_{2r,r} + \alpha_{1z}V'_{z,z} + A_z v'_{2z,z})\\
&\qquad - \frac{\chi_1}{r}H_r - \frac{N}{r}h_{2r} + \frac{2\nu}{r^2}(H_{r,z} - H_{z,r}) + \frac{\nu}{r^2}A_r - \nu\Delta A_r = 0 && \text{in } \Omega^T,\\
&A_{z,t} + v_1' \cdot \nabla A_z + V' \cdot \alpha_{2z}\\
&\qquad - (\alpha_{1r}V'_{z,r} + A_r v'_{2z,r} + \alpha_{1z}V'_{z,z} + A_z v'_{2z,z})\\
&\qquad - \frac{\chi_1}{r}H_z - \frac{N}{r}h_{2z} - \nu\Delta A_z = 0 && \text{in } \Omega^T,\\
&\tau_2 \cdot A' = -\frac{2a_1}{r}W + \frac{\gamma}{r}W && \text{on } S^T,\\
&(\overline{n} \cdot A')_{,n} = \beta_1 H_r + \beta_2 H_z + \beta_3 W_{,r} + \beta_4 W_{,z} + \beta_5 W && \text{on } S^T,\\
&A'|_{t=0} = 0 && \text{in } \Omega.
\end{aligned}
$$

Next N satisfies

$$
(5.8)\quad
\begin{aligned}
&N_{,t} + v_1' \cdot \nabla N + V' \cdot \nabla\chi_2 + (v'_{1r,r} + v'_{1z,z})N + (V'_{r,r} + V'_{z,z})\chi_2\\
&\qquad - \nu\bigg[\bigg(r\bigg(\frac{N}{r}\bigg)_{,r}\bigg)_{,r} + \frac{1}{r^2}N_{,\varphi\varphi} + N_{,zz} + 2\bigg(\frac{N}{r}\bigg)_{,r}\bigg]\\
&\quad = \frac{2\nu}{r}\bigg(-H_{\varphi,z} + \frac{1}{r}H_{z,\varphi}\bigg) - \frac{1}{r}\bigg(w_{1,z}H_r + W_{,z}h_{2r} - w_{1,r}H_z\\
&\qquad - W_{,r}h_{2z} + \frac{w_1}{r}H_z + \frac{W}{r}h_{2z}\bigg) + \frac{2}{r}(w_1 W_{,z} + W w_{2,z}) && \text{in } \Omega^T,\\
&N = 2\bigg(k - \frac{\gamma}{2\nu}\bigg)V' \cdot \overline{\tau}_2 && \text{on } S^T,\\
&N|_{t=0} = 0 && \text{in } \Omega.
\end{aligned}
$$

Finally, $\widetilde{V}$ is a solution to the elliptic problem

$$
(5.9)\qquad \operatorname{rot}\widetilde{V} = A \quad \text{in } \Omega, \qquad \operatorname{div}\widetilde{V} = 0 \quad \text{in } \Omega, \qquad \widetilde{V} \cdot \overline{n} = 0 \quad \text{on } S.
$$

First we obtain an estimate for solutions to problem (5.8). Let us introduce the set Ω_* (see the proof of Lemma 3.1) and let us add the artificial boundary condition $N|_{r=\varepsilon_*} = 0$. Multiplying (5.8) by N/r^2, integrating the result over Ω_* and repeating the considerations leading to the inequality before (3.6) we obtain

$$
(5.10)\quad
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|N|^2_{2,-1,\Omega_*} + \nu\bigg\|\frac{N}{r}\bigg\|^2_{1,\Omega_*} \le c|h_\varphi|^2_{3,-1,\Omega_*}|N|^2_{2,-1,\Omega_*}\\
&\qquad + cK^2|v|^2_{2,S} + c\int_S \overline{n} \cdot \nabla\frac{N}{r}N\,dS\\
&\qquad + \bigg|\int_{\Omega_*}(V' \cdot \nabla\chi_2 + (V'_{r,r} + V'_{z,z})\chi_2)\frac{N}{r^2}\,dx\bigg|
\end{aligned}
$$

$$+c\bigg|\int_{\Omega_*}\bigg(-H_{\varphi,z}+\frac{1}{r}H_{z,\varphi}\bigg)\frac{N}{r^2}\,dx\bigg|$$
$$+c\bigg|\int_{\Omega_*}\bigg(w_{1,z}H_r+W_{,z}h_{2r}-w_{1,r}H_z-W_{,r}h_{2z}+\frac{w_1}{r}H_z+\frac{W}{r}h_{2z}\bigg)\frac{N}{r^2}\,dx\bigg|$$
$$+c\bigg|\int_{\Omega_*}(w_1W_{,z}+Ww_{2,z})\frac{N}{r^2}\,dx\bigg|.$$

The fourth term on the r.h.s. of (5.10) we estimate by

$$\varepsilon_1\bigg|\frac{N}{r}\bigg|^2_{6,\Omega_*}+c(1/\varepsilon_1)\bigg(\bigg|\frac{V'}{r}\nabla\chi_2\bigg|^2_{6/5,\Omega_*}+\bigg|\nabla'V'\frac{\chi_2}{r}\bigg|^2_{6/5,\Omega_*}\bigg),$$

where $\varepsilon_1\in(0,1)$ and

$$\bigg|\frac{V'}{r}\nabla\chi_2\bigg|^2_{6/5,\Omega_*}\le|\nabla\chi_2|^2_{2,-(1-\varepsilon),\Omega_*}|V'|^2_{3,-\varepsilon,\Omega_*},\tag{5.11}$$

$$\bigg|\nabla'V'\frac{\chi_2}{r}\bigg|^2_{6/5,\Omega_*}\le|\nabla V'|^2_{3/2,\Omega_*}\bigg|\frac{\chi_2}{r}\bigg|^2_{6,\Omega_*},\tag{5.12}$$

where ε is arbitrary small positive number.

We estimate the r.h.s. of (5.11). First we examine

$$\begin{aligned}|\nabla\chi_2|^2_{2,-(1-\varepsilon),\Omega_*}&\le\bigg|\nabla\frac{\chi_2}{r}r^\varepsilon\bigg|^2_{2,\Omega_*}+\varepsilon^2\bigg|\frac{\chi_2}{r}r^{\varepsilon-1}\bigg|^2_{2,\Omega_*}\\&\le R^{2\varepsilon}\bigg|\nabla\frac{\chi_2}{r}\bigg|^2_{2,\Omega_*}+\frac{\varepsilon^2}{\varepsilon^2}R^{2\varepsilon}\bigg|\nabla\frac{\chi_2}{r}\bigg|^2_{2,\Omega_*}\le 2R^{2\varepsilon}\bigg|\nabla\frac{\chi_2}{r}\bigg|^2_{2,\Omega_*},\end{aligned}$$

where in the second inequality the Hardy inequality was used.

Applying (2.1) we have

$$|V'|^2_{3,-\varepsilon,\Omega_*}\le c|\nabla V'|^2_{3/(2-\varepsilon),\Omega_*}$$

In view of the above estimates inequality (5.11) takes the form

$$\bigg|\frac{V'}{r}\nabla\chi_2\bigg|^2_{6/5,\Omega_*}\le cR^{2\varepsilon}\bigg|\nabla(\chi_2/r)\bigg|^2_{2,\Omega_*}|\nabla V'|^2_{3/2-\varepsilon,\Omega_*},$$

where $\varepsilon>0$ can be chosen arbitrary small.

The fifth term on the r.h.s. of (5.10) is estimated by

$$\varepsilon_2\bigg|\nabla\frac{N}{r}\bigg|^2_{2,\Omega_*}+c(1/\varepsilon_2)|H|^2_{2,-3,\Omega_*},$$

where $\varepsilon_2\in(0,1)$.

The sixth term on the r.h.s. of (5.10) we estimate by

$$\varepsilon_3\bigg|\frac{N}{r}\bigg|^2_{6,\Omega_*}+c(1/\varepsilon_3)(\|w\|^2_{1,0,\Omega_*}|H|^2_{3,-2,\Omega_*}+\|W\|^2_{1,0,\Omega_*}|h|^2_{3,-2,\Omega_*}),$$

where $\varepsilon_3\in(0,1)$.

Finally, the last term on the r.h.s. of (5.10) can be bounded by

$$\varepsilon_4\bigg|\frac{N}{r^{2-\varepsilon}}\bigg|_{2,\Omega_*}^2+c(1/\varepsilon_4)(\|w_2\|_{1,0,\Omega_*}^2|W_{,z}|_{4,-3/4-\varepsilon,\Omega_*}^2+\|W\|_{1,0,\Omega_*}^2|w_{2,z}|_{4,-3/4-\varepsilon,\Omega_*}^2),$$

where $\varepsilon_4\in(0,1)$ and ε can be chosen as arbitrary small positive number.

Using the above estimates in (5.10), and assuming that $\varepsilon_1-\varepsilon_4$ are sufficiently small, we obtain

$$\begin{aligned}(5.13)\quad &\frac{1}{2}\frac{d}{dt}|N|_{2,-1,\Omega_*}^2+\nu\bigg\|\frac{N}{r}\bigg\|_{1,\Omega_*}^2\le c|h_\varphi|_{3,-1,\Omega_*}^2|N|_{2,-1,\Omega_*}^2\\ &+cK^2|V'|_{2,S_*}^2+c\int_{S_*}\overline{n}\cdot\nabla\frac{N}{r}N\,dS\\ &+cR^{2\varepsilon}\bigg|\nabla\frac{\chi_2}{r}\bigg|_{2,\Omega_*}^2|\nabla V'|_{3/(2-\varepsilon),\Omega_*}^2+c|\nabla V'|_{3/2,\Omega_*}^2\bigg|\frac{\chi_2}{r}\bigg|_{6,\Omega_*}^2+c|H|_{2,-3,\Omega_*}^2\\ &+c(\|w\|_{1,0,\Omega_*}^2|H|_{3,-2,\Omega_*}^2+\|W\|_{1,0,\Omega_*}^2|h|_{3,-2,\Omega_*}^2)\\ &+c(\|w_1\|_{1,0,\Omega_*}^2|W_{,z}|_{4,-3/4-\varepsilon,\Omega_*}^2+\|W\|_{1,0,\Omega_*}^2|w_{2,z}|_{4,-3/4-\varepsilon,\Omega_*}^2).\end{aligned}$$

Integrating (5.13) with respect to time, passing with ε_* to 0 and using the inequality

$$\begin{aligned}&K^2|V'|_{2,S^t}^2+\bigg|\int_{S^t}\overline{n}\cdot\nabla\frac{N}{r}\frac{N}{r}\,dS\,dt'\bigg|\\ &\qquad\le cK(K+1)\sum_j\int_{S^t\cap\operatorname{supp}\varphi_j}\bigg(V_{,x}^{'2}+V^{'2}+\bigg|\nabla\frac{H}{r}\bigg|^2\bigg)\,dS\,dt'\end{aligned}$$

we obtain

$$\begin{aligned}(5.14)\quad &|N(t)|_{2,-1,\Omega}^2+\nu\int_0^t\bigg\|\frac{N(t')}{r}\bigg\|_{1,\Omega}^2dt'\le c\exp(c|h_1|_{3,2,-1,\Omega^t}^2)\\ &\cdot\bigg[K(K+1)\sum_j\int_{S^t\cap\operatorname{supp}\varphi_j}\bigg(V_{,x}^{'2}+V^{'2}+\bigg|\nabla\frac{H}{r}\bigg|^2\bigg)\,dS\,dt'\\ &+c(A)\bigg(\sup_t|\nabla V'|_{3/(2-\varepsilon),\Omega}^2+\sup_t|\nabla V'|_{3/2,\Omega}^2\\ &+\int_0^t\|H(t')\|_{2,-1,\Omega}^2\,dt'+\sup_t\|W\|_{1,0,\Omega}^2\\ &+\int_0^t|W_{,z}(t')|_{4,-3/4-\varepsilon,\Omega}^2\,dt'\bigg)\bigg].\end{aligned}$$

Now we estimate the last three norms under the square bracket. Applying Lemma 2.8 to problem (5.5) yields

$$\|H\|_{2,-1,\Omega^t} + \|Q\|_{L_2(0,t;H^1_{-1}(\Omega))} \le \varphi(\|v_1'\|_{\mathfrak{N}_\delta(\Omega^t)}, \|v_2'\|_{\mathfrak{N}_\delta(\Omega^t)})c(A)\|V'\|_{\mathfrak{N}(\Omega^t)} + c(|V'\cdot\nabla h_2|_{2,-(1+\varepsilon_*),\Omega^t} + |h_1\cdot\nabla V'|_{2,-(1+\varepsilon_*),\Omega^t}),$$

where φ is an increasing positive function and the last two terms are estimated by

$$|V'\cdot\nabla h_2|_{2,-(1+\varepsilon_*),\Omega^t} + |h_1\cdot\nabla V'|_{2,-(1+\varepsilon_*),\Omega^t} \le |V'|_{2,\infty,\Omega^t}|\nabla h_2|_{3,2,-(1+\varepsilon_*),\Omega^t} + |h_1|_{3,2,-(1+\varepsilon_*),\Omega^t}|\nabla V'|_{2,\infty,\Omega^t} \le \varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}$$

Hence we have

$$\|H\|_{2,-1,\Omega^t} + \|Q\|_{L_2(0,t;H^1_{-1}(\Omega))} \le \varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}. \tag{5.15}$$

In view of (3.47) we obtain for solutions to problem (5.6) the inequality

$$\|W(t)\|^2_{1,0,\Omega} \le \varphi(A)\bigg[\sup_{t'\le t}|W(t')|_{2,1,\Omega} + |W|^2_{2,\Omega^t} + |Q|^2_{2,\Omega^t} + |H|^2_{2,-1,\Omega^t} + |V'\cdot\nabla w_2|^2_{2,1,\Omega^t} + \bigg|\frac{V'}{r}w_2\bigg|^2_{2,1,\Omega^t}\bigg], \tag{5.16}$$

where by (3.48) applied to (5.6) we have

$$|W(t)|_{2,1,\Omega} \le c\int_0^t \bigg(|Q(t')|_{2,\Omega} + |H(t')|_{2,-1,\Omega} + |V'\cdot\nabla w_2|_{2,1,\Omega} + \bigg|\frac{V'}{r}w_2\bigg|_{2,1,\Omega}\bigg)\,dt'. \tag{5.17}$$

Next,

$$|V'\cdot\nabla w_2|_{2,1,\Omega^t} + \bigg|V'\frac{w_2}{r}\bigg|_{2,1,\Omega^t} \le |V'|_{\infty,2,\Omega^t}\sup_t\|w_2(t)\|_{1,0,\Omega} \le \varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}. \tag{5.18}$$

Finally, inequality (3.36) in the case of problem (5.6) gives

$$|W|^2_{2,\Omega^t} \le \varphi(A)\bigg(|Q|^2_{2,-\mu,\Omega^t} + |H|^2_{2,-(1+\mu),\Omega^t} + |V'\cdot\nabla w_2|^2_{2,1,\Omega^t} + \bigg|\frac{V'}{r}w_2\bigg|^2_{2,1,\Omega^t}\bigg), \tag{5.19}$$

where μ is arbitrary small positive number. Using (5.15) and (5.18) in (5.17) and (5.19) we obtain

$$\sup_t|W(t)|_{2,1,\Omega} \le \varphi(A,t)\|V'\|_{\mathfrak{N}(\Omega^t)} \tag{5.20}$$

$$|W|_{2,\Omega^t} \le \varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}. \tag{5.21}$$

In view of (5.15), (5.20) and (5.21) inequality (5.16) takes the form

$$\sup_t \|W(t)\|_{1,0,\Omega} \le \varphi(A,t)\|V'\|_{\mathfrak{N}(\Omega^t)}.$$

Finally we estimate the last term on the r.h.s. of (5.14). Then instead of (3.7) we consider the expression

$$V_{\varphi,z} = -A_r + \frac{1}{r}H_z.$$

Hence, we have

$$\tag{5.22} \begin{aligned} &\int_0^t |V_{\varphi,z}(t')|^2_{4,-3/4-\varepsilon,\Omega}\,dt' \\ &\quad\le c\int_0^t \|A_r(t')\|^2_{1,-1/2-\varepsilon,\Omega}\,dt' + c\int_0^t \|H(t')\|^2_{1,-3/2-\varepsilon,\Omega}\,dt', \end{aligned}$$

where in view of (5.15) the last integral is bounded by $\varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}$.

Finally, we estimate the first integral on the r.h.s. of (5.22). Applying Lemma 3.2 to problem (5.7) yields

$$\tag{5.23} \begin{aligned} |||A'|||_{2,2,1-\mu,\Omega^t} \le \varphi(A)\bigg[&|A'|_{2,1-\mu,\Omega^t} + \|H\|_{2,-\mu,\Omega^t} \\ &+ \|W\|_{2,1-\mu,\Omega^t} + |A_r|_{2,-(1+\mu),\Omega^t} + |V''\nabla'\alpha'|_{2,1-\mu,\Omega^t} \\ &+ |\alpha'\nabla'V''|_{2,1-\mu,\Omega^t} + \left|\frac{\chi_1}{r}H'\right|_{2,1-\mu,\Omega^t}\bigg], \end{aligned}$$

where $V'' = (V'_r, V'_z)$, $\nabla' = (\partial_r, \partial_z)$, $\alpha' = (\alpha_r, \alpha_z)$.

The last three terms on the r.h.s. of (5.23) are bounded by

$$\begin{aligned} |V''|_{6,\infty,\Omega^t}|\nabla\alpha'|_{3,2,1-\mu,\Omega^t} + |\alpha'|_{2,\infty,1-\mu,\Omega^t}|V''_{,x'}|_{\infty,2,\Omega^t} + \left|\frac{\chi_1}{r}\right|_{10/3,\Omega^t}|H'|_{5,1-\mu,\Omega^t} \\ \le \varphi(A)(\|V'\|_{\mathfrak{N}(\Omega^t)} + \|H'\|_{2,1-\mu,\Omega^t}). \end{aligned}$$

Hence, (5.23) assumes the form

$$\begin{aligned} |||A'|||_{2,2,1-\mu,\Omega^t} \le \varphi(A)[&|A'|_{2,1-\mu,\Omega^t} + |A_r|_{2,-(1+\mu),\Omega^t} \\ &+ \|H\|_{2,-\mu,\Omega^t} + \|W\|_{2,1-\mu,\Omega^t} + \|V'\|_{\mathfrak{N}(\Omega^t)}]. \end{aligned}$$

Lemma 3.3 applied for problem (5.7) gives

$$\tag{5.24} \begin{aligned} &|A'(t)|^2_{2,\Omega} + \nu\int_0^t \|A'(t')\|^2_{1,\Omega}\,dt' + |A_r|^2_{2,-1,\Omega^t} \\ &\quad\le \varphi(A)\bigg[\|H\|^2_{2,-1,\Omega^t} + \|W\|^2_{2,1-\mu,\Omega^t} + |V''\nabla'\alpha'|^2_{6/5,2,\Omega^t} \\ &\qquad + |\alpha'\nabla V''|^2_{6/5,2,\Omega^t} + \left|\frac{\chi_1}{r}H'\right|^2_{6/5,2,\Omega^t}\bigg], \end{aligned}$$

where the last three terms under the square bracket are estimated by

$$|V''|_{3,\infty,\Omega^t}|\nabla\alpha'|_{2,\Omega^t}+|\alpha'|_{2,\infty,\Omega^t}|\nabla V''|_{3,2,\Omega^t}+\left|\frac{\chi_1}{r}\right|_{3,\Omega^t}|H'|_{2,6,\Omega^t} \le \varphi(A)(\|V'\|_{\mathfrak{N}(\Omega^t)}+\|H\|_{2,\Omega^t}).$$

Hence, (5.24) takes the form

$$|A'(t)|_{2,\Omega}^2+\nu\int_0^t\|A'(t')\|_{1,\Omega}^2\,dt'+|A_r|_{2,-1,\Omega^t}^2 \le \varphi(A)[\|H\|_{2,-1,\Omega^t}^2+\|W\|_{2,1-\mu,\Omega^t}^2+\|V'\|_{\mathfrak{N}(\Omega^t)}^2]. \tag{5.25}$$

To apply Lemma 3.4 to problem (5.7) we have to estimate in a different way the term which implies $|F_r|_{6/5,2,-\mu,\Omega^t}$ on the r.h.s. of (3.24). For this purpose we consider the expression which appears instead of the first term on the r.h.s. of (3.25),

$$\left|\int_{\Omega^t}\overline{A}_r(V'\cdot\nabla\alpha_{2r}-\alpha_{1r}V'_{r,r}-\alpha_{1z}V'_{z,z})r^{-2\mu}\,dx\,dt'\right| \le \varepsilon\int_0^t|\overline{A}_r(t')|_{2,-(1+\mu),\Omega}^2\,dt'+c(1/\varepsilon)|V'\cdot\nabla\alpha_{2r}-\alpha_{1r}V'_{r,r}-\alpha_{1z}V'_{z,z}|_{2,\Omega^t}^2.$$

The second norm we estimate by

$$|V'|_{p,\infty,\Omega^t}^2|\nabla\alpha_r|_{p',2,\Omega^t}^2+|V'_{,x'}|_{\infty,2,\Omega^t}^2|\alpha'|_{2,\infty,\Omega^t}^2\le\varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}^2,$$

where p satisfies $3/3'-3/p=1$ where $3'<3$ but arbitrary close to 3 so p is arbitrary large and $p'=p/(p-1)>1$ is close to 1.

In view of the above considerations Lemma 3.4 applied to problem (5.7) gives

$$\nu\int_0^t\|A_r(t')\|_{1,-\mu,\Omega}^2\,dt'\le\varphi(A)[\|H\|_{2,-1,\Omega^t}^2+\|W\|_{2,1-\mu,\Omega^t}^2+\|V'\|_{\mathfrak{N}(\Omega^t)}^2]. \tag{5.26}$$

Considering problem (5.6) we obtain instead of (3.40) the inequality

$$\|W\|_{2,1-\mu,\Omega^t}\le\varphi(A)[|Q|_{2,-\mu,\Omega^t}+|H|_{2,-(1+\mu),\Omega^t}+\|V'\|_{\mathfrak{N}(\Omega^t)}] \tag{5.27}$$

In view of (5.15), (5.22), (5.25)–(5.27) we obtain

$$\|A\|_{V_2^0(\Omega^t)}\le\varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}.$$

Then problem (5.9) implies

$$\|\widetilde{V}\|_{\mathfrak{M}(\Omega^t)}\le\varphi(A)\|V'\|_{\mathfrak{N}(\Omega^t)}.$$

Hence Φ_1 is continuous. This concludes the proof. □

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Wojciech M. Zajączkowski
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warsaw, POLAND
and
Institute of Mathematics and Cryptology
Cybernetics Faculty
Military University of Technology
S. Kaliskiego 2
00-908 Warsaw, POLAND

E-mail address: wz@impan.gov.pl