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# FINDING CRITICAL POINTS WHOSE POLARIZATION IS ALSO A CRITICAL POINT

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Abstract. We show that near any given minimizing sequence of paths for the mountain pass lemma, there exists a critical point whose polarization is also a critical point. This is motivated by the fact that if any polarization of a critical point is also a critical point and the Euler–Lagrange equation is a second-order semi-linear elliptic problem, T. Bartsch, T. Weth and M. Willem (J. Anal. Math., 2005) have proved that the critical point is axially symmetric.

## 1. Introduction

If $u\colon \Omega \to \mathbb{R}$ solves the semi-linear elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

one is interested in determining whether $u$ inherits some symmetry of the domain $\Omega \subset \mathbb{R}^N$ and of the nonlinearity $f$. For example if $\Omega$ and $f$ are invariant under rotations, is $u$ also invariant? Of course, when $u$ is the only solution of (1.1), the answer is positive. By observing the eigenfunctions of the Laplacian, one can see that this is not always the case. B. Gidas, W.-M. Ni and L. Nirenberg have proved that if $\Omega$ is a ball, $f$ is independent of $x$ and Lipschitz-continuous and $u$

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is positive, then $u$ is radially symmetric [9]. The main tool in the proof is the maximum principle for second order elliptic operators. One can try to replace the essential positivity assumption by some other assumption. O. Lopes has proved that if the solution $u$ is a minimizer under a constraint, if $\Omega$ is bounded and smooth and $f$ is smooth enough, then $u$ is radially symmetric [12]. His proof relies on a unique continuation principle.

Another family of methods is based on the symmetrization by rearrangement. The first idea is to associate to any nonnegative measurable function $u:\Omega \to \mathbb{R}$ its Schwarz symmetrization $u^*$ which is a radial function such that the corresponding sub-level sets have the same measure as those of $u$; under this transformations, the $L^2$-norm of the gradient decreases [14], [11]. In particular, it is possible to show that many functionals of the calculus of variations decrease under symmetrization, and therefore that if $u$ is a solution of some variational problem, then $u^*$ is also a solution. However this does not imply that $u$ itself is symmetric. One way to show that $u$ is symmetric is to study the equality cases of symmetrization inequalities [4]; this approach is however limited by some stringent assumptions to apply the results.

In order to study partial symmetry, T. Bartsch, T. Weth and M. Willem, have introduced a nice method which mixes a variational argument with the maximum principle [2] (see also [22]). Given a closed half-space $H \subset \mathbb{R}^N$, define $\sigma_H$ to be the reflection with respect to $\partial H$. If $\sigma_H(\Omega) = \Omega$, define for $u:\Omega \to \mathbb{R}$ its polarization $u^H:\Omega \to \mathbb{R}$ by

$$u^H = \begin{cases} \max\{u, u \circ \sigma_H\} & \text{on } H, \\ \min\{u, u \circ \sigma_H\} & \text{on } \mathbb{R}^N \setminus H. \end{cases}$$

Now assume that $u$ is a minimizer of some functional. Then, if the functional does not increase under polarization with respect to $H$, it follows that $u^H$ is a minimizer too. Since symmetrization can be approximated by rearrangement [3] (see also [21]), this is stronger than requiring that the functional does not increase under symmetrization. The new ingredient that T. Bartsch, T. Weth and M. Willem introduce is that if $\Omega$ is a ball and $u^H$ is also a solution for every half-space $H$ such that $\sigma_H(\Omega) = \Omega$, then $u$ is axially symmetric. The method applies to minimizers under constraints and in particular to least energy solutions and least energy nodal solutions of semi-linear equations [2, Theorem 3.2].

It would be nice to extend such results to critical points that are not minimizers under a constraint. One way to construct such critical points is to rely on the Mountain Pass lemma of A. Ambrosetti and P. Rabinowitz [1]. Given a functional $\varphi \in C^1(H^1_0(\Omega))$ such that 0 is a local minimum of $\varphi$, set

$$\Gamma = \{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0 \text{ and } \varphi(\gamma(1)) < 0\},$$

and

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)).$$

Assume also that $\varphi$ satisfies the Palais-Smale condition, that is, if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H_0^1(\Omega)$ such that $(\varphi(u_n))_{n\in\mathbb{N}}$ converges and $\varphi'(u_n) \to 0$ as $n \to \infty$ in $H^{-1}(\Omega)$, then $(u_n)_{n\in\mathbb{N}}$ converges, up to a subsequence. Then there exists $u \in H_0^1(\Omega)$ such that $\varphi'(u) = 0$ and $\varphi(u) = c$. If, in addition, $\Omega$ is a ball and for *every* closed half-space $H \subset \mathbb{R}^N$ such that $\sigma_H(\Omega) = \Omega$ and $u \in H_0^1(\Omega)$, $\varphi(u^H) \le \varphi(u)$, then there exists $u \in H_0^1(\Omega)$ such that $\varphi'(u) = 0$, $\varphi(u) = c$ and $u$ is axially symmetric [19].

In general, it is not difficult to prescribe symmetry to solutions. The remarkable feature of this result is that $u$ is a critical point at a critical level without any symmetry constraint. This result was extended to critical levels defined with the Krasnosel'skiĭ genus [20] and to non-smooth critical point theory [15] (see also [16], [17]). We would like to know when all the solutions obtained by the Mountain Pass lemma are symmetric. To this regard, we recall that the Mountain Pass value $c$ often coincides with the least energy value and for instance, in [5], for a quite general class of autonomous functionals, the authors have recently proved that *any* least energy solution is radially symmetric and with fixed sign. We also point out that symmetry results under assumptions on the Morse index and somewhat restrictive assumptions on the nonlinearity have been obtained in [13], [10]. Going back to the minimax principle, we would like to apply the method of T. Bartsch, T. Weth and M. Willem. The crucial step is to prove that if $u$ is a critical point of $\varphi$ then $u^H$ is also a critical point of $\varphi$. We could not prove this and we also think that this should not be true in general. However, we have something that we think to be the best result in that direction.

To state our result, recall that the critical points $u$ of the Mountain Pass lemma can be localized as follows: if $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence of paths in $\Gamma$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \varphi(\gamma_n(t)) = c, \tag{1.2}$$

then, up to a subsequence,

$$\lim_{n\to\infty} \operatorname{dist}_{H^1}(u, \gamma_n([0,1])) = 0. \tag{1.3}$$

If $\varphi$ does not increase under polarizations with respect to a fixed half-space $H$, based upon a new abstract minimax principle (Proposition 2.1), we prove that for any sequence $(\gamma_n)_{n\in\mathbb{N}}$ satisfying (1.2), there exists a critical point $u$ of $\varphi$ with $\varphi(u) = c$ such that, up to a subsequence, (1.3) holds and, *in addition*, $u^H$ is also a critical point of $\varphi$ at the same level $c$ (Proposition 3.1). This provides some kind of symmetry information of $u$ with respect to $H$, see e.g. [2, Theorem 2.6] for the special situation regarding problem (1.1). One can expect that in many cases, there is at most one critical point $u$ such that $\operatorname{dist}_{H^1}(u, \gamma_n([0,1])) \to 0$ as $n \to \infty$ and $\varphi(u) = c$. In such a case we would have the desired property. Unfortunately, in general, the uniqueness of critical points at the level $c$ and near a family of paths seems quite difficult to establish. The result obtained also extends to continuous functionals in the framework of the non-smooth critical point theory of [8], [7], by exploiting a suitable quantitative deformation theorem [6].

The paper is organized as follows. In Section 2, we prove a new quantitative abstract Minimax Principle. In Section 3, we apply this result in the specific case of the Mountain Pass lemma and the polarization.

## 2. Shadowing minimax principle

In this section we shall prove the following variant of the minimax principle in which two almost critical points related by a function $\Psi$ are found at once.

Proposition 2.1. *Let $(X, \|\cdot\|)$ be a Banach space, $M$ be a metric space and $M_0 \subset M$. Let also consider $\Gamma_0 \subset C(M_0, X)$ and define the set*

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}$$

*If $\varphi \in C^1(X, \mathbb{R})$ satisfies*

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in M} \varphi(\gamma(t)) > \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} \varphi(\gamma_0(t)) = a,$$

*$\Psi \in C(X, X)$ and $\varphi \circ \Psi \leq \varphi$, $\Psi(\Gamma) \subset \Gamma$, then for every $\varepsilon \in ]0, (c-a)/2[$, $\delta > 0$ and $\gamma \in \Gamma$ such that*

$$\sup_M \varphi \circ \gamma \leq c + \varepsilon,$$

*there exist elements $u, v, w \in X$ such that*

- (a.1) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$,
- (a.2) $c - 2\varepsilon \leq \varphi(v) \leq c + 2\varepsilon$,
- (b.1) $\|u - w\| \leq 3\delta$,
- (b.2) $\operatorname{dist}_X(w, \gamma(M)) \leq \delta$,
- (b.3) $\|v - \Psi(w)\| \leq 2\delta$,
- (c.1) $\|\varphi'(u)\|_{X'} < 8\varepsilon/\delta$,
- (c.2) $\|\varphi'(v)\|_{X'} < 8\varepsilon/\delta$.

The proof relies on the following quantitative deformation lemma of M. Willem [23, Lemma 2.3].

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a Banach space, $\varphi \in C^1(X)$, $S \subseteq X$, $c \in \mathbb{R}$, $\varepsilon > 0$ and $\delta > 0$. Assume that for every $u \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon])$ such that $B_{2\delta}(u) \cap S \neq \emptyset$ it holds*

$$\|\varphi'(u)\|_{X'} \geq \frac{8\varepsilon}{\delta}.$$

*Then there exists a homeomorphism* $\eta: X \to X$ *such that* $\varphi \circ \eta \leq \varphi$ *and*

(a) $\eta(u) = u$ *if* $\varphi(u) \notin [c-2\varepsilon, c+2\varepsilon]$ *or* $B_{2\delta}(u) \cap S = \emptyset$*;*
(b) *if* $u \in S$ *and* $\varphi(u) \leq c+\varepsilon$*, then* $\varphi(\eta(u)) \leq c-\varepsilon$*;*
(c) *for every* $u \in X$ *it holds* $\|\eta(u) - u\| \leq \delta$*.*

Proof of Proposition 2.1. Let $\gamma \in \Gamma$, $c > a$, $\varepsilon \in ]0, (c-a)/2[$ and $\delta > 0$ be as in the statement of Proposition 2.1. Aiming to apply the quantitative deformation lemma, we set

$$S := \{w \in \gamma(M) : \text{for every } u \in B_{2\delta}(w) \cap \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]), \\ \text{one has } \|\varphi'(u)\|_{X'} \geq 8\varepsilon/\delta\}.$$

In turn, since $S$ fulfills the assumption of Proposition 2.2, we get a continuous function $\eta: X \to X$ such that $\varphi \circ \eta \leq \varphi$ which satisfies properties (a)–(c). Setting

$$\widetilde{\gamma} := \eta \circ \gamma \in \Gamma,$$

observe that, by virtue of (b), if $t \in M$ and $\varphi(\widetilde{\gamma}(t)) > c-\varepsilon$, then $\gamma(t) \notin S$, namely there exists $u \in B_{2\delta}(\gamma(t))$ such that $c-2\varepsilon \leq \varphi(u) \leq c+2\varepsilon$ and $\|\varphi'(u)\|_{X'} < 8\varepsilon/\delta$. If we now set

$$\widehat{\gamma} := \Psi \circ \widetilde{\gamma} \in \Gamma,$$

we claim that we can find elements $v \in X$ and $t \in M$ with the following properties: $c-2\varepsilon \leq \varphi(v) \leq c+2\varepsilon$, $\|v-\widehat{\gamma}(t)\| \leq 2\delta$, $\varphi(\widehat{\gamma}(t)) > c-\varepsilon$ and $\|\varphi'(v)\|_{X'} < 8\varepsilon/\delta$. In fact, if this was not the case, the assumption of Proposition 2.2 would be fulfilled with the choice $S := \widehat{\gamma}(M) \cap \varphi^{-1}([c-\varepsilon/2, c+\varepsilon])$. We then get a deformation $\widehat{\eta}: X \to X$ such that $\varphi \circ \widehat{\eta} \leq \varphi$ which satisfies properties (a)–(c). Given now an arbitrary element $\tau \in M$, either we have $\varphi(\widehat{\gamma}(\tau)) < c-\varepsilon/2$ or

$$c-\varepsilon/2 \leq \varphi(\widehat{\gamma}(\tau)) = \varphi(\Psi(\widetilde{\gamma}(\tau))) \leq \varphi(\widetilde{\gamma}(\tau)) \leq \varphi(\gamma(\tau)) \leq c+\varepsilon.$$

In any case, by (b) and since $\widehat{\eta} \circ \widehat{\gamma} \in \Gamma$ (by (a)), as $\varphi \circ \widehat{\gamma}|_{M_0} = \varphi \circ \Psi \circ \eta \circ \gamma|_{M_0} \leq \varphi \circ \gamma|_{M_0} \leq a < c-2\varepsilon$),

$$c \leq \sup_M \varphi(\widehat{\eta} \circ \widehat{\gamma}) \leq c-\varepsilon/2,$$

yielding a contradiction and proving the claim.

Setting $w := \widetilde{\gamma}(t) \in X$, since $\varphi(\widetilde{\gamma}(t)) \geq \varphi(\Psi(w)) > c-\varepsilon$, by the first part of the proof there exists an element $u \in X$ with the required properties (a.1) and (c.1). Furthermore, being $u \in B_{2\delta}(\gamma(t))$ and recalling (c), we get

$$\|u-w\| \leq \|u-\gamma(t)\| + \|\eta(\gamma(t)) - \gamma(t)\| \leq 2\delta + \delta,$$

proving (b.1). Analogously, inequalities (b.2) and (b.3) follow. $\square$

Remark 2.3. The minimax principle stated in Proposition 2.1 for $C^1$ smooth functionals continues to hold for continuous functionals in the framework of the non-smooth critical point theory developed in [8], [7] by J.N. Corvellec, M. Degiovanni and M. Marzocchi, where the quantity $\|\varphi'(u)\|$ is replaced by the notion of weak slope $|d\varphi|(u) \in [0,+\infty]$ (see [8, Definition 2.1]). Precisely, the statement of Proposition 2.1 in the continuous case remains the same except the fact that the inequalities $\|\varphi'(u)\|_{X'} < 8\varepsilon/\delta$ and $\|\varphi'(v)\|_{X'} < 8\varepsilon/\delta$ are replaced by $|d\varphi|(u) < 8\varepsilon/\delta$ and $|d\varphi|(v) < 8\varepsilon/\delta$, respectively. In [6, Theorem 2.3] J.N. Corvellec derived a quantitative deformation lemma being the natural non-smooth counterpart of Proposition 2.2. Then, setting

$$A := \{w \in \gamma(M) : \text{for every } u \in B_{2\delta}(w) \cap \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \\ \text{one has } |d\varphi|(u) \geq 8\varepsilon/\delta\}.$$

By applying [6, Theorem 2.3] to the set $A$ (or slightly modifying the argument if $A$ is not closed in $X$) the same conclusion in the first part of the proof of Proposition 2.1 is obtained. In a similar fashion, also the second part of the proof can be proved reusing [6, Theorem 2.3].

For applications of non-smooth critical point theory to various classes of quasi-linear elliptic PDEs, we refer the interested reader to the monograph [18]. In the recent work [15] a symmetric minimax theorem is obtained for a class of lower semi-continuous functionals of the form $\varphi(u) = \int_\Omega j(u, |Du|) - \int_\Omega G(|x|, u)$.

## 3. Application to the Mountain Pass lemma

We will now apply the result of the previous section in order to prove the result announced in the introduction.

Proposition 3.1. *Assume that $\varphi \in C^1(H^1_0(\Omega))$, $0$ is a strict local minimum of $\varphi$ which is not a global minimum and define*

$$\Gamma = \{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0 \text{ and } \varphi(\gamma(1)) < 0\}$$

*and*

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \varphi(\gamma(t)).$$

*Assume that $\varphi$ satisfies the Palais–Smale condition. Let $H$ be a closed half-space with $\sigma_H(\Omega) = \Omega$ and for every $u \in H^1_0(\Omega)$, $\varphi(u^H) \leq \varphi(u)$. If $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $\Gamma$ such that*

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} \varphi(\gamma_n([0,1])) \leq c.$$

*then there exists $u \in H^1_0(\Omega)$ such that $\varphi(u) = \varphi(u^H) = c$, $\varphi'(u) = \varphi'(u^H) = 0$ and*

$$\liminf_{n\to\infty} \operatorname{dist}_{H^1}(u, \gamma_n([0,1])) = 0.$$

PROOF. Notice that the map $\Psi\colon H_0^1(\Omega) \to H_0^1(\Omega)$ defined by $\Psi(u) := u^H$ is continuous by [19, Proposition 2.5], [20, Corollary 2.40]. By assumption, we have $\varphi \circ \Psi \le \varphi$ and $\Psi(\gamma) \in \Gamma$, for all $\gamma \in \Gamma$, where $\Psi(\gamma)(t) := \Psi(\gamma(t))$ for $t \in [0,1]$. Without loss of generality, we can assume that

$$\sup_{t\in[0,1]} \varphi(\gamma_n([0,1])) \le c + \frac{1}{n^2}.$$

Apply now Proposition 2.1 with the choice $M := [0,1]$, $M_0 := \{0,1\}$, $\delta = \delta_n := 1/n$, $\varepsilon = \varepsilon_n := 1/n^2$ and

$$\Gamma_0 := \{\gamma_0 \in C(\{0,1\}, H_0^1(\Omega)) : \gamma_0(0) = 0 \text{ and } \varphi(\gamma_0(1)) < 0\}.$$

One then obtains three sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ such that

$$\lim_{n\to\infty} \varphi(u_n) = \lim_{n\to\infty} \varphi(v_n) = c, \qquad \lim_{n\to\infty} \varphi'(u_n) = \lim_{n\to\infty} \varphi'(v_n) = 0,$$
$$\lim_{n\to\infty} \|u_n - w_n\|_{H^1} = \lim_{n\to\infty} \operatorname{dist}_{H^1}(w_n, \gamma_n([0,1])) = \lim_{n\to\infty} \|v_n - w_n^H\|_{H^1} = 0.$$

Since $\varphi$ satisfies the Palais–Smale condition, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges to some $u \in H_0^1(\Omega)$. Hence, the sequence $(w_n)_{n\in\mathbb{N}}$ also converges to $u$. By continuity of the polarization, $(v_n)_{n\in\mathbb{N}}$ converges to $u^H$. The rest follow by the fact that $\varphi$ is $C^1(H_0^1(\Omega))$. $\square$

REMARK 3.2. As pointed out in Remark 2.3, the shadowing minimax principle in Proposition 2.1 extends to the case of continuous functionals in the framework of the non-smooth critical point theory of [8], [7] replacing $\|\varphi'(u)\|$ by the weak slope $|d\varphi|(u)$ [8, Definition 2.1]. In this setting, the Palais–Smale condition has to be read as follows: if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H_0^1(\Omega)$ such that $(\varphi(u_n))_{n\in\mathbb{N}}$ converges and $|d\varphi|(u_n) \to 0$ as $n \to \infty$, then $(u_n)_{n\in\mathbb{N}}$ converges strongly, up to a subsequence, to some $u$ in $H_0^1(\Omega)$. Therefore, taking into account that $\varphi$ is continuous and the map $H_0^1(\Omega) \ni u \mapsto |d\varphi|(u) \in [0,+\infty]$ is in turn lower semi-continuous [8, Proposition 2.6], Proposition 3.1 holds true for continuous functionals, with essentially the same proof, by replacing the conclusion that $\varphi'(u) = 0$ and $\varphi'(u^H) = 0$ with $|d\varphi|(u) = 0$ and $|d\varphi|(u^H) = 0$, respectively. For many continuous functionals of the Calculus of Variations this implies [18] that $u$ and $u^H$ are distributional solutions of the associated Euler–Lagrange equation.

REMARK 3.3. Up to slight modifications, Proposition 3.1 holds also when the assumption that the closed half-space $H$ is axially symmetric, that is $\sigma_H(\Omega) = \Omega$, is replaced by the more general assumption that $0 \in H$ and $\sigma^H(\Omega) = \Omega$, denoting $\sigma^H(\Omega)$ the polarized domain of $\Omega$, namely the unique domain satisfying $\chi_{\sigma^H(\Omega)} = (\chi_\Omega)^H$. If for instance $0 \notin \partial H$, then $\sigma_H(B_1(0)) \ne B_1(0)$ but instead $\sigma^H(B_1(0)) = B_1(0)$.

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