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MOSER–HARNACK INEQUALITY, KRASNOSEL'SKII TYPE FIXED POINT THEOREMS IN CONES AND ELLIPTIC PROBLEMS

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Abstract. Fixed point theorems of Krasnosel'skiĭ type are obtained for the localization of positive solutions in a set defined by means of the norm and of a semi-norm. In applications to elliptic boundary value problems, the semi-norm comes from the Moser–Harnack inequality for nonnegative superharmonic functions whose use is crucial for the estimations from below. The paper complements and gives a fixed point alternative approach to our similar results recently established in the frame of critical point theory. It also provides a new method for discussing the existence and multiplicity of positive solutions to elliptic boundary value problems.

1. Introduction

The main motivation of this paper comes from the already classical problem of positive solutions for a semi-linear elliptic equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Existence, uniqueness and multiplicity of its solutions have been the subject of many papers in the the last four decades and required different type of arguments such as upper and lower solution method, variational techniques and topological degree method (see e.g. [1]–[3], [7], [8], [14], [21], [24]). In the same time, lots of papers have been produced dealing with positive solutions of two- and multi-point boundary value problems for ordinary differential equations. The tools that have been used in these two directions were the same, or similar, in some cases, and different in others. For instance, the compression-expansion fixed point theorems of Krasnosel'skiĭ [17] have been extensively used as basic tool for the existence and localization of positive solutions to ordinary differential equations (see e.g. [4], [9], [10], [13], [18]–[20], [22], [25], [27], [31]), but almost never applied to partial differential equations, except particular situations which can be reduced to ordinary differential equations, such as the case of radial solutions. It is the aim of this paper to make this technique work for elliptic equations too. The main ingredient is the Moser–Harnack inequality for nonnegative superharmonic functions. We show that this local inequality is enough to produce a suitable cone of functions for that Krasnosel'skiĭ's technique works for the nonlinear operator associated to (1.1).

To make clear the appropriateness of the Krasnosel'skiĭ's results for ordinary differential equations and their limits of applicability to partial differential equations, we first shortly discuss problem (1.1) for $n = 1$, i.e.

$$\begin{cases} Lu := -u'' = f(u) & \text{in } (0,1), \\ u > 0 & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

This problem is equivalent to the fixed point equation $u = Nu$ in $C([0,1], \mathbb{R}_+)$, where $N = L^{-1}F$,

$$\begin{aligned} (Fu)(x) &= f(u(x)), & x \in [0,1],\ u \in C([0,1], \mathbb{R}_+), \\ (L^{-1}h)(x) &= \int_0^1 G(x,y)h(y)\,dy, & x \in [0,1],\ h \in C([0,1], \mathbb{R}_+) \end{aligned}$$

and $G(x,y)$ is the Green function

$$G(x,y) = \begin{cases} x(1-y) & \text{for } 0 \le x \le y \le 1, \\ (1-x)y & \text{for } 0 \le y < x \le 1. \end{cases}$$

The following properties are essential for the applicability of Krasnosel'skiĭ's technique:

(a) $G(x,y) \le G(y,y)$ for all $x, y \in [0,1]$; and

(b) for each subinterval $\Omega_0 = (a,b)$ of $\Omega = (0,1)$, $0 < a < b < 1$, there exists a constant $M > 0$ with

$$G(x,y) \geq MG(y,y) \quad \text{for all } x \in [a,b],\ y \in [0,1].$$

These imply that for each $h \in L^2([0,1], \mathbb{R}_+)$ and all $x \in [a,b]$, $x' \in [0,1]$, one has

$$\begin{aligned}(L^{-1}h)(x) &= \int_0^1 G(x,y)h(y)\,dy \geq M \int_0^1 G(y,y)h(y)\,dy \\ &\geq M \int_0^1 G(x',y)h(y)\,dy = M(L^{-1}h)(x').\end{aligned}$$

This yields the fundamental estimation from bellow

$$(L^{-1}h)(x) \geq M|L^{-1}h|_\infty \quad \text{for all } x \in [a,b]. \tag{1.3}$$

Here by $|\cdot|_\infty$ we have denoted the maximum norm in $C[0,1]$. Based on this estimation one defines the cone

$$K = \{u \in C([0,1], \mathbb{R}_+) : u(x) \geq M|u|_\infty \quad \text{for all } x \in [a,b]\}$$

and one can infer that $N(K) \subset K$. This is the framework where the compression-expansion theorems of Krasnosel'skiĭ's type can be easily applied. For instance, we may use the following version:

Theorem 1.1 (Krasnosel'skiĭ). *Let $(X, |\cdot|)$ be a Banach space, K a cone of X and $N: K \to K$ a completely continuous operator. Assume that for some $\alpha, \beta > 0$, $\alpha \neq \beta$, the following conditions are satisfied:*

$$Nu \not\leq u \quad \textit{for all } u \in K,\ |u| = \alpha, \tag{1.4}$$

$$Nu \not\geq u \quad \textit{for all } u \in K,\ |u| = \beta. \tag{1.5}$$

Then N has a fixed point $u \in K$ with $\min\{\alpha, \beta\} < |u| < \max\{\alpha, \beta\}$.

Assume that function f from (1.1) is nondecreasing. If

$$\frac{f(M\alpha)}{\alpha} > \frac{1}{A}, \tag{1.6}$$

where $A = \int_a^b G(x^*, y)\,dy = (L^{-1}\chi_{\Omega_0})(x^*)$ (χ_{Ω_0} is the characteristic function of Ω_0) and x^* is a chosen point in $[0,1]$, then condition (1.4) holds. Indeed, otherwise, if for some $u \in K$, $|u|_\infty = \alpha$, one has $Nu \leq u$, then

$$\begin{aligned}\alpha &\geq u(x^*) \geq (Nu)(x^*) = L^{-1}(Fu)(x^*) \geq L^{-1}[(Fu)\chi_{\Omega_0}](x^*) \\ &\geq L^{-1}[f(M\alpha)\chi_{\Omega_0}](x^*) = f(M\alpha)(L^{-1}\chi_{\Omega_0})(x^*) = f(M\alpha)A,\end{aligned}$$

a contradiction to (1.6). Also, if

$$\frac{f(\beta)}{\beta} < \frac{1}{B}, \tag{1.7}$$

where $B = \max_{x\in[0,1]} \int_0^1 G(x,y)\,dy = |L^{-1}1|_\infty$, then (1.5) holds. Assume by contradiction that $Nu \geq u$ for some $u \in K$, $|u|_\infty = \beta$. Then, for some $x' \in [0,1]$, one has $|u|_\infty = u(x')$ and

$$\beta = u(x') \leq (Nu)(x') = L^{-1}F(u)(x') \leq f(\beta)(L^{-1}1)(x') \leq f(\beta)B.$$

This contradicts (1.7). Therefore, if f is nondecreasing and satisfies (1.6) and (1.7), then (1.2) has a solution with $\min\{\alpha,\beta\} < |u|_\infty < \max\{\alpha,\beta\}$.

Notice that all the above arguments would be valuable for $n > 1$ and $L = -\Delta$, provided that the global Harnack type inequality (1.3) holds in a subdomain Ω_0 of Ω with $\overline{\Omega}_0 \subset \Omega$ (see [29]). Unfortunately, such a result is not known for $n > 1$ and we may assert that this is the reason for which Krasnosel'skiĭ's fixed point theorems in cones could not be directely applied to partial differential equations. However, instead of global inequality (1.3), a local Moser–Harnack inequality [23], [11], [15], [16] holds for $n > 1$, namely:

Lemma 1.2 (Moser). *Let $n \geq 3$ and $1 \leq p < n/(n-2)$, or $n = 2$ and $1 \leq p < \infty$, and let $R > 0$. Then there exists a constant $M_0 = M_0(n,p,R) > 0$ such that for every nonnegative superharmonic function u in $B_{4R}(x_0)$, the following inequality is satisfied*

$$u(x) \geq M_0|u|_{L^p(B_{2R}(x_0))} \quad \text{for } x \in B_R(x_0). \tag{1.8}$$

In the previous lemma, by $B_\rho(x_0)$, we have mean the open ball in $\mathbb{R}^n$ of centre x_0 and radius ρ and by a *superharmonic* function in a domain $\Omega \subset \mathbb{R}^n$, any function $u \in H^1(\Omega)$ satisfying

$$\Delta u \leq 0 \quad \text{in } \mathcal{D}'(\Omega),$$

that is,

$$\int_\Omega \nabla u \cdot \nabla w \geq 0 \quad \text{for every } w \in C_0^\infty(\Omega) \text{ with } w \geq 0 \text{ in } \Omega.$$

A similar estimation to (1.8) also holds on any bounded subdomain Ω_0 with $\overline{\Omega}_0 \subset \Omega$ (i.e. $\Omega_0 \Subset \Omega$), as shows the following theorem.

Theorem 1.3. *Let $n \geq 3$ and $1 \leq p < n/(n-2)$, or $n = 2$ and $1 \leq p < \infty$, and let $\Omega_0 \Subset \Omega$. Then there exists a constant $M = M(n,p,\Omega,\Omega_0) > 0$ such that for every nonnegative superharmonic function u in Ω, the following inequality holds:*

$$u(x) \geq M|u|_{L^p(\Omega_0)} \quad \text{for } x \in \Omega_0. \tag{1.9}$$

Proof. We fix any number $R > 0$ with $4R < \operatorname{dist}(\Omega_0, \partial\Omega)$ and we consider a finite open cover of the compact $\overline{\Omega}_0$:

$$B_{2R/3}(x_1),\ B_{2R/3}(x_2),\ \ldots, B_{2R/3}(x_m).$$

From (1.8) it follows that there is a constant $M_1 \in (0,1)$ such that for every nonnegative superharmonic function u in Ω,

$$|u|_{L^p(B_{2R/3}(x_\nu))} \geq M_1 |u|_{L^p(B_{2R}(x_\nu))}, \quad \nu = 1,\dots,m. \tag{1.10}$$

We claim that there exists a constant $\mu > 0$ with

$$|u|_{L^p(B_{2R/3}(x_i))} \geq \mu |u|_{L^p(B_{2R/3}(x_j))} \tag{1.11}$$

for all $i,j \in \{1,\dots,m\}$. Indeed, if $i \neq j$, we can choose distinct $i_0 = i$, $i_1, i_2, \dots, i_{k-1}, i_k = j$ from the set $\{1,\dots,m\}$ with $k \leq m$ and $|x_{i_\nu} - x_{i_{\nu-1}}| \leq 4R/3$ for $\nu = 1,\dots,k$. Then, for every $x \in B_{2R/3}(x_{i_1})$, one has $|x - x_{i_1}| < 2R/3$ and since $|x_{i_1} - x_{i_0}| \leq 4R/3$, we infer that $|x - x_{i_0}| \leq |x - x_{i_1}| + |x_{i_1} - x_{i_0}| < 2R$. Hence $B_{2R/3}(x_{i_1}) \subset B_{2R}(x_{i_0})$ and so

$$|u|_{L^p(B_{2R}(x_{i_0}))} \geq |u|_{L^p(B_{2R/3}(x_{i_1}))}.$$

This together with (1.10) yields

$$|u|_{L^p(B_{2R/3}(x_{i_0}))} \geq M_1 |u|_{L^p(B_{2R/3}(x_{i_1}))}.$$

If we repeat successively the above argument we finally obtain

$$|u|_{L^p(B_{2R/3}(x_{i_0}))} \geq M_1^k |u|_{L^p(B_{2R/3}(x_j))}.$$

Since $k \leq m$ and $M_1 < 1$, one has $M_1^k \geq M_1^m$ and so (1.11) holds with $\mu = M_1^m$. Now for every $x \in \Omega_0$, there is $i \in \{1,\dots,m\}$ with $x \in B_{2R/3}(x_i)$. Then

$$u^p(x) \geq M_0^p |u|^p_{L^p(B_{2R}(x_i))} \geq M_0^p |u|^p_{L^p(B_{2R/3}(x_i))} \geq \frac{M_0^p \mu^p}{m} \sum_{j=1}^m |u|^p_{L^p(B_{2R/3}(x_j))}.$$

Since $\Omega_0 \subset \bigcup_{j=1}^m B_{2R/3}(x_j)$, we have

$$\sum_{j=1}^m |u|^p_{L^p(B_{2R/3}(x_j))} \geq |u|^p_{L^p(\Omega_0)}.$$

Hence

$$u(x) \geq \frac{M_0 \mu}{m^{1/p}} |u|_{L^p(\Omega_0)} \quad (x \in \Omega_0),$$

which proves (1.9) with $M = M_0\mu/m^{1/p}$. □

Our goal in this paper is to show that local estimation (1.9) is enough for making Krasnosel'skiĭ's technique applicable to elliptic problems. The main idea, incipiently introduced in [26], [28], is to try to localize solutions in a conical "annulus" jointly defined by the norm and a semi-norm, the last one beeing suggested by the Moser–Harnack inequality. The same idea is used in [30] in the framework of critical point theory.

2. Main abstract results

Let X, Y be normed linear spaces with norm $|\cdot|_X$ and $|\cdot|_Y$, respectively and let $\mathcal{I}: X \to Y$ be a continuous linear map. For any element $u \in X$, we shall denote

$$\|u\| := |\mathcal{I}u|_Y.$$

Clearly $\|\cdot\|$ is a semi-norm on X. In what follows we shall design the norm $|\cdot|_X$ by $|\cdot|$, for simplicity.

Let K be a wedge in X, i.e. a closed convex set with $\lambda K \subset K$ for every $\lambda \in \mathbb{R}_+$, and let $\phi \in K$ with $|\phi| = 1$ be any fixed element. Then for any positive numbers R_0, R_1 with $R_0 < \|\phi\| R_1$, there exists a $\mu > 0$ such that $\|\mu\phi\| > R_0$ and $|\mu\phi| < R_1$. Hence the set $K_{R_0 R_1} = \{u \in K : R_0 < \|u\|,\ |u| < R_1\}$ is nonempty.

Theorem 2.1. *Let $N: K \to K$ be completely continuous and let $h \in K$ with $\|h\| > R_0$. Assume that the following conditions are satisfied:*

$$Nu \neq \lambda u \quad \textit{for } |u| = R_1,\ \lambda \geq 1; \tag{2.1}$$

$$(1-\mu) N\left(\min\left\{ \frac{R_1}{|u|}, 1 \right\} u \right) + \mu h \neq u \qquad \textit{for } 0 \leq \mu \leq 1,\ \|u\| = R_0,\ |u| \leq R_2, \tag{2.2}$$

where $R_2 = \max\left\{ R_1,\ |h|,\ \max\limits_{|u| \leq R_1} |N(u)| \right\}$. Then N has a fixed point u in $K_{R_0 R_1}$.

Proof. Let us denote $C := \{u \in K : |u| \leq R_2\}$ and define $\widetilde{N}: C \to C$,

$$\widetilde{N}u = \begin{cases} Nu & \text{if } |u| \leq R_1, \\ N\left(\dfrac{R_1}{|u|}\, u \right) & \text{if } R_1 < |u| \leq R_2. \end{cases}$$

Clearly C is a convex closed subset of X and $\widetilde{N}$ is a compact map. Let us consider two open sets in C, namely

$$U_1 := \{u \in C : |u| < R_1\}, \qquad U_2 := \{u \in C : \|u\| < R_0\}.$$

From (2.1), (2.2) it follows that $\widetilde{N}$ is fixed point free on ∂U_1 and ∂U_2. Now (2.1) and Theorem 7.3 in [12] guarantee that

$$i(\widetilde{N}, U_1) = 1.$$

Here $i(\widetilde{N}, U_1)$ stands for the fixed point index of $\widetilde{N}$ in U_1. Furthermore we remark that $\partial U_2 = \{u \in K : \|u\| = R_0,\ |u| \leq R_2\}$ and that (2.2) implies for $\widetilde{N}$ the following behavior on ∂U_2:

$$(1-\mu)\widetilde{N}u + \mu h \neq u \quad \text{for } u \in \partial U_2.$$

Then

$$i(\widetilde{N}, U_2) = i(h, U_2) = 0$$

since $h \in C \setminus \overline{U}_2$. From

$$1 = i(\widetilde{N}, U_1) = i(\widetilde{N}, U_1 \setminus \overline{U}_2) + i(\widetilde{N}, U_1 \cap U_2),$$
$$0 = i(\widetilde{N}, U_2) = i(\widetilde{N}, U_2 \setminus \overline{U}_1) + i(\widetilde{N}, U_1 \cap U_2),$$

by substraction we have

$$i(\widetilde{N}, U_1 \setminus \overline{U}_2) - i(\widetilde{N}, U_2 \setminus \overline{U}_1) = 1. \tag{2.3}$$

Notice that if $u \in U_2 \setminus \overline{U}_1$, then $\|u\| < R_0$ and $|u| > R_1$. Hence $\widetilde{N}(u) = N(R_1/|u|u) \neq u$ as shows (2.1). Thus $i(\widetilde{N}, U_2 \setminus \overline{U}_1) = 0$. Then (2.3) implies

$$i(\widetilde{N}, U_1 \setminus \overline{U}_2) = 1,$$

and so $\widetilde{N}$ has a fixed point in $U_1 \setminus \overline{U}_2$. The conclusion now follows if we remark that $U_1 \setminus \overline{U}_2 = K_{R_0 R_1}$, and that $\widetilde{N}$ coincides with N on $U_1 \setminus \overline{U}_2$. □

REMARK 2.2. (a) In particular, if N is a self-mapping of the set $\{u \in K : |u| \leq R_1\}$ and $|h| \leq R_1$, then $R_2 = R_1$ and condition (2.2) reduces to

$$(1-\mu)Nu + \mu h \neq u \quad \text{for } \|u\| = R_0,\ |u| \leq R_1,\ 0 \leq \mu \leq 1.$$

(b) In the classical case $X = Y$, $|\cdot| = \|\cdot\|$ and $I = \mathrm{id}$, we have $R_0 < R_1$and (2.2) reduces to the condition

$$(1-\mu)Nu + \mu h \neq u \quad \text{for } |u| = R_0,\ 0 \leq \mu \leq 1$$

(for some $h \in K$ with $|h| > R_0$), which is independent of R_1 and R_2.

We also have a three solutions existence result:

THEOREM 2.3. *Under the assumptions of Theorem* 2.1*, if in addition there exists a number R_{-1} with $0 < R_{-1} < R_0/|\mathcal{I}|$ and*

$$Nu \neq \lambda u \quad \textit{for } |u| = R_{-1},\ \lambda \geq 1, \tag{2.4}$$

then N has three fixed points u_1, u_2, u_3 with

$$R_0 < \|u_1\|, \quad |u_1| < R_1; \qquad R_{-1} < |u_2| < R_1, \quad \|u_2\| < R_0; \qquad |u_3| < R_{-1}.$$

PROOF. Theorem 2.1 guarantees a fixed point u_1 with $R_0 < \|u_1\|$, $|u_1| < R_1$. Also, (2.4) implies $i(\widetilde{N}, U_3) = 1$, where $U_3 = \{u \in K : |u| < R_{-1}\}$. Hence a second fixed point u_3 exists in U_3. Finally, since $\overline{U}_3 \subset U_2$, we have

$$i(\widetilde{N}, U_2 \setminus \overline{U}_3) = i(\widetilde{N}, U_2) - i(\widetilde{N}, U_3) = 0 - 1 = -1,$$

whence a third fixed point u_2 in $U_2 \setminus \overline{U}_3$. □

3. Application to elliptic boundary value problems

We now return to problem (1.1) assuming that Ω is a bounded regular domain in $\mathbb{R}^n$, $n \geq 2$, and $f\colon \mathbb{R}_+ \to R_+$ is continuous. We seek positive solutions, i.e. $u \in C^1(\overline{\Omega})$, $u(x) > 0$ for all $x \in \Omega$ and u satisfies (1.1), where Δu is considered in the sense of distributions.

We recall (see [5, Lemma 1.1] and [6, p. 317]) that if Ω is a bounded regular domain of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and $g \in L^\infty(\Omega)$, then the weak solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\Delta u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

belongs to $C^1(\overline{\Omega})$. Also the linear solution operator $(-\Delta)^{-1}\colon L^\infty(\Omega) \to C^1(\overline{\Omega})$ assigning to each $g \in L^\infty(\Omega)$, the corresponding solution of (3.1), is continuous, compact and order-preserving.

In order to apply the abstract results from Section 2, let $X = C_0(\overline{\Omega})$,

$$C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\},$$

with norm $|u| = |u|_\infty = \max_{\overline{\Omega}} |u(x)|$. We fix any $\Omega_0 \Subset \Omega$ and we let $Y = L^p(\Omega_0)$, where $p \in [1, n/(n-2))$ if $n > 2$ and $p \in [1,\infty)$ for $n = 2$, with norm

$$\|v\| = \left(\int_{\Omega_0} |v|^p \, dx \right)^{1/p} \quad (v \in L^p(\Omega_0)).$$

In this case we take $\mathcal{I}\colon C_0(\overline{\Omega}) \to L^p(\Omega_0)$, $\mathcal{I}u = u|_{\Omega_0}$ (restriction of u to Ω_0). Since for any $u \in C_0(\overline{\Omega})$, $\|u\| \leq |u|(\operatorname{mes}(\Omega_0))^{1/p}$, we have

$$|\mathcal{I}| \leq (\operatorname{mes}(\Omega_0))^{1/p}.$$

Let $K = \{u \in C_0(\overline{\Omega}; \mathbb{R}_+) : u(x) \geq M\|u\| \text{ for all } x \in \Omega_0\}$, where constant $M > 0$ comes from Moser–Harnack inequality (1.8). Define

$$N\colon C(\overline{\Omega}; \mathbb{R}_+) \to C_0(\overline{\Omega}) \quad \text{by} \quad N(u) = (-\Delta)^{-1} F(u),$$

where

$$F\colon C(\overline{\Omega}; \mathbb{R}_+) \to C(\overline{\Omega}), \quad F(u)(x) = f(u(x)).$$

Since $f \geq 0$, and $(-\Delta)^{-1}$ is positive, we have that N maps the set $C(\overline{\Omega}; \mathbb{R}_+)$ into itself. Also, by the Moser–Harnack inequality, we have $N(K) \subset K$.

In this case we can take ϕ be the positive eigenfunction corresponding to the first eigenvalue λ_1, i.e.

$$\begin{aligned} \Delta\phi + \lambda_1\phi &= 0 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $|\phi| = 1$.

Let χ_{Ω_0} be the characteristic function of Ω_0, i.e. $\chi_{\Omega_0}(x) = 1$ if $x \in \Omega_0$, $\chi_{\Omega_0}(x) = 0$ otherwise, and let $C = |1|_{L^p(\Omega_0)} = (\mathrm{mes}(\Omega_0))^{1/p}$. We note that $MC \le 1$. Indeed, from

$$(-\Delta)^{-1}\chi_{\Omega_0} \ge M\|(-\Delta)^{-1}\chi_{\Omega_0}\| \quad \text{in } \Omega_0,$$

we obtain

$$\|(-\Delta)^{-1}\chi_{\Omega_0}\| \ge MC\|(-\Delta)^{-1}\chi_{\Omega_0}\|,$$

whence $MC \le 1$. Denote

$$A := \frac{1}{MC\|(-\Delta)^{-1}\chi_{\Omega_0}\|} \quad \text{and} \quad B := \frac{1}{|(-\Delta)^{-1}1|}.$$

Theorem 3.1. *Assume that there exist R_0, R_1 with $0 < R_0 < MC\|\phi\|R_1$ such that*

$$\frac{\min_{\tau\in[MR_0,R_1]} f(\tau)}{R_0} > A, \tag{3.2}$$

$$\frac{\max_{\tau\in[0,R_1]} f(\tau)}{R_1} < B. \tag{3.3}$$

Then (1.1) *has at least one solution with* $R_0 < \|u\|$, $|u| < R_1$.

Remark 3.2. If f is nondecreasing on $[0, R_1]$, then (3.2), (3.3) become respectively

$$\frac{f(MR_0)}{R_0} > A, \tag{3.4}$$

$$\frac{f(R_1)}{R_1} < B, \tag{3.5}$$

showing the behavior of nonlinearity f at only two points MR_0 and R_1.

Proof. We shall apply Theorem 2.1. We show that (2.1) holds. In fact we have more, namely that $|N(u)| < R_1$ for all $u \in K$ with $|u| \le R_1$. Indeed, from

$$f(u(x)) \le \max_{\tau\in[0,R_1]} f(\tau),$$

and (3.3), we have

$$|N(u)| = |(-\Delta)^{-1}f(u)| \le \Big|(-\Delta)^{-1} \max_{\tau\in[0,R_1]} f(\tau)\Big| = \max_{\tau\in[0,R_1]} f(\tau)|(-\Delta)^{-1}1| < R_1.$$

Next we show that (2.2) holds for $h := R_1\phi$, when, in view of Remark 2.2(a), $R_2 = R_1$. One has

$$\|h\| = R_1\|\phi\| > \frac{R_0}{MC} \ge R_0.$$

Assume that (2.2) does not hold. Then

$$(1-\mu)N(u) + \mu h = u \tag{3.6}$$

for some u, μ with $\|u\| = R_0$, $|u| \le R_1$, $0 \le \mu \le 1$. Since $(-\Delta)^{-1}$ is order-preserving and $u(x) \ge MR_0$ in Ω_0, we have

$$N(u) = (-\Delta)^{-1} f(u) \ge (-\Delta)^{-1}[f(u)\chi_{\Omega_0}] \ge \min_{\tau \in [MR_0, R_1]} f(\tau)(-\Delta)^{-1}\chi_{\Omega_0}.$$

Then (3.6) implies

$$\begin{aligned} \mathcal{I}u &\ge \mu \mathcal{I}h + (1-\mu) \min_{\tau \in [MR_0, R_1]} f(\tau)\mathcal{I}(-\Delta)^{-1}\chi_{\Omega_0} \\ &\ge \mu M\|h\| + (1-\mu)M \min_{\tau \in [MR_0, R_1]} f(\tau)\|(-\Delta)^{-1}\chi_{\Omega_0}\| \\ &\ge \mu M \frac{R_0}{MC} + (1-\mu)M \min_{\tau \in [MR_0, R_1]} f(\tau)\|(-\Delta)^{-1}\chi_{\Omega_0}\|. \end{aligned}$$

Taking the norm in $L^p(\Omega_0)$ we obtain

$$\begin{aligned} R_0 = \|u\| &\ge \left(\mu M \frac{R_0}{MC} + (1-\mu)M \min_{\tau \in [MR_0, R_1]} f(\tau)\|(-\Delta)^{-1}\chi_{\Omega_0}\|\right)|1|_{L^p(\Omega_0)} \\ &= \mu R_0 + (1-\mu)MC \min_{\tau \in [MR_0, R_1]} f(\tau)\|(-\Delta)^{-1}\chi_{\Omega_0}\|. \end{aligned}$$

Consequently

$$R_0 \ge MC \min_{\tau \in [MR_0, R_1]} f(\tau)\|(-\Delta)^{-1}\chi_{\Omega_0}\|,$$

which contradicts (3.2). Now the conclusion follows from Theorem 2.1. □

Theorem 2.3 yields two and three solutions existence results:

Theorem 3.3. *Assume that there exist R_{-1}, R_0, R_1 with $|\mathcal{I}|R_{-1} < R_0 < MC\|\phi\|R_1$ such that* (3.2), (3.3) *and*

$$\frac{\max_{\tau \in [0, R_{-1}]} f(\tau)}{R_{-1}} < B$$

holds. Then (1.1) *has at least two solutions u_1, u_2 with $R_0 < \|u_1\|$, $|u_1| < R_1$ and $R_{-1} < |u_2| < R_1$, $\|u_2\| < R_0$. A third positive solution u_3 exists with $|u_3| < R_{-1}$ if $f(0) > 0$.*

We obtain multiple solutions if nonlinearity f is oscillating.

Theorem 3.4. *Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function and let $(R_0^i)_{1 \le i \le k}$, $(R_1^i)_{1 \le i \le k}$ be increasing sequences of positive numbers satisfying the following conditions:*

$$\begin{aligned} R_0^i &< MC\|\phi\|R_1^i \quad \text{for } i = 1, \dots, k; \\ |\mathcal{I}|R_1^i &< R_0^{i+1} \qquad\quad \text{for } i = 1, \dots, k-1; \end{aligned}$$

$$\frac{\inf_{\tau \in [MR_0^i, R_1^i]} f(\tau)}{R_0^i} > A \quad \text{and} \quad \frac{\max_{\tau \in [0, R_1^i]} f(\tau)}{R_1^i} < B \quad \text{for } i = 1, \dots, k.$$

Then (1.1) *has at least* k *distinct solutions* u_i *with,* $R_0^i < \|u_i\|$, $|u_i| < R_1^i$, *for* $i = 1, \dots, k$.

By the next result it is guaranteed the existence of positive solutions from the behavior of the nonlinearity at zero and infinity.

Theorem 3.5. *Assume that* $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous and nondecreasing.*

(a) *If*

$$\liminf_{\tau\to\infty} \frac{f(\tau)}{\tau} < B, \qquad \limsup_{\tau\to 0+} \frac{f(\tau)}{\tau} > \frac{A}{M}, \tag{3.7}$$

then (1.1) *has at least one solution.*

(b) *If there exists* $R_0 > 0$ *such that*

$$\frac{f(MR_0)}{R_0} > A$$

and

$$\liminf_{\tau\to 0+} \frac{f(\tau)}{\tau} < B, \qquad \liminf_{\tau\to\infty} \frac{f(\tau)}{\tau} < B,$$

then (1.1) *has at least two solutions.*

(c) *If*

$$\liminf_{\tau\to\infty} \frac{f(\tau)}{\tau} < B, \qquad \limsup_{\tau\to\infty} \frac{f(\tau)}{\tau} > \frac{A}{M}, \tag{3.8}$$

then (1.1) *has a sequence of solutions* u_k *with* $|u_k| \to \infty$ *as* $k \to \infty$.

(d) *If*

$$\liminf_{\tau\to 0+} \frac{f(\tau)}{\tau} < B, \qquad \limsup_{\tau\to 0+} \frac{f(\tau)}{\tau} > \frac{A}{M}, \tag{3.9}$$

then (1.1) *has a sequence of solutions* u_k *with* $|u_k| \to 0$ *as* $k \to \infty$.

Proof. (a) Clearly the first inequality in (3.7) guarantees (3.5) for large enough $R_1 > 0$. Next from the second inequality in (3.7) it follows that (3.4) holds for every $R_0 > 0$ sufficiently small.

(b) Obviously the first limit condition implies that $f(0) = 0$. Thus $u = 0$ is a solution. The conclusion follows from Theorem 2.2.

(c) From (3.8) it follows that there are two increasing sequences $(R_0^i)_{i\ge 1}$, $(R_1^i)_{i\ge 1}$ tending to infinity, with $R_0^i < MC\|\phi\|R_1^i$, $|\mathcal{I}|R_1^i < R_0^{i+1}$,

$$\frac{f(MR_0^i)}{R_0^i} > A \quad \text{and} \quad \frac{f(R_1^i)}{R_1^i} < B. \tag{3.10}$$

The sets $K_{R_0^i R_1^i}$ are disjoint and Theorem 2.1 can be applied in each of them.

(d) From (3.9) it follows that there are two decreasing sequences $(R_0^i)_{i\ge 1}$, $(R_1^i)_{i\ge 1}$ tending to zero satisfying $R_0^i < MC\|\phi\|R_1^i$, $|\mathcal{I}|R_1^i < R_0^{i-1}$ and (3.10).□

References

[1] H. Amann, *Existence of multiple solutions for nonlinear elliptic boundary value problems*, Indiana Univ. Math. J. **21** (1972), 925–935.

[2] ———, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[3] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519–543.

[4] R. Avery, J. Henderson and D. O'Regan, *A dual of the compression-expansion fixed point theorems*, Fixed Point Theory and Applications **2007** (2007), Article ID 90715, 11 pages, doi:10.1155/2007/90715.

[5] C. Azizieh and P. Clement, *A priori estimates and continuation methods for positive solutions of p-Laplace equations*, J. Differential Equations **179** (2002), 213–245.

[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[7] H. Brezis and L. Oswald, *Remarks on sublinear elliptic equations*, IMA Preprint Series, no. 112, 1984.

[8] D.G. de Figueiredo, *Positive Solutions of Semilinear Elliptic Equations*, Lecture Notes in Mathematics, vol. 957, Springer, Berlin, 1982.

[9] L.H. Erbe, S. Hu and H. Wang, *Multiple positive solutions of some boundary value problems*, J. Math. Anal. Appl. **184** (1994), 640–648.

[10] L.H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743–748.

[11] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag, Berlin, 1983.

[12] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, 2003.

[13] J. Henderson and H. Wang, *Positive solutions for nonlinear eigenvalue problems*, J. Math. Anal. Appl. **208** (1997), 252–259.

[14] P. Hess, *Multiple solutions of some asymptotically linear elliptic boundary value problems*, Lecture Notes in Mathematics **703** (1979), 145–151.

[15] J. Jost, *Partial Differential Equations*, Springer, New York, 2007.

[16] M. Kassmann, *Harnack inequalities: an introduction*, Boundary Value Problems **2007**, Article ID 81415, 21 pages, doi:10.1155/2007/81415.

[17] M.A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[18] K. Lan and J.R.L. Webb, *Positive solutions of semilinear differential equations with singularities*, J. Differential Equations **148** (1998), 407–421.

[19] R.W. Leggett and L.R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28** (1979), 673–688.

[20] W.-C. Lian, F.-H. Wong and C.-C. Yeh, *On the existence of positive solutions of nonlinear second order differential equations*, Proc. Amer. Math. Soc. **124** (1996), 1117–1126.

[21] P.L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review **24** (1982), 441–467.

[22] M. Meehan and D. O'Regan, *Positive L^p solutions of Hammerstein integral equations*, Arch. Math. **76** (2001), 366–376.

[23] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **15** (1961), 577–591.

[24] P. Omari and F. Zanolin, *An elliptic problem with arbitrarily small positive solutions*, Electron. J. Differential Equations **5** (2000), 301–308.

[25] D. O'Regan and R. Precup, *Theorems of Leray–Schauder Type and Applications*, Taylor and Francis, London, 2002.

[26] ———, *Compression-expansion fixed point theorem in two norms and applications*, J. Math. Anal. Appl. **309** (2005), 383–391.

[27] D. O'Regan and H. Wang, *Positive periodic solutions of systems of second order ordinary differential equations*, Positivity **10** (2006), 285–298.

[28] R. Precup, *Compression-expansion fixed point theorems in two norms*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity **3** (2005), 157–163.

[29] ———, *Positive solutions of semi-linear elliptic problems via Krasnosel'skiĭ type theorems in cones and Harnack's inequality*, Mathematical Analysis and Applications, AIP Conf. Proc., vol. 835, Amer. Inst. Phys., Melville, NY, 2006, pp. 125–132.

[30] ———, *Critical point theorems in cones and multiple positive solutions of elliptic problems*, Nonlinear Anal. **75** (2012), 834–851.

[31] H. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **109** (1994), 1–7.

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