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GLOBAL EXISTENCE, ASYMPTOTIC BEHAVIOR AND BLOW-UP OF SOLUTIONS FOR A VISCOELASTIC EQUATION WITH STRONG DAMPING AND NONLINEAR SOURCE

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Abstract. This paper deals with the initial-boundary value problem for the viscoelastic equation with strong damping and nonlinear source. Firstly, we prove the local existence of solutions by using the Faedo-Galerkin approximation method and Contraction Mapping Theorem. By virtue of the potential well theory and convexity technique, we then prove that if the initial data enter into the stable set, then the solution globally exists and decays to zero with a polynomial rate, and if the initial data enter into the unstable set, then the solution blows up in a finite time. Moreover, we show that the solution decays to zero with an exponential or polynomial rate depending on the decay rate of the relaxation function.

1. Introduction

In this paper, we are concerned with the following initial-boundary value problem for nonlinear viscoelastic equations with strong damping and nonlinear

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source

$$
\begin{aligned}
& u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds \\
(1.1) \qquad & \qquad\qquad - \omega\Delta u_t + \mu u_t = |u|^{r-2}u, \quad (x,t) \in \Omega \times (0,\infty), \\
& u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \\
& u(x,t) = 0, \qquad\qquad\qquad\qquad\qquad (x,t) \in \partial\Omega \times [0,\infty),
\end{aligned}
$$

where Ω is an open bounded Lipschitz subset of $\mathbb{R}^n$ $(n \geq 1)$. The relaxation function g is a positive and uniformly decaying function. The functions u_0 and u_1 are given initial data satisfying

$$
u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \tag{1.2}
$$

$$
\omega > 0, \qquad \mu > -\omega\lambda_1, \tag{1.3}
$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions, and

$$
2 < r \leq 2^* = 2n/(n-2) \quad \text{if } n \geq 3, \qquad 2 < r < \infty \quad \text{if } n = 1, 2. \tag{1.4}
$$

This type of problem arises in viscoelasticity and in system governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann's model (see [1], [4]).

In the absence of the viscoelastic term (that is, if $g = 0$), the equation in (1.1) reduces to the damped wave equation

$$
u_{tt} - \Delta u - \omega\Delta u_t + \mu u_t = |u|^{r-2}u, \quad (x,t) \in \Omega \times (0,\infty). \tag{1.5}
$$

This equation has been extensively studied by many mathematicians. Sufficient conditions for the existence of nonglobal as well as global solutions in the nondissipative case ($\omega = \mu = 0$) were obtained by L. E. Payne and D. H. Sattinger [23] where they introduced the concepts of stable and unstable sets. For the case $\omega = 0$ and $\mu > 0$, R. Ikehata [9] gave a characterization of the global solutions decaying to zero. He also gave a characterization of the existence of blow-up solutions, but restricted to sufficiently small coefficient μ. The later result was improved by J. A. Esquivel-Avila in [7] for any positive coefficient μ. Recently, F. Gazzola and M. Squassina [8] considered equation (1.5) where strong damping term was included ($\omega > 0$) and proved global existence and finite time blow-up of solutions.

As far as the viscoelastic term $\int_0^t g(t-s)\Delta u(x,s)\,ds$ is concerned, problems related to (1.1) have also been extensively studied and many results concerning existence, decay and blow up have been established. For example, the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau + a(x)u_t + |u|^\gamma u = 0, \quad (x,t) \in \Omega \times (0,\infty)$$

has been considered by Cavalcanti et al [5], where $a\colon \Omega \to \mathbb{R}^+$ is a function, which may be null on a part of Ω. Under the condition that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometric restrictions and $-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t)$, $t \geq 0$ to guarantee $\|g\|_{L^1((0,\infty))}$ is small enough, they proved an exponential rate of decay. S. Berrimi and S. A. Messaoudi [2] improved Cavalcanti's result by introducing a different functional, which allowed them to weaken the conditions on both a and g. In the related work, M. M. Cavalcanti et al. [4] studied solutions of

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau - \gamma \Delta u_t = 0, \quad (x,t) \in \Omega \times (0,\infty),$$

for $\rho > 0$ and proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This result was later extended by Messaoudi and Tatar [20] to a situation where a source term is competing with the strong damping mechanism and the one induced by the viscosity. More recently, S. A. Messaoudi and N.-E. Tatar [21] considered

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(x,s)\, ds = b|u|^{p-2}u, \quad (x,t) \in \Omega \times (0,\infty),$$

where $b > 0$, $p > 2$ are constants, in which the source term competes with the dissipation induced by the viscoelastic term only. By introducing a new functional and using potential well method, they obtained the global existence of solutions and the uniform decay of the energy when the initial data are in some stable set.

Concerning blow-up results, S. A. Messaoudi [18] investigated the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau + a u_t|u_t|^{m-2} = b|u|^{r-2}u, \quad (x,t) \in \Omega \times (0,\infty).$$

He proved that any weak solution with negative initial energy blows up in finite time if $r > m$ and

$$\int_0^\infty g(s)\, ds \leq \frac{r-2}{r-2+1/r}$$

while continue to exist for any initial data in the appropriate space if $m \geq r$. This latter result was improved by the same author in [19] for positive initial energy under suitable conditions on g, m and r. For results of same nature, we refer the reader to H. A. Levine and J. Serrin [14], H. A. Levine and S. R. Park [13], W. J. Liu [16], W. J. Liu and M. X. Wang [17], F. Q. Sun and M. X. Wang [24], [25], E. Vitillaro [26].

Motivated by [3], [4], [8], [9], [18], we intend to study the global existence, asymptotic behavior and blow-up of solutions to the initial boundary value problem (1.1) in the present work. The main difficulties we encounter here arise from the simultaneous appearance of the viscoelastic term, the strong damping term, as well as the nonlinear source term. We will show that if the initial data is in the stable set, the solution is global and decaying to zero when only the nonpositivity of g' is needed. Moreover, we will show that the solution decays to zero with an exponential or polynomial rate depending on the decay rate of the relaxation function g. On the contrary, if the initial data is in the unstable set, the solution will blow up in a finite time. To achieve our goal, we use the potential well theory, Faedo–Galerkin approximation, perturbed energy method and concavity technique.

This paper is organized as follows. In the next section we present some assumptions, notations and state our main results. In Section 3 we prove the local existence of solutions for problem (1.1). In Section 4 we discuss the global existence and asymptotic behavior of solutions. In Section 5 we show the exponential or polynomial decay of the solution. A finite time blow-up result for initial data in the unstable set is obtained in the last section.

2. Preliminaries and main results

In this section we present some assumptions, notations and state our main results. We first make the following assumptions.

(G1) $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function satisfying

$$1 - \int_0^\infty g(s)\,ds = l > 0.$$

(G2) There exists a positive constant ξ such that

$$g'(t) \le -\xi g^p(t), \quad t \ge 0,\ 1 \le p < \frac{3}{2}.$$

Remark 2.1. (G1) is necessary to guarantee the hyperbolicity of the system (1.1).

We use the standard Lebesgue space $L^r(\Omega)$ $(1 \le r \le \infty)$ and Sobolev space $H_0^1(\Omega)$. We denote by $\|u\|_r$ the $L^r(\Omega)$ norm and by $\|\nabla \cdot\|_2$ the Dirichlet norm in $H_0^1(\Omega)$. Moreover, for later use we denote by $\langle\cdot\,,\,\cdot\rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. For all $v, w \in H_0^1(\Omega)$, we put

$$(v,w)_* = \omega \int_\Omega \nabla v \cdot \nabla w\,dx + \mu \int_\Omega vw\,dx, \quad \|v\|_* = (v,v)_*^{1/2}.$$

By (1.3), $\|\cdot\|_*$ is an equivalent norm over $H_0^1(\Omega)$. For r satisfies (1.4), we assume that B is the optimal constant of the embedding inequality

$$\|u\|_r \leq B\|\nabla u\|_2, \quad u \in H_0^1(\Omega).$$

We introduce the following functionals as in [3], [20], [21]:

$$I(t) := I(u(t)) = \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \|u(t)\|_r^r, \tag{2.1}$$

$$\begin{aligned} J(t) &:= J(u(t)) \\ &= \frac{1}{2}\left[\left(1 - \int_0^t g(s)\,ds\right)|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right] - \frac{1}{r}\|u(t)\|_r^r, \end{aligned} \tag{2.2}$$

$$E(t) := E(u(t), u_t(t)) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2, \tag{2.3}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-\tau)||\nabla u(t) - \nabla u(\tau)||_2^2\,d\tau \geq 0.$$

To state our main results we introduce the definition of a weak solution to problem (1.1).

DEFINITION 2.2. A weak solution to the initial-boundary value problem (1.1) over $[0, T]$ is a function

$$u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], H^{-1}(\Omega))$$

with $u_t \in L^2([0, T], H_0^1(\Omega))$ such that $u(x, 0) = u_0 \in H_0^1(\Omega)$, $u_t(x, 0) = u_1 \in L^2(\Omega)$ and

$$\begin{aligned} \langle u_{tt}(t), \phi\rangle + \int_\Omega \nabla u(t) \cdot \nabla\phi\,dx - \int_\Omega \nabla\phi \cdot \int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ + \omega\int_\Omega \nabla u_t(t) \cdot \nabla\phi\,dx + \mu\int_\Omega u_t(t)\phi\,dx = \int_\Omega |u(t)|^{r-2}u(t)\phi\,dx \end{aligned}$$

for all test function $\phi \in H_0^1(\Omega)$ and for almost all $t \in [0, T]$.

We are now in a position to state our main results. Our first theorem establishes the existence and uniqueness of a local weak solution to problem (1.1).

THEOREM 2.3. *Assume that* (1.2)–(1.4) *and* (G1) *hold. Then problem* (1.1) *has a unique weak solution over* $[0, T]$ *for some* T *small enough. If*

$$T_{\max} = \sup\{T > 0 : u = u(t) \text{ exists on } [0, T]\} < \infty,$$

then $\|u(t)\|_r \to \infty$ *as* $t \nearrow T_{\max}$.

THEOREM 2.4. *Assume that* (1.2)–(1.4) *and* (G1) *hold and let* u *be the unique local solution to problem (1.1). In addition, assume that* u_0, u_1 *satisfy*

$$E(0) < d_1 := \frac{r-2}{2r}\left(\frac{l}{B^2}\right)^{r/(r-2)}, \tag{2.4}$$

$$I(0) > 0. \tag{2.5}$$

Then the solution is global and bounded. Moreover, there exists a constant $M > 0$ *such that*

$$E(t) \le \frac{M}{1+t}, \quad \text{for all } t \in [0,\infty), \tag{2.6}$$

$$\lim_{t\to\infty}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)\right] = \lim_{t\to\infty}\|u_t(t)\|_2 = 0. \tag{2.7}$$

Observe that in the previous theorem, only the non-positivity of g' was needed and μ can be negetive. We have established the polynomial decay result (see (2.6)) for the global solutions of problem (1.1). In the next section, we shall prove further decay result by strengthening the conditions on g and μ.

THEOREM 2.5. *Assume that* (1.2)–(1.4), (G1) *and* (G2) *hold,* $\mu > 0$ *and let* u *be the global solution to problem* (1.1). *Then, for each* $t_0 > 0$*, there exist positive constants* K *and* k *such that the solution of* (1.1) *satisfies, for all* $t \ge t_0$,

$$E(t) \le \begin{cases} Ke^{-kt} & \text{if } p = 1, \\ K(1+t)^{-1/(p-1)} & \text{if } 1 < p < 3/2. \end{cases} \tag{2.8}$$

THEOREM 2.6. (1.2)–(1.4), (G1) *hold and let* u *be the unique local solution to problem* (1.1). *For any fixed* $\delta < 1$*, assume that* u_0, u_1 *satisfy*

$$E(0) = \delta d_1, \tag{2.9}$$

$$I(0) < 0. \tag{2.10}$$

Suppose that

$$\int_0^\infty g(s)\,ds \le \frac{r-2}{r-2+1/[(1-\widehat{\delta})^2 r + 2\delta(1-\widehat{\delta})]} \tag{2.11}$$

where $\widehat{\delta} = \max\{0,\delta\}$, *and suppose further that* $\int_\Omega u_0 u_1\,dx > 0$ *for* $0 \le E(0) < d_1$, *then* $T_{\max} < \infty$.

For $t \ge 0$, we define $d(t) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \sup_{\lambda\ge 0} J(\lambda u)$. Then, the following lemma shows that d_1 is the lower bound of $d(t)$.

Lemma 2.7. *For $t \geq 0$, we have $0 < d_1 \leq d(t) \leq d_2(u) = \sup_{\lambda\geq 0} J(\lambda u)$, where*

$$d_2(u) = \frac{r-2}{2r}\left[\frac{(1-\int_0^t g(s)\,ds)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)}{\|u(t)\|_r^2}\right]^{r/(r-2)}.$$

Proof. Since

$$J(\lambda u) = \frac{\lambda^2}{2}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)\right] - \frac{\lambda^r}{r}\|u\|_r^r,$$

we get

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)\right] - \lambda^{r-1}\|u\|_r^r$$

and

$$\frac{d^2}{d\lambda^2}J(\lambda u) = \left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)\right] - (r-1)\lambda^{r-2}\|u\|_r^r.$$

Let $\frac{d}{d\lambda}J(\lambda u) = 0$, which implies

$$\overline{\lambda}_1 = 0, \quad \overline{\lambda}_2 = \left[\frac{(1-\int_0^t g(s)\,ds)\|\nabla u\|_2^2 + (g\circ\nabla u)}{\|u\|_r^r}\right]^{1/(r-2)}.$$

An elementary calculation shows

$$\frac{d^2}{d\lambda^2}J(\overline{\lambda}_1 u) > 0 \quad \text{and} \quad \frac{d^2}{d\lambda^2}J(\overline{\lambda}_2 u) < 0.$$

So, we have

$$\begin{aligned}\sup_{\lambda\geq 0} J(\lambda u) = J(\overline{\lambda}_2 u) &= \frac{r-2}{2r}\left[\frac{(1-\int_0^t g(s)\,ds)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)}{\|u(t)\|_r^2}\right]^{r/(r-2)} \\ &\geq \frac{r-2}{2r}\left(\frac{l}{B^2}\right)^{r/(r-2)} > 0.\end{aligned}$$

We conclude the result. □

3. Local existence of solutions

To obtain the local existence of solutions for problem (1.1), we consider firstly a related linear problem. Then, we use the well-known Contraction Mapping Theorem to prove the existence of solutions to the nonlinear problem. Let us now consider, for v given, the linear problem

$$\begin{aligned}&u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds \\ &\qquad\qquad - \omega\Delta u_t + \mu u_t = |v|^{r-2}v, && (x,t)\in\Omega\times(0,T], \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x\in\Omega, \\ &u(x,t) = 0, && (x,t)\in\partial\Omega\times[0,T],\end{aligned} \tag{3.1}$$

where u is the sought solution.

For a given $T>0$, consider the space $\mathcal{H}=C([0,T],H_0^1(\Omega))\cap C^1([0,T],L^2(\Omega))$ endowed with the norm

$$\|u\|_{\mathcal{H}}^2=\max_{t\in[0,T]}(\|u_t(t)\|_2^2+l\|\nabla u(t)\|_2^2).$$

We first prove the following

Lemma 3.1. *Assume that* (1.3), (1.4) *and* (G1) *hold. For every $T>0$, every $v\in\mathcal{H}$ and every initial data u_0,u_1 satisfying* (1.2), *there exists a unique*

$$u\in\mathcal{H}\cap C^2([0,T],H^{-1}(\Omega))\quad\text{such that } u_t\in L^2([0,T],H_0^1(\Omega))$$

which solves the linear problem (3.1).

Proof. We use the Faedo–Galerkin approximation method. For every $h\geq 1$, let $W_h=\operatorname{Span}\{w_1,\dots,w_h\}$, where $\{w_j\}$ is the orthogonal complete system of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ such that $\|w_j\|_2=1$ for all j. Then $\{w_j\}$ is orthogonal and complete in $L^2(\Omega)$ and in $H_0^1(\Omega)$; denote by $\{\lambda_j\}$ the related eigenvalues repeated according to their multiplicity. Let

$$u_0^h=\sum_{j=1}^h\bigg(\int_\Omega\nabla u_0\cdot\nabla w_j\,dx\bigg)w_j\quad\text{and}\quad u_1^h=\sum_{j=1}^h\bigg(\int_\Omega u_1w_j\,dx\bigg)w_j, \tag{3.2}$$

so that $u_0^h\in W_h$, $u_1^h\in W_h$, $u_0^h\to u_0$ in $H_0^1(\Omega)$ and $u_1^h\to u_1$ in $L^2(\Omega)$ as $h\to\infty$. We seek approximate solutions $u_h(t)$ to the problem (3.1) of the form

$$u_h(t)=\sum_{j=1}^h\gamma_j^h(t)w_j,\quad h=1,2,\dots \tag{3.3}$$

where the coefficients $\gamma_j^h(t)$ satisfy $\gamma_j^h(t)=\int_\Omega u_h(t)w_h dx$ with

$$\begin{gathered}\int_\Omega\bigg[\ddot u_h(t)-\Delta u_h(t)+\int_0^t g(t-s)\Delta u_h(s)\,ds\\ -\omega\Delta\dot u_h(t)+\mu\dot u_h(t)-|v(t)|^{r-2}v(t)\bigg]\eta\,dx=0,\\ u_h(0)=u_0^h,\quad \dot u_h(0)=u_1^h,\end{gathered} \tag{3.4}$$

for every $\eta\in W_h$ and $t\geq 0$. For $j=1,\cdots,h$, taking $\eta=w_j$ in (3.4) yields the following Cauchy problem for a linear ordinary differential equation with unknown γ_j^h.

$$\begin{gathered}\ddot\gamma_j^h(t)+(\omega\lambda_j+\mu)\dot\gamma_j^h(t)+\lambda_j\gamma_j^h(t)\\ -\lambda_j\int_0^t g(t-s)\gamma_j^h(s)\,ds=\int_\Omega|v(t)|^{r-2}v(t)w_j\,dx,\\ \gamma_j^h(0)=\int_\Omega u_0w_j\,dx,\quad\dot\gamma_j^h(0)=\int_\Omega u_1w_j\,dx.\end{gathered} \tag{3.5}$$

For all j, the Cauchy problem (3.5) yields a unique local solution $\gamma_j^h \in C^2[0,t_m]$. In turn, this gives a unique $u_h(t)$ in an interval $[0,t_m]$ defined by (3.3) and satisfying (3.4). In particular, (3.3) implies that $\dot{u}_h(t) \in H_0^1(\Omega)$ for every $t \in [0,t_m]$ so that Sobolev inequality entails

$$\|\dot{u}_h(t)\|_{2^*} \le C\|\nabla \dot{u}_h(t)\|_2, \quad \text{for all } t \in [0,t_m]. \tag{3.6}$$

In the next step, we obtain the a priori estimate for the solution $u_h(t)$ so that it can be extended to the whole interval $[0,T]$ according to the extension theorem.

Step 1. (a priori estimate) Taking $\eta = \dot{u}_h(t)$ in (3.4), we have

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u_h(t)\|_2^2 + \|\dot{u}_h(t)\|_2^2) + 2\|\dot{u}_h(t)\|_*^2 \\ &\quad - 2\int_0^t g(t-s)\int_\Omega \nabla \dot{u}_h(t)\cdot\nabla u_h(s)\,dx\,ds = 2\int_\Omega |v(t)|^{r-2}v(t)\dot{u}_h(t)\,dx. \end{aligned} \tag{3.7}$$

For the last term on the left hand side of (3.7) we have

$$\begin{aligned} &2\int_0^t g(t-s)\int_\Omega \nabla \dot{u}_h(t)\cdot\nabla u_h(s)\,dx\,ds \\ &= 2\int_0^t g(t-s)\int_\Omega \nabla \dot{u}_h(t)\cdot[\nabla u_h(s)-\nabla u_h(t)]\,dx\,ds \\ &\quad + 2\int_0^t g(t-s)\int_\Omega \nabla \dot{u}_h(t)\cdot\nabla u_h(t)\,dx\,ds \\ &= -\int_0^t g(t-s)\frac{d}{dt}\int_\Omega |\nabla u_h(s)-\nabla u_h(t)|^2\,dx\,ds \\ &\quad + \int_0^t g(s)\left(\frac{d}{dt}\int_\Omega |\nabla u_h(t)|^2\,dx\right)ds \\ &= -\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega |\nabla u_h(s)-\nabla u_h(t)|^2\,dx\,ds\right] \\ &\quad + \frac{d}{dt}\left[\int_0^t g(s)\left(\int_\Omega |\nabla u_h(t)|^2\,dx\right)ds\right] \\ &\quad + \int_0^t g'(t-s)\int_\Omega |\nabla u_h(s)-\nabla u_h(t)|^2\,dx\,ds - g(t)\int_\Omega |\nabla u_h(t)|^2\,dx. \end{aligned} \tag{3.8}$$

Inserting (3.8) into (3.7) and integrating over $[0,t]\subset[0,T]$, we obtain

$$\begin{aligned} &\left(1-\int_0^t g(s)\,ds\right)\|\nabla u_h(t)\|_2^2 + \|\dot{u}_h(t)\|_2^2 + 2\int_0^t \|\dot{u}_h(s)\|_*^2\,ds \\ &\quad - \int_0^t (g'\circ\nabla u_h)(s)\,ds + (g\circ\nabla u_h)(t) + \int_0^t\int_\Omega g(s)|\nabla u_h(s)|^2\,dx\,ds \\ &\qquad = \|\nabla u_0^h\|_2^2 + \|u_1^h\|_2^2 + 2\int_0^t\int_\Omega |v(s)|^{r-2}v(s)\dot{u}_h(s)\,dx\,ds, \end{aligned} \tag{3.9}$$

for every $h \geq 1$. We estimate the last term in the right-hand side of (3.9) thanks to Hölder, Sobolev and Young inequalities (recall $r \leq 2^*$, (3.6) and $v \in C([0,T], H_0^1(\Omega))$:

$$(3.10)\quad 2\int_0^t \int_\Omega |v(s)|^{r-2} v(s) \dot{u}_h(s)\,dx\,ds \leq 2\int_0^t \|v(s)\|_r^{r-1} \|\dot{u}_h(s)\|_r\,ds$$
$$\leq C_1 \int_0^t \|v(s)\|_r^{2(r-1)}\,ds + \int_0^t \|\dot{u}_h(s)\|_r^2\,ds \leq CT + \int_0^t \|\dot{u}_h(s)\|_*^2\,ds.$$

By using (G1), (3.10) and the fact that

$$-\int_0^t (g' \circ \nabla u_h)(s)\,ds + (g \circ \nabla u_h)(t) + \int_0^t \int_\Omega g(s)|\nabla u_h(s)|^2\,dx\,ds \geq 0,$$

estimate (3.9) yields

$$l\|\nabla u_h(t)\|_2^2 + \|\dot{u}_h(t)\|_2^2 + \int_0^t \|\dot{u}_h(s)\|_*^2\,ds \leq \|\nabla u_0^h\|_2^2 + \|u_1^h\|_2^2 + CT.$$

Taking the convergence in (3.2) into consideration, we arrive at

$$\|u_h\|_{\mathcal{H}}^2 + \int_0^T \|\dot{u}_h(s)\|_*^2\,ds \leq C_T$$

for every $h \geq 1$, where $C_T > 0$ is independent of h. By this uniform estimate and (3.4), we have:

$\{u_h\}$ is bounded in $L^\infty([0,T], H_0^1(\Omega))$,
$\{\dot{u}_h\}$ is bounded in $L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega))$,
$\{\ddot{u}_h\}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$.

Step 2. (passage to the limit) From the above boundedness, we can extract a subsequence from $\{u_h\}$, still denoted by $\{u_h\}$, such that

$u_h \to u$ weakly star in $L^\infty([0,T], H_0^1(\Omega))$,
$\dot{u}_h \to \dot{u}$ weakly in $L^2([0,T], H_0^1(\Omega))$,
$\dot{u}_h \to \dot{u}$ weakly star in $L^\infty([0,T], L^2(\Omega))$,
$\ddot{u}_h \to \ddot{u}$ weakly in $L^2([0,T], H^{-1}(\Omega))$.

Considering that the imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and using the Aubin-Lions compactness lemma (see J. L. Lions [15]), we deduce that

$u_h \to u$ strongly in $L^2(Q_T)$,
$\dot{u}_h \to \dot{u}$ strongly in $L^2(Q_T)$, where $Q_T := \Omega \times [0,T]$.

The proof of uniqueness is easy and we omit it here. □

Now, we are ready to prove the local existence result.

PROOF OF THEOREM 2.3. For $M > 0$ large and $T > 0$, we define a class of functions as

$$\mathcal{M}_T = \{u \in \mathcal{H} : u(0) = u_0,\ u_t(0) = u_1 \text{ and } \|u\|_{\mathcal{H}} \leq M\}.$$

By the trace theorem, $\mathcal{M}_T$ is nonempty if M is large enough, We also define the map f from $\mathcal{M}_T$ into $\mathcal{H}$ by $u := f(v)$, where u is the unique solution of the linear problem (3.1). We would like to show that f is a contraction map satisfying $f(\mathcal{M}_T) \subseteq \mathcal{M}_T$ for a suitable $T > 0$.

Firstly, we show $f(\mathcal{M}_T) \subseteq \mathcal{M}_T$. For this, we make use of the energy identity (3.9) which yields

$$\begin{aligned}
l\|\nabla u(t)\|_2^2 &+ \|u_t(t)\|_2^2 + 2\int_0^t \|u_t(s)\|_*^2\,ds \\
&\le \|\nabla u_0\|_2^2 + \|u_1\|_2^2 + 2\int_0^t\int_\Omega |v(s)|^{r-2}v(s)u_t(s)\,dx\,ds \\
&\le \|\nabla u_0\|_2^2 + \|u_1\|_2^2 + C\int_0^T \|v(s)\|_{2^*}^{r-1}\|u_t(s)\|_{2^*}\,ds \\
&\le \|\nabla u_0\|_2^2 + \|u_1\|_2^2 + C\int_0^T \|v(s)\|_{*}^{r-1}\|u_t(s)\|_{2^*}\,ds \\
&\le \|\nabla u_0\|_2^2 + \|u_1\|_2^2 + CTM^{2(r-1)} + 2\int_0^T \|u_t(s)\|_{2^*}^2\,ds,
\end{aligned}$$

for all $t \in (0, T]$. This leads to

$$\|u\|_H^2 \le \|\nabla u_0\|_2^2 + \|u_1\|_2^2 + CTM^{2(r-1)}.$$

By choosing M large enough so that $\|\nabla u_0\|_2^2 + \|u_1\|_2^2 \le M^2/2$ then choose T sufficiently small so that $CTM^{2(r-1)} \le M^2/2$, we get $\|u\|_{\mathcal{H}} \le M$, which shows that f maps $\mathcal{M}_T$ into itself.

Next, we verify that f is a contraction. To this end we take v_1 and v_2 in $\mathcal{M}_T$ and set $U = u_1 - u_2$, $V = v_1 - v_2$, where $u_1 = f(v_1)$ and $u_2 = f(v_2)$. It's straightforward to verify that U satisfies

$$\begin{aligned}
&U_{tt} - \Delta U + \int_0^t g(t-s)\Delta U(s)\,ds \\
&\qquad - \omega\Delta U_t + \mu U_t = |v_1|^{r-2}v_1 - |v_2|^{r-2}v_2, && (x,t) \in \Omega \times (0,T], \\
&U(x,0) = U_t(x,0) = 0, && x \in \Omega, \\
&U(x,t) = 0, && (x,t) \in \partial\Omega \times [0,T].
\end{aligned} \tag{3.11}$$

By multiplying the differential equation in (3.11) by U_t and integrating over $\Omega \times (0,t)$, we arrive at

$$\begin{aligned}
&\left(1 - \int_0^t g(s)\,ds\right)\|\nabla U(t)\|_2^2 + \|U_t(t)\|_2^2 + 2\int_0^t \|U_t(s)\|_*^2\,ds \\
&\quad - \int_0^t (g' \circ \nabla U)(s)\,ds + (g \circ \nabla U)(t) + \int_0^t\int_\Omega g(s)|\nabla U(s)|^2\,dx\,ds \\
&\qquad = 2\int_0^t\int_\Omega (|v_1(s)|^{r-2}v_1(s) - |v_2(s)|^{r-2}v_2(s))U_t(s)\,dx\,ds.
\end{aligned}$$

Similar to the discussion in [8], we can get

$$\|f(v_1) - f(v_2)\|_{\mathcal{H}}^2 = \|U\|_{\mathcal{H}}^2 \le CTM^{2(r-2)} \|v_1 - v_2\|_{\mathcal{H}}^2. \tag{3.12}$$

By choosing T so small that $CTM^{2(r-2)} < 1$, (3.12) shows that f is a contraction. The Contraction Mapping Principle then guarantee the existence of a unique u satisfying $u = f(u)$. Obviously it is a (weak) solution of (1.1).

The last statement of Theorem 2.3 can be proved similarly as in [8], therefore we omit it. □

4. Global existence and decay of solutions

In this section, for the initial data in the stable set, we show that the solution is global and decaying to zero. We need the following lemmas.

LEMMA 4.1. *Assume that* (1.2)–(1.4) *and* (G1) *hold. If u is the solution of* (1.1), *then*

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 - \|u_t(t)\|_*^2 \le -\|u_t(t)\|_*^2 \le 0, \tag{4.1}$$

for almost every $t \in [0, T_{\max})$.

PROOF. In view of (G1), multiplying the differential equation in (1.1) by u_t and integrating by parts over Ω, we obtain the result. □

LEMMA 4.2. *Under the same assumptions as in Theorem* 2.4, *one has $I(t) > 0$ for all $t \in [0, T_{\max})$.*

PROOF. Since $I(0) > 0$, there exists a $T_* \le T_{\max}$ such that $I(t) \ge 0$ for all $t \in [0, T_*)$. This implies that

$$\begin{aligned} J(t) = \frac{r-2}{2r}\bigg[\bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u\|_2^2 + (g \circ \nabla u)\bigg] + \frac{1}{r} I(t) \\ \ge \frac{r-2}{2r}\bigg[\bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u\|_2^2 + (g \circ \nabla u)\bigg], \end{aligned}$$

for all $t \in [0, T_*)$. Thus, by (G1), (4.2) and Lemma 4.1, we have

$$l\|\nabla u\|_2^2 \le \bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u\|_2^2 \le \frac{2r}{r-2} J(t) \le \frac{2r}{r-2} E(t) \le \frac{2r}{r-2} E(0),$$

for all $t \in [0, T_*)$. This combines with the Sobolev imbedding, (G1) and (2.4), implies that

$$\begin{aligned} \|u\|_r^r \le B^r \|\nabla u\|_2^r \le \frac{B^r}{l} \|\nabla u\|_2^{r-2} l \|\nabla u\|_2^2 \\ \le \frac{B^r}{l}\bigg(\frac{2r}{(r-2)l} E(0)\bigg)^{(r-2)/2} l\|\nabla u\|_2^2 < \bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u\|_2^2, \end{aligned}$$

for all $t \in [0, T_*)$. Therefore, $I(t) > 0$, for all $t \in [0, T_*)$. By repeating this procedure and using the fact that

$$\lim_{t \to T_*} \frac{B^r}{l} \left(\frac{2r}{(r-2)l} E(t) \right)^{(r-2)/2} \le \frac{B^r}{l} \left(\frac{2r}{(r-2)l} E(0) \right)^{(r-2)/2} < 1,$$

T_* is extended to $T_{\max}$. □

LEMMA 4.3. *Under the same assumptions as in Theorem 2.4, the solution is global in time. Moreover, there is a real number $M > 0$, such that*

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \le M, \tag{4.3}$$

$$\int_0^t \|u_t(s)\|_2^2 \, ds \le \frac{d_1}{\lambda_1 \omega + \mu}, \tag{4.4}$$

for all $t \in [0, \infty)$.

PROOF. From (4.1), (4.2) and (2.3), we get

$$\begin{aligned} E(0) &\ge E(t) + \int_0^t \|u_t(s)\|_*^2 \, ds = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 + \int_0^t \|u_t(s)\|_*^2 \, ds \\ &\ge \frac{r-2}{2r} \left[\left(1 - \int_0^t g(s)\, ds \right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \right] \\ &\quad + \frac{1}{2}\|u_t(t)\|_2^2 + \int_0^t \|u_t(s)\|_*^2 \, ds \\ &\ge \frac{r-2}{2r} l \|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 + \int_0^t \|u_t(s)\|_*^2 \, ds. \end{aligned} \tag{4.5}$$

Therefore,

$$l\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \le CE(0), \qquad \int_0^t \|u_t(s)\|_*^2 \, ds \le d_1,$$

where C is a positive constant, which depends only on r and l. Hence, $u(t)$ is bounded in $\mathcal{H}$ for all $t > 0$ and, by Theorem 2.3, global. Moreover, by Poincáre inequality, we get

$$\int_0^t \|u_t(s)\|_2^2 \, ds \le \frac{d_1}{\lambda_1 \omega + \mu}, \quad \text{for all } t \in [0, \infty).$$ □

LEMMA 4.4. *Under the same assumptions as in Theorem 2.4, there is a constant $M > 0$, such that, for all $t \in [0, \infty)$,*

$$\int_0^t I(u(s))\, ds \le M, \quad \int_0^t \left[\left(1 - \int_0^\tau g(s)\, ds \right) \|\nabla u(\tau)\|_2^2 + (g \circ \nabla u)(\tau) \right] d\tau \le M.$$

PROOF. Multiplying the differential equation in (1.1) by u and integrating over Ω, using integration by parts and (2.1), we have

$$\frac{d}{dt}\langle u_t(t), u(t) \rangle - \|u_t(t)\|_2^2 + I(u(t)) + \frac{1}{2}\frac{d}{dt}\|u(t)\|_*^2 = 0.$$

By integrating the above inequality it follows that

$$\int_0^t I(u(s))\,ds+\frac{1}{2}\|u(t)\|_*^2 \le \frac{1}{2}\|u_0\|_*^2+\int_0^t\|u_t(s)\|_2^2\,ds+\|u_1\|_2\|u_0\|_2+\|u_t(t)\|_2\|u(t)\|_2.$$

Therefore, from (4.4) and Ponicáre inequality, we have

$$\int_0^t I(u(s))\,ds \le \frac{1}{2}\|u_0\|_*^2+\frac{d_1}{\lambda_1\omega+\mu}+\|u_1\|_2\|u_0\|_2+C\|u_t(t)\|_2\|\nabla u(t)\|_2. \tag{4.6}$$

Since Lemma 4.2 implies

$$\eta\bigg[\bigg(1-\int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\bigg]\le I(u(t)),$$

where

$$\eta=1-\frac{B^r}{l}\bigg(\frac{2r}{(r-2)l}E(0)\bigg)^{(r-2)/2}>0,$$

it follows from (4.6) and (4.3) that

$$\eta\int_0^t\bigg[\bigg(1-\int_0^\tau g(s)\,ds\bigg)\|\nabla u(\tau)\|_2^2+(g\circ\nabla u)(\tau)\bigg]\,d\tau\le\int_0^t I(u(\tau))\,d\tau\le M,$$

with a constant $M>0$. □

Proof of Theorem 2.4. We only need to prove (2.6) and (2.7). In fact, note that the following inequality holds

$$\frac{d}{dt}((1+t)E(t))\le E(t).$$

By integrating this inequality over $[0,t]$, we have

$$(1+t)E(t)\le E(0)+\frac{1}{2}\int_0^t\|u_t(s)\|_2^2\,ds+\int_0^t J(s)\,ds.$$

Since

$$rJ(t)=I(t)+\frac{r-2}{2}\bigg[\bigg(1-\int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\bigg],$$

the above inequality gives

$$(1+t)E(t)\le E(0)+\frac{1}{2}\int_0^t\|u_t(s)\|_2^2\,ds+\frac{1}{r}\int_0^t I(u(s))\,ds + \frac{r-2}{2r}\int_0^t\bigg[\bigg(1-\int_0^\tau g(s)\,ds\bigg)\|\nabla u(\tau)\|_2^2+(g\circ\nabla u)(\tau)\bigg]\,d\tau.$$

Finally, by using Lemma 4.3 and Lemma 4.4 we obtain (2.6).

From Lemma 4.2 and (4.2), we get $J(u(t))\ge 0$ and $I(u(t))>0$. (2.6) implies $\lim_{t\to\infty}E(t)=0$. Therefore, we have

$$\|u_t(t)\|\to 0\quad\text{and}\quad J(u(t))\to 0,\quad\text{as } t\to\infty. \qquad \square$$

5. Exponential or polynomial decay of solutions

In this section we state and prove our exponential or polynomial decay result which depend on the rate of the decay of the relaxation function g. For this purpose, we adapt the method of S. Berrimi and S. A. Messaoudi in [3]. Since the damping terms are included in our problem (1.1), we use the following "modified" functional

$$F(t) := E(t) + \varepsilon_1 G(t) + \varepsilon_2 H(t), \tag{5.1}$$

where ε_1 and ε_2 are positive constants and

$$\begin{aligned} G(t) &:= \int_\Omega (u(t)u_t(t) + \frac{\mu}{2}|u(t)|^2)\, dx, \\ H(t) &:= -\int_\Omega (\mu u(t) + u_t(t)) \int_0^t g(t-s)(u(t)-u(s))\, ds\, dx. \end{aligned}$$

LEMMA 5.1. *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t) \tag{5.2}$$

holds for two positive constants α_1 and α_2.

PROOF. Straightforward computations, in addition with Poincáre inequality and (4.5), lead to

$$\begin{aligned} F(t) &\le E(t) + \frac{\varepsilon_1}{2}\int_\Omega |u_t(t)|^2\, dx + \frac{(\mu+1)\varepsilon_1}{2}\int_\Omega |u(t)|^2\, dx \\ &\quad + \frac{\varepsilon_2}{2}\int_\Omega (\mu u(t) - u_t(t))^2\, dx \\ &\quad + \frac{\varepsilon_2}{2}\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\, ds\right)^2 dx \\ &\le E(t) + \left(\frac{\varepsilon_1}{2} + \varepsilon_2\right)\int_\Omega |u_t(t)|^2\, dx \\ &\quad + \lambda_1^{-1}\left(\frac{(\mu+1)\varepsilon_1}{2} + \mu^2\varepsilon_2\right)\int_\Omega |\nabla u(t)|^2\, dx \\ &\quad + \frac{\varepsilon_2}{2}\lambda_1^{-1}(1-l)(g\circ\nabla u)(t) \le \frac{1}{\alpha_1}E(t) \end{aligned}$$

and

$$\begin{aligned} F(t) &\ge E(t) - \left(\frac{\varepsilon_1}{2} + \varepsilon_2\right)\int_\Omega |u_t(t)|^2\, dx \\ &\quad - \lambda_1^{-1}\left(\frac{(\mu+1)\varepsilon_1}{2} + \mu^2\varepsilon_2\right)\int_\Omega |\nabla u(t)|^2\, dx - \frac{\varepsilon_2}{2}\lambda_1^{-1}(1-l)(g\circ\nabla u)(t) \\ &\ge \left(\frac{1}{2} - \frac{\varepsilon_1}{2} - \varepsilon_2\right)\int_\Omega |u_t(t)|^2\, dx \end{aligned}$$

$$+\left[l-\lambda_1^{-1}\left(\frac{(\mu+1)\varepsilon_1}{2}+\mu^2\varepsilon_2\right)\right]\int_\Omega |\nabla u(t)|^2\,dx$$
$$+\left[1-\frac{\varepsilon_2}{2}\lambda_1^{-1}(1-l)\right](g\circ\nabla u)(t)\geq \frac{1}{\alpha_2}E(t),$$

for ε_1 and ε_2 small enough. □

LEMMA 5.2. *Assume that* (1.2)–(1.4), (G1) *and* (G2) *hold. Then the functional $G(t)$ satisfies, along the solution of* (1.1),

$$G'(t)\leq \|u_t(t)\|_2^2-\frac{l}{2}\|\nabla u(t)\|_2^2-\omega\|\nabla u_t(t)\|_2^2 + \frac{1}{2l}\left[\int_0^t g^{2-p}(s)\,ds\right](g^p\circ\nabla u)(t)+\|u(t)\|_r^r. \tag{5.3}$$

PROOF. By using the differential equation in (1.1), we easily see that

$$\begin{aligned} G'(t) &= \int_\Omega [u(t)u_{tt}(t)+u_t^2(t)+\mu u(t)u_t(t)]\,dx \\ &= \|u_t(t)\|_2^2-\|\nabla u(t)\|_2^2-\omega\|\nabla u_t(t)\|_2^2 \\ &\quad+\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx+\|u(t)\|_r^r. \end{aligned}\tag{5.4}$$

Young's inequality, (G1) and direct calculation (see [3]) yield

$$\begin{aligned} \int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx &\leq \frac{1}{2}\|\nabla u(t)\|_2^2 \\ &\quad+\frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)\,ds\right)^2 dx \\ &\leq \frac{1}{2}\|\nabla u(t)\|_2^2+\frac{1}{2}(1+\eta)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)|\,ds\right)^2 dx \\ &\quad+\frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|\,ds\right)^2 dx \\ &\leq \frac{1}{2}\|\nabla u(t)\|_2^2+\frac{1}{2}(1+\eta)(1-l)^2\|\nabla u(t)\|_2^2 \\ &\quad+\frac{1}{2}\left(1+\frac{1}{\eta}\right)\left[\int_0^t g^{2-p}(s)\,ds\right] \\ &\quad\cdot\int_\Omega\int_0^t g^p(t-s)|\nabla u(s)-\nabla u(t)|^2\,ds\,dx, \end{aligned}\tag{5.5}$$

for any $\eta>0$. By choosing $\eta=l/(1-l)$ and combining (5.4)–(5.5), we arrive at (5.3). □

LEMMA 5.3. *Assume that* (1.2)–(1.4), (G1) *and* (G2) *hold. Then the functional* $H(t)$ *satisfies,*

$$
\begin{aligned}
(5.6)\quad H'(t) \le &\left\{\delta\left[1+2(1-l)^2+B^{2(r-1)}\left(\frac{2rE(0)}{(r-2)l}\right)^{r-2}\right]+\frac{\mu(1-l)}{2\lambda_1}\right\}\|\nabla u(t)\|_2^2\\
&+\left\{2\delta+\frac{1}{4\delta}\left(2+\frac{1}{\lambda_1}+\omega\right)\right\}\left[\int_0^t g^{2-p}(s)\,ds\right](g^p\circ\nabla u)(t)\\
&+\frac{(1+\mu)g(0)}{4\delta\lambda_1}(-g'\circ\nabla u)(t)+\omega\delta\|\nabla u_t(t)\|_2^2\\
&+\left\{\delta-\left(\int_0^t g(s)\,ds\right)\left(1-\frac{\mu}{2}\right)\right\}\|u_t(t)\|_2^2,
\end{aligned}
$$

for any $\delta > 0$.

PROOF. Direct calculations give

$$
\begin{aligned}
(5.7)\quad H'(t) = &-\int_\Omega(\mu u_t(t)+u_{tt}(t))\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega(\mu u(t)+u_t(t))\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\\
&-\left(\int_0^t g(s)\,ds\right)\int_\Omega(\mu u(t)+u_t(t))u_t(t)\,dx\\
=&\int_\Omega\nabla u(t)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)dx\\
&-\int_\Omega\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)\\
&\cdot\left(\int_0^t g(t-s)\nabla u(s)\,ds\right)dx\\
&-\int_\Omega|u(t)|^{r-2}u(t)\left(\int_0^t g(t-s)(u(t)-u(s))\,ds\right)dx\\
&-\int_\Omega u_t(t)\left(\int_0^t g'(t-s)(u(t)-u(s))\,ds\right)dx\\
&-\left(\int_0^t g(s)\,ds\right)\|u_t(t)\|_2^2\\
&+\omega\int_\Omega\nabla u_t(t)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)dx\\
&-\mu\int_\Omega u(t)\left(\int_0^t g'(t-s)(u(t)-u(s))\,ds\right)dx\\
&-\mu\left(\int_0^t g(s)\,ds\right)\int_\Omega u(t)u_t(t)\,dx.
\end{aligned}
$$

We now estimate the right side of (5.7). For $\delta > 0$, similar as in [3], we have the estimates of the first to the fourth terms:

$$\int_\Omega \nabla u(t) \cdot \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \leq \delta \|\nabla u(t)\|_2^2 + \frac{1}{4\delta} \left[\int_0^t g^{2-p}(s)\, ds \right] (g^p \circ \nabla u)(t), \tag{5.8}$$

$$\int_\Omega \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) \left(\int_0^t g(t-s) \nabla u(s)\, ds \right) dx \leq 2\delta(1-l)^2 \|\nabla u(t)\|_2^2 + \left(2\delta + \frac{1}{4\delta} \right) \left[\int_0^t g^{2-p}(s)\, ds \right] (g^p \circ \nabla u)(t), \tag{5.9}$$

$$\begin{aligned} &\int_\Omega |u(t)|^{r-2} u(t) \left(\int_0^t g(t-s)(u(t) - u(s))\, ds \right) dx \\ &\leq \delta \|u(t)\|_{2(r-1)}^{2(r-1)} + \frac{1}{4\delta\lambda_1} \left[\int_0^t g^{2-p}(s)\, ds \right] (g^p \circ \nabla u)(t) \\ &\leq \delta B^{2(r-1)} \left(\frac{2rE(0)}{(r-2)l} \right)^{r-2} \|\nabla u(t)\|_2^2 \\ &\quad + \frac{1}{4\delta\lambda_1} \left[\int_0^t g^{2-p}(s)\, ds \right] (g^p \circ \nabla u)(t), \end{aligned} \tag{5.10}$$

and

$$\int_\Omega u_t(t) \left(\int_0^t g'(t-s)(u(t) - u(s))\, ds \right) dx \leq \delta \|u_t(t)\|_2^2 + \frac{g(0)}{4\delta\lambda_1} (-g' \circ \nabla u)(t). \tag{5.11}$$

For the sixth term

$$\omega \int_\Omega \nabla u_t(t) \cdot \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \leq \omega\delta \|\nabla u_t(t)\|_2^2 + \frac{\omega}{4\delta} \left[\int_0^t g^{2-p}(s)\, ds \right] (g^p \circ \nabla u)(t). \tag{5.12}$$

The seventh term

$$\mu \int_\Omega u(t) \left(\int_0^t g'(t-s)(u(t) - u(s))\, ds \right) dx \leq \frac{\mu\delta}{\lambda_1} \|\nabla u(t)\|_2^2 + \frac{\mu g(0)}{4\delta\lambda_1} (-g' \circ \nabla u)(t). \tag{5.13}$$

The last term

$$
\begin{aligned}
(5.14)\quad &\mu\left(\int_0^t g(s)\,ds\right)\int_\Omega u(t)u_t(t)\,dx \\
&\qquad\le \frac{\mu}{2\lambda_1}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2+\frac{\mu}{2}\left(\int_0^t g(s)\,ds\right)\|u_t(t)\|_2^2.
\end{aligned}
$$

A combination of (5.7)–(5.14) yields (5.6). □

PROOF OF THEOREM 2.5. Since g is continuous and $g(0)>0$, then for any $t_0>0$ we have

$$
(5.15)\qquad \int_0^t g(s)\,ds \ge \int_0^{t_0} g(s)\,ds := g_0>0, \quad \text{for all } t\ge t_0.
$$

By using (4.1), (5.1), (5.3), (5.6) and (5.15), we obtain

$$
\begin{aligned}
F'(t)\le &-\left\{\mu+\varepsilon_2\left[\left(1-\frac{\mu}{2}\right)g_0-\delta\right]-\varepsilon_1\right\}\|u_t(t)\|_2^2+\varepsilon_1\|u(t)\|_r^r \\
&-\left\{\frac{\varepsilon_1 l}{2}-\varepsilon_2\left[\delta\left(1+2(1-l)^2+\frac{\mu}{\lambda_1}\right.\right.\right. \\
&\left.\left.\left.+B^{2(r-1)}\left(\frac{2rE(0)}{(r-2)l}\right)^{r-2}\right)+\frac{\mu(1-l)}{2\lambda_1}\right]\right\}\|\nabla u(t)\|_2^2 \\
&+\left\{\frac{\varepsilon_1}{2l}+\varepsilon_2\left[2\delta+\frac{1}{4\delta}\left(2+\frac{1}{\lambda_1}+\omega\right)\right]\right\} \\
&\cdot\left[\int_0^t g^{2-p}(s)\,ds\right](g^p\circ\nabla u)(t) \\
&+\left(\frac{1}{2}-\varepsilon_2\frac{(1+\mu)g(0)}{4\delta\lambda_1}\right)(g'\circ\nabla u)(t)-\omega(1+\varepsilon_1-\delta\varepsilon_2)\|\nabla u_t(t)\|_2^2.
\end{aligned}
$$

At this point we fix $\delta>0$ and choose ε_1 and ε_2 satisfying

$$
\begin{aligned}
(5.16)\quad k_2=\frac{\varepsilon_1 l}{2}-\varepsilon_2\left[\delta\left(1+2(1-l)^2+\frac{\mu}{\lambda_1}\right.\right.& \\
\left.\left.+B^{2(r-1)}\left(\frac{2rE(0)}{(r-2)l}\right)^{r-2}\right)+\frac{\mu(1-l)}{2\lambda_1}\right]&>0.
\end{aligned}
$$

We then pick ε_1 and ε_2 so small that (5.2) and (5.16) remain valid and

$$
\begin{aligned}
k_1 &= \mu+\varepsilon_2\left[\left(1-\frac{\mu}{2}\right)g_0-\delta\right]-\varepsilon_1>0, \\
k_3 &= \left(\frac{1}{2}-\varepsilon_2\frac{(1+\mu)g(0)}{4\delta\lambda_1}\right) \\
&\quad -\frac{1}{\xi}\left\{\frac{\varepsilon_1}{2l}+\varepsilon_2\left[2\delta+\frac{1}{4\delta}\left(1+\frac{1}{\lambda_1}+\omega\right)\right]\right\}\left[\int_0^\infty g^{2-p}(s)\,ds\right]\ge 0, \\
k_4 &= \omega(1+\varepsilon_1-\delta\varepsilon_2)\ge 0.
\end{aligned}
$$

Therefore, using the assumption $g'(t) \le -\xi g^p(t)$ in (G2), we have, for some $\beta > 0$,

$$F'(t) \le -\beta[\|u_t(t)\|_2^2 - \|u(t)\|_r^r + \|\nabla u(t)\|_2^2 + (g^p \circ \nabla u)(t)] \tag{5.17}$$

for all $t \ge t_0$.

Case 1. $p = 1$.

By virtue of the choice of ε_1, ε_2 and δ, estimate (5.17) yields, for some constant $\alpha > 0$,

$$F'(t) \le -\alpha E(t), \quad \text{for all } t \ge t_0. \tag{5.18}$$

Hence, with the help of the left hand side inequality in (5.2) and (5.18), we find

$$F'(t) \le -\alpha\alpha_1 F(t), \quad \text{for all } t \ge t_0. \tag{5.19}$$

A simple integration of (5.19) over (t_0, t) leads to

$$F(t) \le F(t_0)e^{\alpha\alpha_1 t_0}e^{-\alpha\alpha_1 t}, \quad \text{for all } t \ge t_0.$$

Therefore, (2.8) is established by virtue of (5.2) again.

Case 2. $1 < p < 3/2$.

Similar as the discussion in [3], we can obtain

$$(g^p \circ \nabla u) \ge C_1(g \circ \nabla u)^p, \tag{5.20}$$

for some constant $C_1 > 0$. Consequently, a combination of (5.20) and (5.17) yields

$$F'(t) \le -C_2[\|u_t(t)\|_2^2 - \|u(t)\|_r^r + \|\nabla u(t)\|_2^2 + (g \circ \nabla u)^p], \quad \text{for all } t \ge t_0,$$

for some constant $C_2 > 0$. On the other hand, since $E(t)$ is uniformly bounded (by $E(0)$), we have

$$E^p(t) \le C_3[E^{p-1}(0)(\|u_t(t)\|_2^2 - \|u(t)\|_r^r + \|\nabla u(t)\|_2^2) + (g \circ \nabla u)^p],$$

for all $t \ge t_0$ and some constant $C_3 > 0$. Combining the last two inequalities and (5.2), we obtain

$$F'(t) \le -C_4 F^p(t), \quad \text{for all } t \ge t_0, \tag{5.21}$$

for some constant $C_4 > 0$. A simple integration of (5.21) over (t_0, t) gives

$$F(t) \le K(1 + t)^{-1/(p-1)}, \quad \text{for all } t \ge t_0.$$

Therefore, (2.8) is obtained by virtue of (5.2). □

6. Finite time blow-up of solutions

In this section, we prove a finite time blow-up result for initial data in the unstable set. We need the following lemmas.

LEMMA 6.1. *Under the same assumptions as in Theorem* 2.6, *one has* $I(t) < 0$ *and*

$$d_1 < \frac{r-2}{2r}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)\right] < \frac{r-2}{2r}\|u\|_r^r, \tag{6.1}$$

for all $t \in [0, T_{\max})$.

PROOF. By Lemma 4.1 and (2.9), we have $E(t) \le \delta d_1$ for all $t \in [0, T_{\max})$. Furthermore, we can obtain $I(t) < 0$ for all $t \in [0, T_{\max})$. In fact, if it is not true, then there exists some $t^* \in [0, T_{\max})$ such that $I(t^*) = 0$. Thus $I(t) < 0$ for all $0 \le t < t^*$, i.e.

$$\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u) < \|u\|_r^r, \quad 0 \le t < t^*. \tag{6.2}$$

By the proof of Lemma 2.7, we get

$$d < \frac{r-2}{2r}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)\right], \quad 0 \le t < t^*. \tag{6.3}$$

It follows from (6.2) and (6.3) that

$$\|u(t)\|_r^r > \frac{2r}{r-2}d_1 > 0, \quad 0 \le t < t^*.$$

By the continuity of $t \mapsto \|u(t)\|_r^r$, we get $u(t^*) \ne 0$. By Lemma 2.7 and (2.2), we obtain

$$d_1 \le \frac{r-2}{2r}\|u(t^*)\|_r^r = J(u(t^*)),$$

which contradicts to $J(u(t^*)) \le E(t^*) < d_1$. By using Lemma 2.7 again, we obtain (6.1). □

LEMMA 6.3 ([11], [12]). *Let* $L(t)$ *be a positive, twice differentiable function, which satisfies, for* $t > 0$, *the inequality*

$$L(t)L''(t) - (1+\alpha)L'(t)^2 \ge 0$$

with some $\alpha > 0$. *If* $L(0) > 0$ *and* $L'(0) > 0$, *then there exists a time* $T^* \le L(0)/[\alpha L'(0)]$ *such that* $\lim_{t\to T^{*-}} L(t) = \infty$.

PROOF OF THEOREM 2.6. We prove this result by adapting and modifying the method used in [8] where a characterization of the blow-up solutions is proved for problem (1.1) without the viscoelastic term: $\int_0^t g(t-s)\Delta u(x,s)\,ds$. However, the authors have ignored the condition "$L'(0) > 0$" (see Lemma 6.3) when exploiting the convexity technique. Therefore, we should modify the definition of the auxiliary function here.

Assume by contradiction that the solution u is global. Then, we consider $L: [0,T] \to \mathbb{R}_+$ defined by

$$L(t) = \|u(t)\|_2^2 + \int_0^t \|u(\tau)\|_*^2 \, d\tau + (T-t)\|u_0\|_*^2 + b(t+T_0)^2,$$

where T and T_0 are positive constants to be chosen later, $b > 0$ if $E(0) < 0$ and $b = 0$ if $E(0) \geq 0$. Then $L(t) > 0$ for all $t \in [0,T]$. Furthermore,

$$\begin{aligned} L'(t) &= 2\int_\Omega u(t)u_t(t)\, dx + \|u(t)\|_*^2 - \|u_0\|_*^2 + 2b(t+T_0) \\ &= 2\int_\Omega u(t)u_t(t)\, dx + 2\int_0^t (u(\tau), u_\tau(\tau))_* \, d\tau + 2b(t+T_0) \end{aligned} \tag{6.4}$$

and, consequently,

$$L''(t) = 2\langle u_{tt}(t), u(t)\rangle + 2\|u_t(t)\|_2^2 + 2(u(t), u_t(t))_* + 2b,$$

for almost every $t \in [0,T]$. Testing the equation in (1.1) with u and plugging the result into the expression of L'' we obtain

$$\begin{aligned} L''(t) = 2\bigg[&\|u_t(t)\|_2^2 - \bigg(1 - \int_0^t g(s)\, ds\bigg)\|\nabla u(t)\|_2^2 \\ &- \int_\Omega \nabla u(t) \cdot \bigg(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds\bigg)\, dx + \|u(t)\|_r^r + b\bigg], \end{aligned}$$

for almost every $t \in [0,T]$. Therefore, we get

$$\begin{aligned} &L(t)L''(t) - \frac{r+2}{4}L'(t)^2 = L(t)L''(t) \\ &+ (r+2)\bigg[\eta(t) - \Big(L(t) - (T-t)\|u_0\|_*^2\Big)\bigg(\|u_t(t)\|_2^2 + \int_0^t \|u_\tau(\tau)\|_*^2\, d\tau + b\bigg)\bigg], \end{aligned}$$

where $\eta: [0,T] \to \mathbb{R}_+$ is the function defined by

$$\begin{aligned} \eta(t) = &\bigg(\|u(t)\|_2^2 + \int_0^t \|u(\tau)\|_*^2\, d\tau + b(t+T_0)^2\bigg)\bigg(\|u_t(t)\|_2^2 + \int_0^t \|u_t(\tau)\|_*^2\, d\tau + b\bigg) \\ &- \bigg(\int_\Omega u(t)u_t(t)\, dx + \int_0^t (u(t), u_t(t))_*\, d\tau + b(t+T_0)\bigg)^2 \geq 0. \end{aligned}$$

As a consequence, we read the following differential inequality

$$L(t)L''(t) - \frac{r+2}{4}L'(t)^2 \geq L(t)\xi(t), \tag{6.5}$$

for almost every $t \in [0,T]$, where $\xi\colon [0,T] \to \mathbb{R}_+$ is the map defined by

$$\begin{aligned}\xi(t) = &-r\|u_t(t)\|_2^2 - 2\bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|_2^2 - (r+2)\int_0^t \|u_t(\tau)\|_*^2\,d\tau \\ &- 2\int_\Omega \nabla u(t)\cdot\bigg(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\bigg)\,dx + 2\|u(t)\|_r^r - rb \\ = &-2rE(t) + r(g\circ\nabla u)(t) + (r-2)\bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|_2^2 \\ &- 2\int_\Omega \nabla u(t)\cdot\bigg(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\bigg)\,dx \\ &- (r+2)\int_0^t \|u_t(\tau)\|_*^2\,d\tau - rb.\end{aligned}$$

By (4.1), for all $t \in [0,T]$, we may also write

$$\begin{aligned}\xi(t) \geq &-2rE(0) + r(g\circ\nabla u)(t) + (r-2)\bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|_2^2 \\ &- 2\int_\Omega \nabla u(t)\cdot\bigg(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\bigg)\,dx \\ &+ (r-2)\int_0^t \|u_t(\tau)\|_*^2\,d\tau - rb.\end{aligned} \tag{6.6}$$

By using the Young's inequality, we get

$$\begin{aligned}2\int_\Omega &\nabla u(t)\cdot\bigg(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\bigg)\,dx \\ &\leq \frac{1}{\varepsilon}\int_0^t g(s)\,ds\|\nabla u(t)\|_2^2 + \varepsilon(g\circ\nabla u)(t),\end{aligned} \tag{6.7}$$

for any $\varepsilon > 0$. Substitute (6.7) for the fourth term of the righthand side of (6.6), we obtain

$$\begin{aligned}\xi(t) \geq &-2rE(0) + \bigg[(r-2) - \bigg(r - 2 + \frac{1}{\varepsilon}\bigg)\int_0^t g(s)\,ds\bigg]\|\nabla u(t)\|_2^2 \\ &+ (r-\varepsilon)(g\circ\nabla u)(t) + (r-2)\int_0^t \|u_t(\tau)\|_*^2\,d\tau - rb.\end{aligned} \tag{6.8}$$

If $\delta < 0$, i.e. $E(0) < 0$, we choose $\varepsilon = r$ in (6.8) and b small enough such that $b \leq -2E(0)$. Then, by (2.11), we have

$$\begin{aligned}\xi(t) \geq &\bigg[(r-2) - \bigg(r - 2 + \frac{1}{r}\bigg)\int_0^t g(s)\,ds\bigg]\|\nabla u(t)\|_2^2 \\ &+ (r-2)\int_0^t \|u_t(\tau)\|_*^2\,d\tau + r(-2E(0) - b)\end{aligned} \tag{6.9}$$

$$\geq \left[(r-2) - \left(r - 2 + \frac{1}{r} \right) \int_0^t g(s)\, ds \right] \|\nabla u(t)\|_2^2 + (r-2) \int_0^t \|u_t(\tau)\|_*^2 \, d\tau \geq 0.$$

If $0 \leq \delta < 1$, i.e. $0 \leq E(0) = \delta d_1 < d_1$, we choose $\varepsilon = (1-\delta)r + 2\delta$ in (6.8). Then, we get

$$\xi(t) \geq -2rE(0) + \left[(r-2) - \left(r - 2 + \frac{1}{(1-\delta)r + 2\delta} \right) \int_0^t g(s)\, ds \right] \|\nabla u(t)\|_2^2 + \delta(r-2)(g \circ \nabla u)(t) + (r-2) \int_0^t \|u_t(\tau)\|_*^2 \, d\tau.$$

By (2.11), we have

$$(r-2) - \left(r - 2 + \frac{1}{(1-\delta)r + 2\delta} \right) \int_0^t g(s)\, ds \geq \delta(r-2) \left(1 - \int_0^t g(s)\, ds \right),$$

and therefore, by (6.1), we get

$$\xi(t) \geq -2rE(0) + \delta(r-2) \left[\left(1 - \int_0^t g(s)\, ds \right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \right] + (r-2) \int_0^t \|u_t(\tau)\|_*^2 \, d\tau \geq 2r(\delta d_1 - E(0)) + (r-2) \int_0^t \|u_t(\tau)\|_*^2 \, d\tau \geq 0. \tag{6.10}$$

Therefore, by (6.5), (6.9) and (6.10), we obtain

$$L(t)L''(t) - \frac{r+2}{4} L'(t)^2 \geq 0,$$

for almost every $t \in [0, T]$. By (6.4), if $E(0) < 0$, we then choose T_0 sufficiently large such that $L'(0) = 2\int_\Omega u_0 u_1 \, dx + 2bT_0 > 0$. If $0 \leq E(0) < d_1$, the condition $\int_\Omega u_0 u_1 \, dx > 0$ also ensure that $L'(0) > 0$. As $(r+2)/4 > 1$, letting $\alpha = (r-2)/4$, by using the concavity argument, we get $\lim_{t \to T^{*-}} L(t) = \infty$, which implies that $\lim_{t \to T^{*-}} \|\nabla u(t)\|_2^2 = \infty$. $\square$

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