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WEAK SOLUTIONS OF QUASILINEAR ELLIPTIC SYSTEMS VIA THE COHOMOLOGICAL INDEX

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Abstract. In this paper we study a class of quasilinear elliptic systems of the type

$$\begin{cases} -\mathrm{div}(a_1(x,\nabla u_1,\nabla u_2)) = f_1(x,u_1,u_2) & \text{in } \Omega, \\ -\mathrm{div}(a_2(x,\nabla u_1,\nabla u_2)) = f_2(x,u_1,u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω bounded domain in $\mathbb{R}^N$. We assume that $A\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $F\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ exist such that $a = (a_1, a_2) = \nabla A$ satisfies the so called Leray–Lions conditions and $f_1 = \partial F/\partial u_1$, $f_2 = \partial F/\partial u_2$ are Carathéodory functions with *subcritical growth*.

The approach relies on variational methods and, in particular, on a cohomological local splitting which allows one to prove the existence of a nontrivial solution.

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1. Introduction

In this paper we investigate the existence of solutions for the quasilinear elliptic system with homogeneous Dirichlet boundary conditions

$$\text{(1.1)}\qquad \begin{cases} -\mathrm{div}(a_1(x,\nabla u_1,\nabla u_2)) = f_1(x,u_1,u_2) & \text{in } \Omega,\\ -\mathrm{div}(a_2(x,\nabla u_1,\nabla u_2)) = f_2(x,u_1,u_2) & \text{in } \Omega,\\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary $\partial\Omega$, $a_1, a_2\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and f_1, $f_2\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (i.e. measurable in $x \in \Omega$ for all $\xi = (\xi_1,\xi_2) \in \mathbb{R}^{2N}$, respectively $u = (u_1,u_2) \in \mathbb{R}^2$, and continuous in ξ, respectively u, for almost all $x \in \Omega$).

We assume that $a(x,\xi) = (a_1(x,\xi), a_2(x,\xi))$ satisfies the Leray–Lions conditions:

(A_1) (growth condition) there exist $p_j > 1$, $j = 1,2$, and $\alpha_1 > 0$ such that

$$|a_1(x,\xi)| \leq \alpha_1(|\xi_1|^{p_1-1} + |\xi_2|^{p_2/p_1'} + 1), \quad |a_2(x,\xi)| \leq \alpha_1(|\xi_1|^{p_1/p_2'} + |\xi_2|^{p_2-1} + 1),$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^{2N}$, where $1/p_j + 1/p_j' = 1$, $j = 1,2$;

(A_2) (coercivity condition) there exists $\alpha_2 > 0$ such that

$$a(x,\xi)\cdot\xi \geq \alpha_2(|\xi_1|^{p_1} + |\xi_2|^{p_2}) \quad \text{for a.a. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^{2N};$$

(A_3) (monotonicity condition)

$$[a(x,\xi) - a(x,\xi')]\cdot(\xi - \xi') > 0$$

for almost all $x \in \Omega$, all $\xi, \xi' \in \mathbb{R}^{2N}$ such that $\xi \neq \xi'$.

Furthermore, we suppose that there exist two Carathéodory functions $A\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi_1,\xi_2)$, and $F\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F = F(x,u_1,u_2)$, such that

$$a_1(x,\xi_1,\xi_2) = \nabla_{\xi_1} A(x,\xi_1,\xi_2), \quad a_2(x,\xi_1,\xi_2) = \nabla_{\xi_2} A(x,\xi_1,\xi_2),$$

hence, $a(x,\xi) = \nabla_\xi A(x,\xi)$, and

$$f_1(x,u_1,u_2) = \frac{\partial F}{\partial u_1}(x,u_1,u_2), \quad f_2(x,u_1,u_2) = \frac{\partial F}{\partial u_2}(x,u_1,u_2).$$

Thus, note that under suitable assumptions (1.1) is the Euler–Lagrange equation of the functional $\Phi\colon W \to \mathbb{R}$ defined as

$$\text{(1.2)}\qquad \Phi(u_1,u_2) = \int_\Omega A(x,\nabla u_1,\nabla u_2)\,dx - \int_\Omega F(x,u_1,u_2)\,dx,$$

$u = (u_1,u_2) \in W$, where $W = W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ is the product space of the usual Sobolev spaces. Whence, our problem reduces to the study of critical points of Φ in W and, if problem (1.1) admits the trivial solution $u_1 \equiv u_2 \equiv 0$, our aim is proving the existence of at least one nontrivial weak solution.

A model function which satisfies (A_1)–(A_3) is

$$\overline{A}(x,\xi) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2}, \quad \xi = (\xi_1,\xi_2) \in \mathbb{R}^{2N}, \tag{1.3}$$

with $p_j > 1$, $j = 1,2$, or more generally,

$$\widetilde{A}(x,\xi) = M(x)|\xi_1|^{p_1} + N(x)|\xi_2|^{p_2}, \quad \xi = (\xi_1,\xi_2) \in \mathbb{R}^{2N}, \tag{1.4}$$

where $M, N\colon \Omega \to [d_1, d_2]$, $0 < d_1 < d_2$, are measurable functions.

Considering $\overline{A}$ as in (1.3), problem (1.1) reduces to the corresponding simpler problem

$$\begin{cases} -\Delta_{p_1} u_1 = f_1(x,u_1,u_2) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 = f_2(x,u_1,u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Quasilinear elliptic operators such as those in problem (1.1), that satisfy the hypotheses (A_1)–(A_3), were first studied in [14] and are known in the literature as Leray–Lions operators. Since then, several existence results for problems involving such operators have been obtained via monotonicity methods and, in particular, by using a truncation technique (see [4], [5], [7] and references therein). More recently, an abstract cohomological local splitting theory has been developed in [15]–[17] and has been applied in order to obtain some existence results in the scalar case (see [9]). Here, our aim is to use a similar approach extending the known results to the quasilinear elliptic system (1.1).

On the other hand, many authors have studied problem (1.5) (see, e.g. [2], [6], [11], [12], [17], [20]), and have obtained several existence results under hypotheses of sublinear, superlinear, and resonant type on the nonlinearity F (for nonexistence results of nontrivial bounded solution see [20]). In [4], by assuming a hypothesis of monotonicity on F, a quasilinear elliptic system involving operators of Leray–Lions type similar to (1.1) was studied. Our results in this paper are motivated by theirs and use some ideas from [9], [17].

The rest of this paper is organized as follows. In Section 2, we introduce the complete set of hypotheses on A and F and their partial derivatives, then we describe the variational setting involving the functional Φ and point out some of its properties. In Section 3 we give some abstract results involving a cohomological local splitting. In Section 4 we prove that the functional Φ satisfies the Palais–Smale condition. Finally, in Section 5 we conclude the paper with the complete statements of our results and their proofs.

2. Hypotheses and variational setting

Throughout this paper, we use the following notations:

- meas$(\,\cdot\,)$ is the Lebesgue measure in $\mathbb{R}^N$;
- $|\cdot|$ is the standard norm on any Euclidean space (no ambiguity arises as the dimension of the vector is clear);
- $L^p(\Omega)$ is the space of Lebesgue–measurable functions $u\colon\Omega \to \mathbb{R}$ with finite norm $|u|_p = (\int_\Omega |u|^p\,dx)^{1/p}$ if $p \in [1,\infty[$;
- $L^\infty(\Omega)$ is the space of Lebesgue–measurable and essentially bounded functions $u\colon\Omega \to \mathbb{R}$ with norm $|u|_\infty = \operatorname{ess\,sup}_\Omega |u|$;
- $(W_0^{1,p}(\Omega), \|\cdot\|_p)$ is the classical Sobolev space with $\|u\|_p = |\nabla u|_p$ if $p \geq 1$.

From now on, assume that A and its partial derivatives a_1, a_2, satisfy the hypotheses (A_1)–(A_3). Hence, taking $p_1, p_2 \geq 1$ as in (A_1), let us denote $(W, \|\cdot\|)$ the product space

$$W = W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega), \quad \text{with } \|u\| = (\|u_1\|_{p_1}^2 + \|u_2\|_{p_2}^2)^{1/2},\ u = (u_1,u_2) \in W.$$

Since both $(W_0^{1,p_j}(\Omega), \|\cdot\|_{W_0^{1,p_j}(\Omega)})$, $j = 1,2$, are reflexive Banach spaces, so is $(W, \|\cdot\|)$. Moreover, denote with $(W', \|\cdot\|_{W'})$ its dual space.

According to classical results on this subject, let us introduce the following further conditions on A:

(A_4) there exist $0 < \alpha \leq \beta$ such that

$$\alpha\left(\frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2}\right) \leq A(x,\xi) \leq \beta\left(\frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2}\right),$$

for almost all $x \in \Omega$, all $\xi \in \mathbb{R}^{2N}$;

(A_5) there exist $\alpha_3, R, \mu > 0$ such that

$$\alpha_3 a(x,\xi)\cdot\xi \leq \mu A(x,\xi) - a(x,\xi)\cdot\xi$$

for almost all $x \in \Omega$ if $|\xi| \geq R$.

Remark 2.1. If conditions (A_1), (A_4) and (A_5) hold, then a constant $\alpha_4 \geq 0$ exists such that

$$a(x,\xi)\cdot\xi \leq \mu A(x,\xi) + \alpha_4 \quad \text{for a.a. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^{2N}. \tag{2.1}$$

Remark 2.2. Let us point out that hypothesis (A_5) is a kind of "coercivity condition" used in [3], [8], [9]. As we see in Section 4, this hypothesis is crucial to managing Palais–Smale sequences.

Example 2.3. Direct computations allow one to prove that $\widetilde{A}$ as in (1.4), hence $\overline{A}$ in (1.3), satisfies also conditions (A_4) and (A_5).

On the other hand, for the function F and its partial derivatives f_1, f_2, let us introduce the following conditions:

(F_1) $f_1(x,0,0) \equiv 0$, $f_2(x,0,0) \equiv 0$ in Ω, and, for simplicity, $F(x,0,0) \equiv 0$;

(F_2) there exist $s_j \in (1,p_j^*)$, $q_j \in (1,q_j^*)$, $j=1,2$, and $\sigma > 0$ such that

$$\begin{aligned} |f_1(x,u_1,u_2)| &\le \sigma(|u_1|^{s_1-1} + |u_2|^{q_1-1} + 1), \\ |f_2(x,u_1,u_2)| &\le \sigma(|u_1|^{q_2-1} + |u_2|^{s_2-1} + 1), \end{aligned}$$

for almost all $x \in \Omega$ and for all $(u_1,u_2) \in \mathbb{R}^2$, where we assume

$$p_j^* = \begin{cases} Np_j/(N-p_j) & \text{if } p_j < N, \\ \text{any real number strictly greater than 1} & \text{if } p_j \ge N, \end{cases} \qquad j=1,2,$$

and $q_1^* = 1 + p_2^*(p_1^*-1)/p_1^*$, $q_2^* = 1 + p_1^*(p_2^*-1)/p_2^*$;

(F_3) there exists $\theta \ge \mu$ such that $\theta > \max\{p_1,p_2\}$ and

$$0 < \theta F(x,u_1,u_2) \le f_1(x,u_1,u_2)u_1 + f_2(x,u_1,u_2)u_2,$$

for almost all $x \in \Omega$ if $|(u_1,u_2)| \ge R$, where μ and R are as in (A_5).

Remark 2.4. Without loss of generality, in (F_1) we can assume $F(x,0,0) \equiv 0$ almost everywhere in Ω. In fact, if $F(\,\cdot\,,0,0) \in L^1(\Omega)$, then we have just to add a constant to the functional Φ and its differential does not change.

Remark 2.5. By means of the Mean Value Theorem condition (F_2) and direct computations imply that

$$\begin{aligned} (2.2) \qquad |F(x,u_1,u_2) - F(x,0,0)| \le{}& \sigma(|u_1|^{s_1} + |u_2|^{s_2} + |u_1||u_2|^{q_1-1} \\ &+ |u_1|^{q_2-1}|u_2| + |u_1| + |u_2|) \end{aligned}$$

for almost all $x \in \Omega$ and for all $(u_1,u_2) \in \mathbb{R}^2$.

Remark 2.6. If hypotheses (F_1)–(F_3) hold, (2.2) and direct computations imply that there exists $C_0 \ge 0$ such that

$$(2.3) \qquad f_1(x,u_1,u_2)u_1 + f_2(x,u_1,u_2)u_2 \ge \theta F(x,u_1,u_2) - C_0$$

for almost all $x \in \Omega$, all $(u_1,u_2) \in \mathbb{R}^2$.

Remark 2.7. Note that hypothesis (F_3) can be weakened if we replace (A_2) with the stronger coerciveness condition $a_j(x,\xi)\cdot\xi_j \ge \alpha_5|\xi_j|^{p_j}$, $j=1,2$, for some $\alpha_5 > 0$.

Lemma 2.8. *If $F(\,\cdot\,,0,0) \equiv 0$ and* (2.2), (F_3) *hold, then there exist $C \ge 0$ and $h \in L^\infty(\Omega)$, $h(x) > 0$ for almost all $x \in \Omega$, such that*

$$(2.4) \qquad F(x,u_1,u_2) \ge h(x)|(u_1,u_2)|^\theta - C \quad \textit{for a.a. } x \in \Omega, \textit{ all } (u_1,u_2) \in \mathbb{R}^2.$$

Proof. Taking $(u_1,u_2) \in \mathbb{R}^2$, two cases may occur: either $|(u_1,u_2)| \ge R$ or $|(u_1,u_2)| < R$.

If $|(u_1, u_2)| \geq R$, denote

$$(\widetilde{u}_1, \widetilde{u}_2) = R\frac{(u_1, u_2)}{|(u_1, u_2)|} \quad \text{and} \quad \underline{t} = \left(\frac{|(u_1, u_2)|}{R}\right)^{\theta}.$$

In general, taking $t \geq 1$ condition (F_3) implies

$$\begin{aligned}\frac{d}{dt}(F(x, t^{1/\theta}\widetilde{u}_1, t^{1/\theta}\widetilde{u}_2)) &= \frac{1}{\theta t} f_1(x, t^{1/\theta}\widetilde{u}_1, t^{1/\theta}\widetilde{u}_2) t^{1/\theta}\widetilde{u}_1 \\ &\quad + \frac{1}{\theta t} f_2(x, t^{1/\theta}\widetilde{u}_1, t^{1/\theta}\widetilde{u}_2)\ t^{1/\theta}\widetilde{u}_2 \geq \frac{1}{t} F(x, t^{1/\theta}\widetilde{u}_1, t^{1/\theta}\widetilde{u}_2).\end{aligned}$$

Since $\underline{t} \geq 1$, by integrating we get $F(x, u_1, u_2) \geq \underline{t}F(x, \widetilde{u}_1, \widetilde{u}_2)$ which implies $F(x, u_1, u_2) \geq h(x)|(u_1, u_2)|^{\theta}$, with $h(x) = R^{-\theta} \min\{F(x, u_1, u_2) > 0 : |(u_1, u_2)| = R\}$ for almost all $x \in \Omega$, where $h \in L^{\infty}(\Omega)$ follows from (2.2).

On the other hand, from (2.2) and assuming

$$C = 2\left|\sup_{|(u_1,u_2)|\leq R} F(x, u_1, u_2)\right|_{\infty},$$

direct computations imply

$$F(x, u_1, u_2) \geq h(x)|(u_1, u_2)|^{\theta} - C \quad \text{for a.a. } x \in \Omega \text{ if } |(u_1, u_2)| < R.$$

Hence, the proof is complete. □

Remark 2.9. If conditions (F_1)–(F_3) hold, from (2.2) and (2.4) it follows

$$\min\{s_1, s_2\} \geq \theta > \max\{p_1, p_2\}.$$

As (A_4) implies $a(x, 0, 0) \equiv 0$, then from (F_1) it follows that problem (1.1) always admits the trivial solution $u_1 \equiv u_2 \equiv 0$. Thus, in order to obtain a non-trivial weak solution, we impose an additional condition on F involving a suitable "eigenvalue problem" (for a similar condition, see [6, pp. 312]).

More precisely, let $\mathcal{G}: \mathbb{R}^2 \to [0, \infty)$ be a given even C^1-function such that

$$\mathcal{G}(t^{1/p_1}u_1, t^{1/p_2}u_2) = t\mathcal{G}(u_1, u_2) \quad \text{for all } t \geq 0,\ (u_1, u_2) \in \mathbb{R}^2, \tag{2.5}$$

$$\mathcal{G}(u_1, u_2) \leq \alpha_6(|u_1|^{p_1} + |u_2|^{p_2}) \quad \text{for all } (u_1, u_2) \in \mathbb{R}^2, \text{ for some } \alpha_6 > 0, \tag{2.6}$$

and consider the related nonlinear "eigenvalue problem"

$$\begin{cases} -\Delta_{p_1} u_1 = \lambda \dfrac{\partial \mathcal{G}}{\partial u_1}(u_1, u_2) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 = \lambda \dfrac{\partial \mathcal{G}}{\partial u_2}(u_1, u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Remark 2.10. Examples of functions which satisfy conditions (2.5)–(2.6) are:

(a) $\mathcal{G}(u_1, u_2) = (c_1/p_1)|u_1|^{p_1} + (c_2/p_2)|u_2|^{p_2}$ for some c_1, $c_2 > 0$;
(b) $\mathcal{G}(u_1, u_2) = c_3|u_1|^{r_1}|u_2|^{r_2}$ for some $c_3 > 0$, where $r_1/p_1 + r_2/p_2 = 1$,

and the related eigenvalue problems are

$$\begin{cases} -\Delta_{p_1} u_1 = \lambda c_1 |u_1|^{p_1-2} u_1 & \text{in } \Omega, \\ -\Delta_{p_2} u_2 = \lambda c_2 |u_2|^{p_2-2} u_2 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{in case (a)},$$

$$\begin{cases} -\Delta_{p_1} u_1 = \lambda c_3 r_1 |u_1|^{r_1-2} u_1 |u_2|^{r_2} & \text{in } \Omega, \\ -\Delta_{p_2} u_2 = \lambda c_3 r_2 |u_1|^{r_1} |u_2|^{r_2-2} u_2 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{in case (b)}.$$

Via the cohomological index Perera et al. [17, Theorem 4.6] prove that (2.7) admits a sequence of eigenvalues $\lambda_k \nearrow \infty$ with some "good" properties (see Proposition 3.3).

Thus, we can consider the following assumption:

(F_4) there exist $\varrho > 0$, $k \geq 1$, and $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}$ with $\lambda_k < \underline{\lambda} \leq \overline{\lambda} < \lambda_{k+1}$ such that

$$\beta \underline{\lambda} \mathcal{G}(u_1, u_2) \leq F(x, u_1, u_2) \leq \alpha \overline{\lambda} \mathcal{G}(u_1, u_2),$$

for almost all $x \in \Omega$ if $|(u_1, u_2)| \leq \varrho$.

Lemma 2.11. *Assume that* (F_1), (F_4) *and* (2.2) *hold. Then, there exists* $C_1 > 0$ *such that*

$$\begin{aligned} (2.8) \quad -C_1(|u_1|^{p_1^*} + |u_2|^{p_2^*}) + \underline{\lambda}\beta \mathcal{G}(u_1, u_2) &\leq F(x, u_1, u_2) \\ &\leq \overline{\lambda}\alpha \mathcal{G}(u_1, u_2) + C_1(|u_1|^{p_1^*} + |u_2|^{p_2^*}) \end{aligned}$$

for almost all $x \in \Omega$, *all* $(u_1, u_2) \in \mathbb{R}^2$.

Proof. For almost all $x \in \Omega$, two cases may occur: either $|(u_1, u_2)| > \varrho$ or $|(u_1, u_2)| \leq \varrho$.

If $|(u_1, u_2)| > \varrho$, it is $|u_1| > \varrho/2$ or $|u_2| > \varrho/2$. Then, (2.2) and direct computations imply that

$$|F(x, u_1, u_2)| \leq \widetilde{\sigma}(|u_1|^{p_1^*} + |u_2|^{p_2^*})$$

for some $\widetilde{\sigma} > 0$. Hence, this last estimate and (2.6) imply (2.8) is satisfied for a suitable $C_1 > 0$.

On the contrary, if $|(u_1, u_2)| \leq \varrho$, (2.8) is a direct consequence of (F_4). □

Now, let us consider the functional $\Phi: W \to \mathbb{R}$ defined as in (1.2). Classical arguments allow one to prove the following regularity result.

Lemma 2.12. *The conditions* (A_1), (A_4) *and* (F_2) *imply* $\Phi \in C^1(W, \mathbb{R})$ *with differential operator*

$$d\Phi(u_1, u_2)[(\varphi_1, \varphi_2)] = \sum_{j=1}^{2} \int_\Omega (a_j(x, \nabla u_1, \nabla u_2) \cdot \nabla \varphi_j - f_j(x, u_1, u_2)\varphi_j) dx,$$

for all (u_1, u_2), $(\varphi_1, \varphi_2) \in W$. *Hence, the critical points of* Φ *in* W *are the weak solutions of* (1.1).

Finally, we conclude this section establishing some geometric properties of Φ that we use later. To this aim, denoting

$$\Phi^a = \{(u_1, u_2) \in W \colon \Phi(u_1, u_2) \le a\} \quad \text{for any } a \in \mathbb{R},$$

and reasoning as in [17, Lemma 10.20], the following lemma can be proved.

Lemma 2.13. *Under the hypotheses* (A_1), (A_4), (A_5) *and* (F_1)–(F_3), *there is an* $a_0 \le 0$ *such that for all* $a < a_0$, Φ^a *is homotopic to the unit sphere*

$$S_1 = \{u = (u_1, u_2) \in W : \|(u_1, u_2)\| = 1\}.$$

Proof. Fix $(u_1, u_2) \in S_1$. Taking $t > 0$, from (A_4) and Lemma 2.8 it follows that

$$\Phi(tu_1, tu_2) \le \beta \sum_{j=1}^{2} \frac{t^{p_j}}{p_j} \int_\Omega |\nabla u_j|^{p_j}\, dx - t^\theta \int_\Omega h(x)|(u_1, u_2)|^\theta\, dx + C\text{meas}(\Omega).$$

Since $\theta > \max\{p_1, p_2\}$ and $\int_\Omega h(x)|(u_1, u_2)|^\theta\, dx > 0$, we have

$$\Phi(tu_1, tu_2) \to -\infty \quad \text{as } t \to \infty. \tag{2.9}$$

On the other hand, using (2.1) and (2.3), with $\theta \ge \mu$, if $t > 0$ we obtain

$$\begin{aligned}
\frac{d}{dt}(\Phi(tu_1, tu_2)) &= \int_\Omega \Big(a_1(x, t\nabla u_1, t\nabla u_2) \cdot \nabla u_1 + a_2(x, t\nabla u_1, t\nabla u_2) \cdot \nabla u_2\Big)\, dx \\
&\quad - \int_\Omega (f_1(x, tu_1, tu_2)u_1 + f_2(x, tu_1, tu_2)u_2)\, dx \\
&\le \frac{\mu}{t} \int_\Omega \Big(A(x, tu_1, tu_2) - F(x, tu_1, tu_2)\Big)\, dx + \frac{\alpha_4 + C_0}{t}\text{meas}(\Omega) \\
&= \frac{\mu}{t}\Big(\Phi(tu_1, tu_2) - a_0\Big),
\end{aligned}$$

where $a_0 = -(\alpha_4 + C_0)\text{meas}(\Omega)/\mu \le 0$. Hence, if $\Phi(tu_1, tu_2) \le a$ for some $a < a_0$, then

$$\frac{d}{dt}(\Phi(tu_1, tu_2)) < 0.$$

Thus, since (A_4) and (F_1) imply $\Phi(0, 0) = 0$, taking any $a < a_0$ from (2.9) it follows that there exists a unique $t_a = t_a(u_1, u_2) > 0$ such that $\Phi(t_a u_1, t_a u_2) = a$ and

$$\Phi(tu_1, tu_2) > a \quad \text{for all } 0 \le t < t_a, \qquad \Phi(tu_1, tu_2) < a \quad \text{for all } t > t_a.$$

Consequently, $\Phi^a = \{(tu_1, tu_2) : (u_1, u_2) \in S_1,\ t \ge t_a(u_1, u_2)\}$, where, by the Implicit Function Theorem, $t_a \colon (u_1, u_2) \in S_1 \mapsto t_a(u_1, u_2) \in (0, +\infty)$ is a C^1 map. □

Corollary 2.14. *Assume that the hypotheses of Lemma* 2.13 *hold and take any* $a < a_0$. *Then, using the same notations as in the proof of Lemma* 2.13, *we have that* Φ^a *is a deformation retract of* $W\setminus\{0\}$ *via* $H\colon [0,1]\times(W\setminus\{0\}) \to W\setminus\{0\}$ *defined by*

$$H(t,(u_1,u_2)) = \begin{cases} (1-t)(u_1,u_2) + t\, t_a(u_1,u_2)(u_1,u_2) & \text{if } (u_1,u_2) \in (W\setminus\{0\})\setminus\Phi^a, \\ (u_1,u_2) & \text{if } (u_1,u_2)\in\Phi^a. \end{cases}$$

3. Cohomological local splitting

Let us first recall the notion of cohomological local splitting introduced in [17, Definition 3.33] (see also [15]). In what follows i denotes the Fadell–Rabinowitz cohomological index (see [13]) and for a subset C of a Banach space W we write

$$IC = \{tu : u \in C,\ t \in [0,1]\}.$$

Definition 3.1. We say that a C^1-functional $\Phi\colon W \to \mathbb{R}$, defined on a Banach space W, has a cohomological local splitting near zero in dimension q, $1 \le q < +\infty$, if there are

(a) a bounded symmetric subset $\mathcal{M}$ of $W\setminus\{0\}$ that is radially homeomorphic to the unit sphere in W, and disjoint symmetric subsets $A_0 \neq \emptyset$ and B_0 of $\mathcal{M}$ such that

$$i(A_0) = i(\mathcal{M}\setminus B_0) = q;$$

(b) a homeomorphism h from $I\mathcal{M}$ onto a neighborhood U of zero containing no other critical points, such that $h(0) = 0$ and

$$\Phi|_A \le 0 < \Phi|_{B\setminus\{0\}}$$

where $A = h(IA_0)$ and $B = h(IB_0)\cup\{0\}$.

On the other hand, denoting by $H^*(\,\cdot\,,\cdot\,)$ the Alexander–Spanier cohomology with $\mathbb{Z}_2$-coefficients (see [19]), the cohomological critical groups of Φ at an isolated critical point u_0 are defined by

$$C^q(\Phi,u_0) = H^q(\Phi^c\cap U, \Phi^c\cap U\setminus\{u_0\}), \quad \text{if } q\ge 0, \tag{3.1}$$

where $c = \Phi(u_0)$ is the corresponding critical value and U is a neighborhood of u_0 containing no other critical point of Φ (see e.g. [10]).

The following result can be stated.

Proposition 3.2 [17, Proposition 3.34]). *If Φ has a cohomological local splitting near zero in dimension k, then $C^k(\Phi, 0) \neq 0$.*

Here, we want to apply the previous theory to our setting.

First of all, let us recall some results concerning the nonlinear eigenvalue problem (2.7) proved in [17]. To this aim, define

$$I(u_1, u_2) = \frac{1}{p_1} \int_\Omega |\nabla u_1|^{p_1} \, dx + \frac{1}{p_2} \int_\Omega |\nabla u_2|^{p_2} \, dx, \quad (u_1, u_2) \in W.$$

Clearly, $I \in C^1(W, \mathbb{R})$ is such that

$$I(t^{1/p_1} u_1, t^{1/p_2} u_2) = t I(u_1, u_2) \quad \text{for all } t \geq 0, \ (u_1, u_2) \in W. \tag{3.2}$$

Furthermore, by [17, Lemma 10.6], the set

$$\mathcal{M} := \{u = (u_1, u_2) \in W : I(u_1, u_2) = 1\}$$

is radially homeomorphic to the unit sphere S_1 in W.

Now, taking the function $\mathcal{G}$ as in the hypothesis (F_4), define

$$J(u_1, u_2) = \int_\Omega \mathcal{G}(u_1, u_2) \, dx \quad \text{and} \quad \Psi(u_1, u_2) = \frac{I(u_1, u_2)}{J(u_1, u_2)} \quad \text{if } J(u_1, u_2) \neq 0.$$

Conditions (2.5)–(2.6) imply that $J \in C^1(W, \mathbb{R})$ and

$$J(t^{1/p_1} u_1, t^{1/p_2} u_2) = t J(u_1, u_2) \quad \text{for all } t \geq 0, \ (u_1, u_2) \in W. \tag{3.3}$$

Moreover, the set $\mathcal{M}^+ := \{u = (u_1, u_2) \in \mathcal{M} : J(u_1, u_2) > 0\}$ is a symmetric open submanifold of $\mathcal{M}$ and $\widetilde{\Psi} = \Psi|_{\mathcal{M}^+}$ is a C^1 function on $\mathcal{M}^+$.

For simplicity, for each $\lambda \in \mathbb{R}$ denote

$$\begin{aligned} \widetilde{\Psi}^\lambda &= \{u = (u_1, u_2) \in \mathcal{M}^+ : \widetilde{\Psi}(u_1, u_2) \leq \lambda\}, \\ \widetilde{\Psi}_\lambda &= \{u = (u_1, u_2) \in \mathcal{M}^+ : \widetilde{\Psi}(u_1, u_2) \geq \lambda\}, \end{aligned}$$

and, if $\mathcal{F}$ is the class of symmetric subsets of $\mathcal{M}^+$, let $\mathcal{F}_k = \{M \in \mathcal{F} : i(M) \geq k\}$ for each $k \in \mathbb{N}$ and

$$\lambda_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \widetilde{\Psi}(u_1, u_2). \tag{3.4}$$

Proposition 3.3 ([17, Theorem 10.10]). *Each λ_k in (3.4) is an eigenvalue of (2.7). Furthermore, $\lambda_k \nearrow +\infty$ and, if $\lambda_k < \lambda < \lambda_{k+1}$, then*

$$i(\widetilde{\Psi}^\lambda) = k = i(\mathcal{M}^+ \setminus \widetilde{\Psi}_{\lambda_{k+1}}).$$

Considering $\underline{\lambda}$, $\overline{\lambda}$ as in (F_4) and fixing $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$, let

$$A_0 = \widetilde{\Psi}^\lambda \quad \text{and} \quad B_0 = \widetilde{\Psi}_{\lambda_{k+1}} \cup (\mathcal{M} \setminus \mathcal{M}^+).$$

Obviously, by the previous definitions we have

$$A_0 = \left\{ u = (u_1, u_2) \in \mathcal{M}^+ : I(u_1, u_2) \le \lambda \int_\Omega \mathcal{G}(u_1, u_2)\, dx \right\},$$
$$B_0 = \left\{ u = (u_1, u_2) \in \mathcal{M}^+ : I(u_1, u_2) \ge \lambda_{k+1} \int_\Omega \mathcal{G}(u_1, u_2)\, dx \right\} \cup \{(u_1, u_2) \in \mathcal{M} : J(u_1, u_2) = 0\}.$$

Moreover, for each $\rho > 0$ define the map

$$h_\rho(tu_1, tu_2) = ((t\rho)^{1/p_1} u_1, (t\rho)^{1/p_2} u_2), \quad t \in [0,1],\ (u_1, u_2) \in \mathcal{M},$$

which is a homeomorphism between $I\mathcal{M}$ and the neighbourhood of zero

$$U_\rho = \{(t^{1/p_1}\overline{u}_1, t^{1/p_2}\overline{u}_2) : (\overline{u}_1, \overline{u}_2) \in \mathcal{M},\ 0 \le t \le \rho\}.$$

For simplicity, we denote $B_\rho = h_\rho(IB_0) \cup \{0\}$ and $A_\rho = h_\rho(IA_0)$ for any $\rho > 0$.

In order to show that Φ has a cohomological local splitting near zero, it suffices to prove that the following statement holds.

Lemma 3.4 (Splitting geometry). *If* (A_4)*,* (F_1)*,* (F_2) *and* (F_4) *hold, there exists* $\rho^* > 0$ *such that*

(a) $\Phi(u_1, u_2) > 0$ *if* $(u_1, u_2) \in B_{\rho^*} \setminus \{0\}$,
(b) $\Phi(u_1, u_2) \le 0$ *if* $(u_1, u_2) \in A_{\rho^*}$.

Proof. Taking any $\rho > 0$, note that $B_\rho = \{(t^{1/p_1}\overline{u}_1, t^{1/p_2}\overline{u}_2) : (\overline{u}_1, \overline{u}_2) \in B_0,\ 0 \le t \le \rho\} \cup \{0\}$. Then, taking $(u_1, u_2) \in B_\rho$, we have $(u_1, u_2) = (t^{1/p_1}\overline{u}_1, t^{1/p_2}\overline{u}_2)$ for some $(\overline{u}_1, \overline{u}_2) \in B_0$ and $0 \le t \le \rho$. Clearly, by definition we have $I(u_1, u_2) \le \rho$.

Moreover, the Sobolev Imbedding Theorem and direct computations imply

$$|u_1|_{p_1^*}^{p_1^*} \le C_2 \|u_1\|_{p_1}^{p_1^*} \le C_3 (I(u_1, u_2))^{p_1^*/p_1},$$
$$|u_2|_{p_2^*}^{p_2^*} \le C_2 \|u_1\|_{p_2}^{p_2^*} \le C_3 (I(u_1, u_2))^{p_2^*/p_2},$$

for some $C_2, C_3 > 0$. Together with the second inequality in (2.8), these estimates imply that

$$\int_\Omega F(x, u_1, u_2)\, dx \le \overline{\lambda}\alpha \int_\Omega \mathcal{G}(u_1, u_2)\, dx + \epsilon(\rho) I(u_1, u_2) = \overline{\lambda}\alpha J(u_1, u_2) + \epsilon(\rho) I(u_1, u_2), \tag{3.5}$$

where $\epsilon(\rho) = C_1 C_3 (\rho^{p_1^*/p_1 - 1} + \rho^{p_2^*/p_2 - 1}) \to 0$ as $\rho \to 0$. Hence, (3.5) and (A_4) imply that

$$\Phi(u_1, u_2) \ge (\alpha - \epsilon(\rho)) I(u_1, u_2) - \overline{\lambda}\alpha J(u_1, u_2). \tag{3.6}$$

Now, two cases may occur: either $(\overline{u}_1, \overline{u}_2) \in \widetilde{\Psi}_{\lambda_{k+1}}$ or $(\overline{u}_1, \overline{u}_2) \in \mathcal{M} \setminus \mathcal{M}^+$.

If $(\overline{u}_1, \overline{u}_2) \in \widetilde{\Psi}_{\lambda_{k+1}}$, (3.2) and (3.3) imply

$$I(u_1, u_2) \geq \lambda_{k+1} J(u_1, u_2),$$

thus, if $\rho > 0$ is small enough, from (3.6) it follows

$$\Phi(u_1, u_2) \geq \left(\alpha\left(1 - \frac{\overline{\lambda}}{\lambda_{k+1}}\right) - \epsilon(\rho)\right) I(u_1, u_2) > 0.$$

On the other hand, if $(\overline{u}_1, \overline{u}_2) \in \mathcal{M} \setminus \mathcal{M}^+$, we have $J(u_1, u_2) \leq 0$ so, if $\rho > 0$ is small enough, (3.6) implies

$$\Phi(u_1, u_2) \geq (\alpha - \epsilon(\rho)) I(u_1, u_2) > 0.$$

Whence, (a) holds.

In order to prove (b), note that the first inequality in (2.8) gives

$$-\int_\Omega F(x, u_1, u_2)\, dx \leq \epsilon(\rho) I(u_1, u_2) - \beta \underline{\lambda} \int_\Omega \mathcal{G}(u_1, u_2)\, dx,$$

which, together with (A_4), implies

$$\begin{aligned} \Phi(u_1, u_2) &\leq \beta I(u_1, u_2) - \beta \underline{\lambda} \int_\Omega \mathcal{G}(u_1, u_2) dx + \epsilon(\rho) I(u_1, u_2) \\ &\leq \left(\beta\left(1 - \frac{\underline{\lambda}}{\lambda}\right) + \epsilon(\rho)\right) I(u_1, u_2) \leq 0 \end{aligned}$$

if $(u_1, u_2) \in A_\rho$, for ρ sufficiently small. This completes the proof. □

Proposition 3.5. *If the hypotheses* (A_1)*,* (A_4)*,* (F_1)*,* (F_2) *and* (F_4) *hold, then* Φ *has a cohomological local splitting near zero in dimension* k*, where* k *is as in* (F_4)*. Hence,* $C^k(\Phi, 0) \neq 0$*.*

Proof. By Lemma 2.12 the functional Φ is C^1 in W. Furthermore, considering k as in (F_4) and $\mathcal{M}$, A_0, B_0 as in the first part of this section with $\lambda_k < \underline{\lambda} \leq \lambda \leq \overline{\lambda} < \lambda_{k+1}$, from $\mathcal{M} \setminus B_0 = \mathcal{M}^+ \setminus \widetilde{\Psi}_{\lambda_{k+1}}$ and Proposition 3.3 it follows

$$i(A_0) = k = i(\mathcal{M} \setminus B_0).$$

Then Lemma 3.4 and Proposition 3.2 complete the proof. □

4. A compactness condition

From now on, assume that (A_1), (A_4) and (F_2) hold. Thus, Φ is a C^1 functional on W (see Lemma 2.12).

Briefly, we say that $(u_n)_n \subset W$, $u_n = (u_{1,n}, u_{2,n})$, is a *Palais–Smale sequence at level* c, $c \in \mathbb{R}$, if

$$\Phi(u_{1,n}, u_{2,n}) \xrightarrow{n} c, \qquad \|d\Phi(u_{1,n}, u_{2,n})\|_{W'} \xrightarrow{n} 0. \tag{4.1}$$

Recall that the functional Φ satisfies the *Palais–Smale condition at level* c in W ($(\mathrm{PS})_c$ for short) if every Palais–Smale sequence at level c has a subsequence that converges in the norm of W.

In order to show that Φ satisfies $(\mathrm{PS})_c$ for each $c \in \mathbb{R}$, some lemmas are needed.

Lemma 4.1. *Assume that also the hypotheses* (A_2), (A_5), (F_1) *and* (F_3) *hold. Then, taking any* $c \in \mathbb{R}$, *each* $(\mathrm{PS})_c$ *sequence is bounded.*

Proof. Let $(u_n)_n \subset W$, $u_n = (u_{1,n}, u_{2,n})$, be such that (4.1) holds. Whence, we have

$$\begin{aligned} \Phi(u_{1,n}, u_{2,n}) &= c + o(1), \\ d\Phi(u_{1,n}, u_{2,n})[(u_{1,n}, 0)] &= o(1)\|u_{1,n}\|_{p_1}, \\ d\Phi(u_{1,n}, u_{2,n})[(0, u_{2,n})] &= o(1)\|u_{2,n}\|_{p_2}, \end{aligned}$$

with $o(1)$ any infinitesimal sequence of real numbers.

Since $\mu \le \theta$, by using (A_2) and (A_5) we get

$$\theta A(x,\xi) - a(x,\xi)\cdot\xi \ge \alpha_2\alpha_3(|\xi_1|^{p_1} + |\xi_2|^{p_2}) \quad \text{for a.a. } x \in \Omega \text{ if } |\xi| \ge R.$$

Thus, from (F_3) it follows

$$\begin{aligned} &\theta c + o(1) + o(1)\|u_n\| \\ &= \theta\Phi(u_{1,n}, u_{2,n}) - d\Phi(u_{1,n}, u_{2,n})[(u_{1,n}, 0)] - d\Phi(u_{1,n}, u_{2,n})[(0, u_{2,n})] \\ &= \int_\Omega (\theta A(x, \nabla u_{1,n}, \nabla u_{2,n}) - a(x, \nabla u_{1,n}, \nabla u_{2,n}) \cdot \nabla u_n)\, dx \\ &\quad - \int_\Omega (\theta F(x, u_{1,n}, u_{2,n}) - f_1(x, u_{1,n}, u_{2,n})u_{1,n} - f_2(x, u_{1,n}, u_{2,n})u_{2,n})\, dx \\ &\ge \alpha_2\alpha_3(\|u_{1,n}\|_{p_1}^{p_1} + \|u_{2,n}\|_{p_2}^{p_2}) - \alpha_2\alpha_3 \int_{\Omega^R(\nabla u_n)} (|\nabla u_{1,n}|^{p_1} + |\nabla u_{2,n}|^{p_2})\, dx \\ &\quad + \int_{\Omega^R(\nabla u_n)} (\theta A(x, \nabla u_{1,n}, \nabla u_{2,n}) - a(x, \nabla u_{1,n}, \nabla u_{2,n}) \cdot \nabla u_n)\, dx \\ &\quad - \int_{\Omega^R(u_n)} (\theta F(x, u_{1,n}, u_{2,n}) - f_1(x, u_{1,n}, u_{2,n})u_{1,n} - f_2(x, u_{1,n}, u_{2,n})u_{2,n})\, dx, \end{aligned}$$

with

$$\Omega^R(\nabla u_n) = \{x \in \Omega : |\nabla u_n(x)| \le R\}, \quad \Omega^R(u_n) = \{x \in \Omega : |u_n(x)| \le R\}. \tag{4.2}$$

But direct computations and definitions (4.2) imply that they are bounded not only

$$\bigg(\int_{\Omega^R(\nabla u_n)} (|\nabla u_{1,n}|^{p_1} + |\nabla u_{2,n}|^{p_2})\, dx\bigg)_n,$$
$$\bigg(\int_{\Omega^R(\nabla u_n)} A(x, \nabla u_{1,n}, \nabla u_{2,n})\, dx\bigg)_n,$$
$$\bigg(\int_{\Omega^R(\nabla u_n)} a(x, \nabla u_{1,n}, \nabla u_{2,n}) \cdot \nabla u_n)\, dx\bigg)_n,$$

(by using conditions (A_1), (A_4)) but also

$$\bigg(\int_{\Omega^R(u_n)} F(x, u_{1,n}, u_{2,n})\, dx\bigg)_n,$$
$$\bigg(\int_{\Omega^R(u_n)} (f_1(x, u_{1,n}, u_{2,n})u_{1,n} + f_2(x, u_{1,n}, u_{2,n})u_{2,n})\, dx\bigg)_n,$$

(by using conditions (F_2) and (2.2)). Thus, $(u_n)_n$ has to be bounded in W, too. $\square$

Now, we prove the following *compactness result* by using an argument similar to that in [1, Lemma 3.2] (see also [4]). But first, as useful in the following, let us recall a suitable version of the Young's Inequality: fixing any $\varepsilon > 0$ there exists $\gamma_{\varepsilon,p_j} > 0$, i.e. a constant γ_{ε,p_j} depending only on ε and p_j, such that

$$\eta_1\eta_2 \le \varepsilon\eta_1^{p_j} + \gamma_{\varepsilon,p_j}\eta_2^{p_j'} \quad \text{for all } \eta_1,\ \eta_2 \ge 0. \tag{4.3}$$

Lemma 4.2. *Assume that* (A_2), (A_3) *also hold. If* $(u_n)_n \subset W$, $u_n = (u_{1,n}, u_{2,n})$, *and* $u = (u_1, u_2) \in W$ *are such that*

$$u_{j,n} \rightharpoonup u_j \quad \textit{weakly in } W_0^{1,p_j}(\Omega),\ j = 1, 2, \tag{4.4}$$

$$\int_\Omega (a(x, \nabla u_{1,n}, \nabla u_{2,n}) - a(x, \nabla u_1, \nabla u_2)) \cdot (\nabla u_n - \nabla u)\, dx \to 0, \tag{4.5}$$

then $u_{j,n} \to u_j$ *strongly in* $W_0^{1,p_j}(\Omega)$, $j = 1, 2$.

Proof. For simplicity, assume

$$D_n(x) = (a(x, \nabla u_{1,n}(x), \nabla u_{2,n}(x)) - a(x, \nabla u_1(x), \nabla u_2(x))) \cdot (\nabla u_n(x) - \nabla u(x)),$$

for $x \in \Omega$. Since the imbedding $W_0^{1,p_j}(\Omega) \hookrightarrow L^1(\Omega)$ is compact and $D_n \to 0$ in $L^1(\Omega)$, up to a subsequence we may assume that

$$u_{j,n}(x) \to u_j(x) \quad \text{a.e. in } \Omega,\ j = 1, 2, \quad \text{and} \quad D_n(x) \to 0 \quad \text{a.e. in } \Omega.$$

Hence, there exists a set $N \subset \Omega$, meas$(N) = 0$, such that for all $j = 1, 2$ it is

$$\begin{aligned} &|u_j(x)|, |\nabla u_j(x)| < \infty, \quad u_{j,n}(x) \to u_j(x) \\ &\text{and} \quad D_n(x) \to 0 \quad \text{for all } x \in \Omega \setminus N. \end{aligned} \tag{4.6}$$

Now, fixing $x \in \Omega \setminus N$, let $\xi_n = (\xi_{1,n}, \xi_{2,n})$, with $\xi_{j,n} = \nabla u_{j,n}(x)$ $(j = 1, 2)$, and $\xi = (\xi_1, \xi_2)$, with $\xi_j = \nabla u_j(x)$ $(j = 1, 2)$.

From one hand, using (A_2) we have

$$a(x, \xi_n) \cdot \xi_n \geq \alpha_2(|\xi_{1,n}|^{p_1} + |\xi_{2,n}|^{p_2}). \tag{4.7}$$

On the other hand, fixing any $\varepsilon > 0$, from (A_1), the Young's Inequality (4.3) and direct computations it follows

$$\begin{aligned} a(x, \xi_n) \cdot \xi &= a_1(x, \xi_n) \cdot \xi_1 + a_2(x, \xi_n) \cdot \xi_2 \\ &\leq \alpha_1(|\xi_{1,n}|^{p_1-1} + |\xi_{2,n}|^{p_2/p_1'} + 1)|\xi_1| \\ &\quad + \alpha_1(|\xi_{1,n}|^{p_1/p_2'} + |\xi_{2,n}|^{p_2-1} + 1)|\xi_2| \\ &\leq 2\alpha_1\varepsilon(|\xi_{1,n}|^{p_1} + |\xi_{2,n}|^{p_2}) + h^*(\varepsilon, \xi), \\ a(x, \xi) \cdot \xi_n &= a_1(x, \xi) \cdot \xi_{1,n} + a_2(x, \xi) \cdot \xi_{2,n} \\ &\leq \alpha_1(|\xi_1|^{p_1-1} + |\xi_2|^{p_2/p_1'} + 1)|\xi_{1,n}| + \alpha_1(|\xi_1|^{p_1/p_2'} + |\xi_2|^{p_2-1} + 1)|\xi_{2,n}| \\ &\leq 3\alpha_1\varepsilon(|\xi_{1,n}|^{p_1} + |\xi_{2,n}|^{p_2}) + h^{**}(\varepsilon, \xi), \end{aligned}$$

where both $h^*(\varepsilon, \xi)$ and $h^{**}(\varepsilon, \xi)$ are suitable positive expressions depending only on ε and ξ.

Thus, these last estimates and (4.7) imply

$$D_n(x) \geq (\alpha_2 - 5\alpha_1\varepsilon)(|\xi_{1,n}|^{p_1} + |\xi_{2,n}|^{p_2}) + a(x, \xi) \cdot \xi - h^*(\varepsilon, \xi) - h^{**}(\varepsilon, \xi);$$

hence, choosing ε small enough, from (4.6) we have that $(\xi_{1,n})_n$, $(\xi_{2,n})_n$ are bounded sequences in $\mathbb{R}^N$ and so is $(\xi_n)_n$ in $\mathbb{R}^{2N}$.

Thus, we can consider ξ^* as a cluster point of $(\xi_n)_n$. Obviously, we have $|\xi^*| < \infty$ and, by the continuity of $a(x, \cdot)$, (4.6) implies

$$(a(x, \xi^*) - a(x, \xi)) \cdot (\xi^* - \xi) = 0.$$

Whence, from (A_3) we have $\xi^* = \xi$. So, for the uniqueness of the cluster point, we have $\xi_n \to \xi$. Hence, $\nabla u_n(x) \to \nabla u(x)$ for all $x \in \Omega \setminus N$, i.e. almost everywhere in Ω.

Now, in order to complete the proof, it is enough following the same arguments developed in the the last part of the proof of [7, Lemma 5]. □

Proposition 4.3. *Assume that* (A_1)–(A_5) *and* (F_1)–(F_3) *hold. Then* Φ *satisfies the* $(PS)_c$ *condition for all* $c \in \mathbb{R}$.

Proof. Fixing $c \in \mathbb{R}$, let $(u_n)_n \subset W$, $u_n = (u_{1,n}, u_{2,n})$, be a $(PS)_c$ sequence, so (4.1) holds. Then, from Lemma 4.1 it follows that it is bounded and $u \in W$, $u = (u_1, u_2)$, exists such that, passing to a subsequence if necessary, (4.2) holds. Whence,

$$u_{j,n} \to u_j \quad \text{in } L^r(\Omega) \text{ for all } 1 \leq r < p_j^*,\ j = 1, 2. \tag{4.8}$$

Now, in order to complete the proof by applying Lemma 4.2, we need (4.5). So, firstly let us remark that (4.4) implies

$$\int_\Omega a(x, \nabla u_1, \nabla u_2) \cdot \nabla(u_n - u)\, dx \to 0. \tag{4.9}$$

Furthermore, from (4.1) it follows

$$\begin{aligned}\int_\Omega a(x, \nabla u_{1,n}, \nabla u_{2,n}) \cdot \nabla(u_n - u)\, dx = o(1) \\ + \int_\Omega f_1(x, u_{1,n}, u_{2,n})(u_{1,n} - u_1)\, dx + \int_\Omega f_2(x, u_{1,n}, u_{2,n})(u_{2,n} - u_2)\, dx.\end{aligned} \tag{4.10}$$

We claim that

$$\int_\Omega f_j(x, u_{1,n}, u_{2,n})(u_{j,n} - u_j)\, dx \to 0 \quad \text{for both } j = 1 \text{ and } j = 2. \tag{4.11}$$

In fact, from (F_2) it follows

$$\begin{aligned}&\left| \int_\Omega f_1(x, u_{1,n}, u_{2,n})(u_{1,n} - u_1)\, dx \right| \\ &\qquad \le \sigma \int_\Omega (|u_{1,n}|^{s_1-1}|u_{1,n} - u_1| + |u_{2,n}|^{q_1-1}|u_{1,n} - u_1| + |u_{1,n} - u_1|)\, dx,\end{aligned}$$

where the Cauchy–Schwarz inequality implies

$$\begin{aligned}\int_\Omega |u_{1,n}|^{s_1-1}|u_{1,n} - u_1|\, dx &\le \left(\int_\Omega |u_{1,n}|^{s_1}\, dx\right)^{(s_1-1)/s_1} |u_{1,n} - u_1|_{s_1}, \\ \int_\Omega |u_{2,n}|^{q_1-1}|u_{1,n} - u_1|\, dx &\le \left(\int_\Omega |u_{1,n}|^{(q_1-1)p_1/(p_1-1)}\, dx\right)^{(p_1-1)/p_1} |u_{1,n} - u_1|_{p_1}.\end{aligned}$$

Thus, (4.8) implies (4.11) if $j = 1$. Similar arguments allow one to obtain (4.11) also if $j = 2$. So, (4.9)–(4.11) imply (4.5), so the conclusion follows from Lemma 4.2. □

5. Main results

The main result of this paper can be stated as follows.

Theorem 5.1. *If* (A_1)–(A_5) *and* (F_1)–(F_4) *hold, then problem* (1.1) *has a nontrivial weak solution in* W.

Proof. Arguing by contradiction, suppose that the origin is the unique critical point of Φ in W. As in this case (3.1) becomes

$$C^q(\Phi, 0) = H^q(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\}), \quad q \ge 0,$$

where U is a neighborhood of $(0, 0)$ containing no other critical points of Φ, we can take $U = W$ and obtain

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}), \quad q \ge 0.$$

Since Φ satisfies the $(\mathrm{PS})_c$ condition at each level $c \in \mathbb{R}$, by the Deformation Lemma (see [18]) Φ^a is a deformation retract of $\Phi^0 \setminus \{0\}$ for any $a < \Phi(0,0) = 0$ and Φ^0 is a deformation retract of W. Thus, we conclude that

$$C^q(\Phi, 0) = H^q(W, \Phi^a) \quad \text{for any } a < 0.$$

On the other hand, Lemma 2.13 implies that Φ^a is contractible for all $a < a_0$. Therefore,

$$C^q(\Phi, 0) = 0 \quad \text{for all } q \geq 0.$$

This contradicts Proposition 3.5 and proves the theorem. □

Corollary 5.2. *If* (F_1)–(F_4) *hold, then system* (1.5) *has a nontrivial weak solution.*

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