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THE SIZE OF SOME CRITICAL SETS BY MEANS OF DIMENSION AND ALGEBRAIC φ-CATEGORY

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Abstract. Let M^n, N^n, $n \geq 2$, be compact connected manifolds. We first observe that mappings of zero degree have high dimensional critical sets and show that the only possible degree is zero for maps $f\colon M \to N$, under the assumption on the index $[\pi_1(N) : \mathrm{Im}(f_*)]$ to be infinite. By contrast with the described situation one shows, after some estimates on the algebraic φ-category of some pairs of finite groups, that a critical set of smaller dimension keeps the degree away from zero.

1. Introduction and preliminary results

The critical set and the set of critical values were of constant interest over the last decades, mostly through their size, as they play important roles in many theories such as Morse theory, Lusternik–Schnirelmann theory, variational calculus etc. [12]. While the evaluation tool for size the critical sets within the mentioned approaches is usually the cardinality, the remarkable Sard theorem ensures us that the sets of critical values have zero measures and indirectly points out that the measure cannot distinguish the sets of critical values of *different* maps. For example the set of critical values of a constant map as well as the set of critical values of the projection $p\colon S^n \to \mathbb{R}^n$, $p(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$

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have both zero measure, yet one of them is a zero dimensional manifold while the other one is the $(n-1)$-dimensional sphere.

On the other hand the cardinality might not be a suitable tool for evaluating the critical sets as soon as one knows that every map between two given manifolds has infinitely many critical points/values. However the *topological dimension* may play some role in this evaluation process, the series of works by P. T. Church and J. G. Timourian from the mid-sixties to mid-seventies being good arguments in this respect. We only mention here three of them, namely [3]–[5].

In this work we also employ the topological dimension to provide some examples of maps with large critical sets, showing that the main tool (algebraic φ-category of pairs of fundamental groups) and the technique of [13] can be used more effectively. Some different examples of maps with high dimensional critical sets are provided by [14], where we use a different approach involving top volume forms of the target oriented manifolds as the key tools.

In this section we show that the C^1 maps of zero degree have high dimensional critical sets and those with nonempty set of regular points which are not onto have high dimensional sets of critical values. In this respect we first recall a classical non-separation result and study the relations between the critical sets and the sets of critical values of mappings related by a commutative diagram with local diffeomorphisms on two parallel sides.

In the next section we prove that mappings acting between manifolds with infinite algebraic φ-category of their fundamental groups have all zero degree and, consequently, high dimensional critical sets.

Finally, in the last section, we first provide some estimates on the algebraic φ-category of some pairs of finite groups. We next observe that the algebraic φ-category of the pair of fundamental groups of two compact connected smooth manifolds is a lower bound for the absolute degree of some mappings between the two manifolds.

Theorem 1.1 ([10, p. 48]). *Every connected manifold M is a Cantor manifold. More precisely, no subset of M of dimension $\leq n-2$ separates M, where $n = \dim M$.*

Let M, N be smooth manifolds and let $f\colon M \to N$ be a C^1 mapping. Denote by $C(f)$ the critical set of f which consists in those points $p \in M$ with the property that $\operatorname{rank}_p f < \min\{\dim M, \dim N\}$ and denote by $B(f)$ the set $f(C(f))$ of critical values of f. Recall that $C(f)$ is closed and the set $R(f) = M \setminus C(f)$ of regular points of f is consequently open.

Lemma 1.2. *Let M, M', N, N' be n-dimensional smooth manifolds. If $\alpha\colon M \to M'$, $\beta\colon N \to N'$ are surjective local diffeomorphisms and $f'\colon M' \to N'$,*

$f\colon M \to N$ are such that the following diagram is commutative

$$\begin{array}{ccc} M' & \xrightarrow{f'} & N' \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ M & \xrightarrow[f]{} & N \end{array} \qquad \beta \circ f' = f \circ \alpha$$

then $C(f) = \alpha(C(f'))$ and $B(f) = \beta(B(f'))$.

Proof. Indeed $(d\beta)_{f'(x)} \circ (df')_x = (df)_{\alpha(x)} \circ (d\alpha)_x$, namely

$$(df')_x = [(d\beta)_{f'(x)}]^{-1} \circ (df)_{\alpha(x)} \circ (d\alpha)_x$$

for every $x \in M'$, which shows that $\operatorname{rank}_x(f') = \operatorname{rank}_{\alpha(x)}(f)$ for all $x \in M'$ as well as the equality $C(f) = \alpha(C(f'))$. For the second equality we have successively

$$\begin{aligned} \beta(B(f')) &= \beta(f'(C(f'))) = (\beta \circ f')(C(f')) \\ &= (f \circ \alpha)(C(f')) = f(\alpha(C(f')) = f(C(f)) = B(f). \end{aligned} \qquad \square$$

Remark 1.3. If M', M, N', N and α, β are as in Lemma 1.2, then, according to R. E. Hodel [9], one gets

$$\dim[C(f)] = \dim[C(f')] \quad \text{and} \quad \dim[B(f)] = \dim[B(f')].$$

Proposition 1.4 ([2]). *Let M, N be smooth manifolds such that $\dim M \geq \dim N \geq 2$. If N is additionally connected and $f\colon M \to N$ is a closed non-surjective C^1 mapping such that $C(f) \neq M$, then $\dim[B(f)] = \dim(N) - 1$.*

Corollary 1.5. *Let M^m, N^n, $m = n \geq 2$ be smooth manifolds such that M is compact.*

(a) *If M, N are orientable and $f\colon M \to N$ has zero degree, then either $C(f) = M$ or the set $R(f) = M \setminus C(f)$ is not connected. Consequently, $\dim[C(f)] \geq n - 1$. In particular $\dim[C(f)] \geq n - 1$ for all $f \in C^1(M, N)$, whenever N is orientable but not compact.*

(b) *If N is compact orientable and M non-orientable, then $\dim[C(f)] \geq n - 1$ for all $f \in C^1(M, N)$.*

Proof. (a) We first recall that $\operatorname{sign}(df)_x$, $x \in R(f)$, is defined to be $+1$ or -1 as $(df)_x$ preserves or reverses the orientation and observe that the function $R(f) \to \mathbb{Z}$ is locally constant, i.e. it is actually constant on each component of $R(f)$. Recall that the *degree* $\deg(f)$ of f is defined to be

$$\sum_{x \in f^{-1}(y)} \operatorname{sign}(df)_x, \tag{1.1}$$

where $y \in \mathrm{Im}(f)$ is a regular value of f, as the sum (1.1) is independent of $y \in N \setminus B(f)$ ([11, p. 28]). On the other hand the equalities

$$0 = \deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{sign}(df)_x,$$

show that the sign map $\mathrm{sgn}(df)_{(\,\cdot\,)}$ takes both values ± 1. Consequently, $R(f) = M \setminus C(f)$ is not connected, which shows, by using Theorem 1.1, that

$$\dim[C(f)] \geq n - 1.$$

(b) Consider the orientable double cover $p: \widetilde{M} \to M$ of M and recall that $H_n(M, \mathbb{Z}) \cong 0$, which shows that $H_n(f \circ p): H_n(\widetilde{M}, \mathbb{Z}) \to H_n(N, \mathbb{Z})$ is zero.

Consequently $\deg(f \circ p) = 0$, which implies, according to Remark 1.3, that $\dim[C(f)] = \dim[C(f \circ p)] \geq n - 1$. □

The proof of Corollary 1.5(a) was suggested to the author by Andrzej Weber and the statement (b) of Corollary 1.5 was proved before, in a slightly more general context, by P. T. Church in [4] (see Remark 2.9).

Lemma 1.6. *If M^m, N^n are smooth manifolds and $f: M \to N$ is a C^1 mapping, then the following inequality holds* $\dim[\mathrm{Im}(f)] \leq \dim(M)$.

Proof. If $m \geq n$, the required inequalities are obvious. Otherwise, consider the C^1 mapping $g: M \times \mathbb{R}^{n-m} \to N, g(x, y) = f(x)$ and observe that $C(g) = M \times \mathbb{R}^{n-m}$. Indeed,

$$\mathrm{rank}_{(x,y)} g = \mathrm{rank}_x f \leq m < n = \dim(N) = \dim(M \times \mathbb{R}^{n-m}).$$

Thus

$$\mathrm{Im}(f) = \mathrm{Im}(g) = g(C(g)) = B(g).$$

By using [1, Proposition 2.2], one gets that $\dim[\mathrm{Im}(f)] = \dim[B(g)] \leq m$. □

2. Mappings of zero degree

In this section we provide sufficient conditions, in terms of fundamental groups and induced group homomorphisms, for high dimensional critical sets and sets of critical values.

Theorem 2.1. *If M^m, N^n are compact connected smooth manifolds and $f: M \to N$ is a C^1 map such that the index $[\pi_1(N) : \mathrm{Im}(f_*)]$ is infinite, then the following statements hold:*

(a) $\dim[B(f)] = n - 1$, *whenever $m \geq n$ and $C(f) \neq M$.*
(b) $\deg(f) = 0$ *whenever $m = n$ and M, N are orientable.*

PROOF. Using the theory of covering maps, there exists a covering mapping $p\colon \widetilde{N} \to N$ such that $\mathrm{Im}(p_*) = \mathrm{Im}(f_*)$ and a lifting $\widetilde{f}\colon M \to \widetilde{N}$ of f, namely $p \circ \widetilde{f} = f$. Because the number of sheets of the covering mapping $p\colon \widetilde{N} \to N$ is the infinite index $[\pi_1(N) : \mathrm{Im}(f_*)]$, where $f_*\colon \pi_1(M) \to \pi_1(N)$ is the induced group homomorphism, it follows that $\widetilde{N}$ is not compact. Since p is a local diffeomorphism, we also get, by using Lemma 1.2 for $\alpha = id_M$ and $\beta = p$, the equalities $C(f) = C(\widetilde{f})$ and $p(B(\widetilde{f})) = B(f)$.

(a) We just need to apply Corollary 1.4 and use the equality $\dim[p(B(\widetilde{f}))] = \dim[B(\widetilde{f})]$, which occur since p is open and has zero dimensional fibers [9].

(b) Indeed, since M is compact and $\widetilde{N}$ is not compact, it follows that $\deg(\widetilde{f}) = 0$, that is $\deg(f) = \deg(p \circ \widetilde{f}) = \deg(p)\deg(\widetilde{f}) = 0$. □

We are next interested in pairs (M^n, N^n) of connected orientable manifolds with the property that $\deg(f) = 0$ for all $f \in C^1(M, N)$. As we have already seen in Theorem 2.1(b) , this is the case if $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$, where $\varphi_{\mathrm{alg}}(G, H)$, for an arbitrary pair of groups (G, H), stands for the so called, *algebraic φ-category* of the pair (G, H), defined by $\min\{[H : \mathrm{Im}(f)] \mid f \in \mathrm{Hom}(G, H)\}$. If $[K : \mathrm{Im}(f)]$ is infinite for all $f \in \mathrm{Hom}(G, H)$, then the notation $\varphi_{\mathrm{alg}}(G, K) = \infty$ is used.

THEOREM 2.2. *If M^n, N^n $(n \geq 2)$ are compact connected manifolds, then $\dim[C(f)] \geq n - 1$, for all $f \in C^1(M, N)$, in each of the following situations:*

(a) *$\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$ and N is orientable.*

(b) *$\pi_1(M)$ is finite and $\pi_1(N)$ is infinite.*

PROOF. (a) Let $f\colon M \to N$ be a C^1 map and $f_*\colon \pi_1(M) \to \pi_1(N)$ be the induced homomorphism. Because $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$, it follows that the index $[\pi_1(N) : \mathrm{Im}(f_*)]$ is infinite.

- *M is not orientable.* This case was treated in Corollary 1.5(b).
- *M is orientable*: We just need to apply Theorem 2.1(b) and Corollary 1.5(b).

(b) We first observe that $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$ for all $f \in C^1(M, N)$, so that we only need to consider the case N-non-orientable, since the case N-orientable was treated in (a). The universal covering $\widetilde{M}$ of M is obviously compact and, for every C^1 map $f\colon M \to N$, there exists a C^1 mapping $\underline{f}\colon \widetilde{M} \to \overline{N}$ making the following diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\underline{f}} & \overline{N} \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[f]{} & N \end{array} \qquad \pi_N \circ \underline{f} = f \circ p_M,$$

commutative, where $\overline{N}$ is the orientable double cover of N. Since $\pi_1(\overline{N})$ is a subgroup of index 2 of $\pi_1(N)$ and $\pi_1(N)$ is infinite, it follows that $\pi_1(\overline{N})$ is infinite, which, combined with the simply connectedness of $\widetilde{M}$, shows that $\varphi_{\mathrm{alg}}(\pi_1(\widetilde{M}), \pi_1(\overline{N})) = \infty$. Therefore the following relations hold

$$\dim[C(\underline{f})] \geq \dim(\overline{N}) - 1 = \dim(N) - 1.$$

The above diagram shows that $C(f) = p_M(C(\underline{f}))$, such that we have successively

$$\dim[C(f)] = \dim[p_M(C(\underline{f}))] = \dim[C(\underline{f})] \geq \dim(\overline{N}) - 1 = \dim(N) - 1. \quad \square$$

For any group G we denote by $[G,G]$ and by $t(G)$ its *commutator* and *torsion* subgroups respectively. Recall that both of them are normal subgroups of G.

LEMMA 2.3 ([13]). *If G, K are finitely generated abelian groups such that the inequality* $\mathrm{rank}(G/t(G)) < \mathrm{rank}(K/t(K))$ *holds, then* $\varphi_{\mathrm{alg}}(G,K) = \infty$.

LEMMA 2.4. *For every groups G, K, the following inequalities hold:*

(a) $\varphi_{\mathrm{alg}}(G,K) \geq \varphi_{\mathrm{alg}}\left(\dfrac{G}{[G,G]}, \dfrac{K}{[K,K]}\right)$.

(b) $\varphi_{\mathrm{alg}}(G,K) \geq \varphi_{\mathrm{alg}}\left(\dfrac{G}{t(G)}, \dfrac{K}{t(K)}\right)$.

PROOF. Indeed, for any group homomorphism $f\colon G \to K$ there exists a group homomorphism

$$[f]\colon \frac{G}{[G,G]} \to \frac{K}{[K,K]},$$

whose image is $\mathrm{Im}([f]) = (\mathrm{Im}(f))[K,K]$, such that the following diagram is commutative

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & K \\ {\scriptstyle p_G}\big\downarrow & & \big\downarrow{\scriptstyle p_K} \\ \dfrac{G}{[G,G]} & \xrightarrow[{[f]}]{} & \dfrac{K}{[K,K]} \end{array}$$

Moreover, for every $f \in \mathrm{Hom}(G,K)$, we have successively:

$$\begin{aligned}[K : \mathrm{Im}(f)] &\geq [K : (\mathrm{Im}(f))[K,K]] = \left[\frac{K}{[K,K]} : \frac{(\mathrm{Im}(f))[K,K]}{[K,K]}\right] \\ &= \left[\frac{K}{[K,K]} : \mathrm{Im}([f])\right] \geq \varphi_{\mathrm{alg}}\left(\frac{G}{[G,G]}, \frac{K}{[K,K]}\right),\end{aligned}$$

which shows that, indeed,

$$\varphi_{\mathrm{alg}}(G,K) \geq \varphi_{\mathrm{alg}}\left(\frac{G}{[G,G]}, \frac{K}{[K,K]}\right).$$

The second inequality can be justified in a similar way. $\square$

COROLLARY 2.5. *Let X, Y be pathwise connected spaces and $\beta_1(X)$, $\beta_1(Y)$ their first Betti numbers.*

(a) $\varphi_{\mathrm{alg}}(\pi_1(X), \pi_1(Y)) \geq \varphi_{\mathrm{alg}}(H_1(X), H_1(Y))$.
(b) *If X, Y are compact* ENR *such that $\beta_1(X) < \beta_1(Y)$, then*

$$\varphi_{\mathrm{alg}}(\pi_1(X), \pi_1(Y)) = \infty.$$

In particular $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$ for every pair M, N of compact manifolds such that $\beta_1(M) < \beta_1(N)$.

PROOF. (a) If follows immediately from Lemma 2.4(a) by using the isomorphisms

$$H_1(X) \simeq \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}, H_1(Y) \simeq \frac{\pi_1(Y)}{[\pi_1(Y), \pi_1(Y)]}.$$

(b) Indeed $\beta_1(X) < \beta_1(Y)$ if and only if

$$\mathrm{rank}\left[\frac{H_1(X)}{t(H_1(X))}\right] < \mathrm{rank}\left[\frac{H_1(Y)}{t(H_1(Y))}\right],$$

the last inequality shows, by using Lemmas 2.3 and 2.4(b), that

$$\varphi_{\mathrm{alg}}(H_1(M), H_1(N)) = \infty. \qquad \square$$

If M is a differentiable manifold, we denote by $\sharp_r M$ the connected sum $M \sharp M \sharp \ldots \sharp M$ of r copies of M. Recall that the connected sum $\sharp_g T^2$ of g copies of the torus T^2 is also denoted by T_g and the connected sum $\sharp_g \mathbb{R}P^2$ of g copies of the projective plane $\mathbb{R}P^2$ by P_g.

EXAMPLES 2.6.

(a) If $g < g'$, then $\varphi_{\mathrm{alg}}(\pi_1(T_g), \pi_1(T_{g'})) = \infty$.
(b) If $g < g'$, then $\varphi_{\mathrm{alg}}(\pi_1(P_g), \pi_1(P_{g'})) = \infty$.
(c) If $g < 2g' + 1$, then $\varphi_{\mathrm{alg}}(\pi_1(P_g), \pi_1(T_{g'})) = \infty$.
(d) If $2g < g' - 1$, then $\varphi_{\mathrm{alg}}(\pi_1(T_g), \pi_1(P_{g'})) = \infty$.

Indeed, the statements (a)–(d) follow because $\beta_1(T_g) = 2g$ and $\beta_1(P_g) = g - 1$.

REMARK 2.7. Observe that the Examples 2.6(a), (b) can be generalized to more general manifolds M^m, N^n ($m, n \geq 3$ and M orientable) and their connected sums $\sharp_r M$, $\sharp_s N$, where r, s are chosen to satisfy the inequality $r\beta_1(M) < s\beta_1(N)$. Indeed $\varphi_{\mathrm{alg}}(\pi_1(\sharp_r M), \pi_1(\sharp_s N)) = \infty$ since $\beta_1(\sharp_r M) = r\beta_1(M)$ and $\beta_1(\sharp_r N) \geq s\beta_1(N)$, as the homology group $H_1(\sharp_r M)$ is isomorphic to the direct sum of r copies of $H_1(M)$ and $H_1(\sharp_s N)$ has a subgroup isomorphic to the direct sum of s copies of $H_1(N)$ ([6, p. 258]).

EXAMPLES 2.8.

(a) If $g < g'$, then $\dim[C(f)] \geq 1$ for every C^1 map $f\colon T_g \to T_{g'}$.
(b) If $g < 2g' + 1$, then $\dim[C(f)] \geq 1$ for every C^1 map $f\colon P_g \to T_{g'}$.
(c) If $g \geq 2$, then $\dim[C(f)] \geq 1$ for every C^1 map $f\colon P^2 \to P_g$.
(d) If $g < g'$, then $\dim[B(f)] = 1$ for every C^1 non-constant map $f\colon P_g \to P_{g'}$.
(e) If $2g < g' - 1$, then $\dim[B(f)] = 1$ for every non-constant C^1 map $f\colon T_g \to P_{g'}$.
(f) If $n > k \geq 1$ and r is an arbitrary positive integer, then $\dim[C(f)] \geq n - 1$, for every C^1 map $f\colon \sharp_r(T^k \times S^{n-k}) \to \sharp_r T^n$.
(g) If r, s are arbitrary positive integers, then the inequality $\dim[C(f)] \geq 2$ holds, for every C^1 map $f\colon \sharp_r\mathbb{R}P^3 \to \sharp_s(S^1 \times \mathbb{R}P^2)$.
(h) If r, s are arbitrary positive integers, then the inequality $\dim[C(f)] \geq 3$ holds, for every C^1 map $f\colon \sharp_r\mathbb{C}P^2 \to \sharp_s(T^2 \times \mathbb{R}P^2)$.

COROLLARY 2.9. *If M^m, N^n are compact connected manifolds such that $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$ and $m \geq n \geq 2$, then no submanifold of M of dimension less than or equal to $n - 2$ is the critical set of any differentiable mapping $f\colon M \to N$.*

PROOF. Assume that $C(f)$ is a submanifold of M and $\dim[C(f)] \leq n - 2$, for some $f\colon M \to N$. Combining Lemma 1.6 with Theorem 2.2(a) one gets the following relations

$$\dim[C(f)] \geq \dim[f(C(f))] = \dim[B(f)] \geq \dim(N) - 1. \qquad \square$$

REMARK 2.10. (a) Corollary 2.9 still works if the submanifolds are replaced with images of arbitrary differentiable mappings. For example, no union of finitely many differential images of circles in $\sharp_r\mathbb{R}P^3$ is the critical set for any C^1 map $f\colon \sharp_r\mathbb{R}P^3 \to \sharp_s(S^1 \times \mathbb{R}P^2)$. Similarly, no union of finitely many differential images of P_g's and/or of T_g's in $\sharp_r\mathbb{R}P^3$ can be the critical set for any C^1 mapping $f\colon \sharp_r\mathbb{C}P^2 \to \sharp_s(T^2 \times \mathbb{R}P^2)$.

(b) If M, E, N are three manifolds such that $\dim M \geq \dim N$ and $p\colon E \to N$ is a submersion, observe that a mapping $f\colon M \to E$ does intersect transversally the fiber $\mathcal{F}_{f(x)} := p^{-1}(p(f(x)))$ of p through $f(x)$, for some $x \in M$, if and only if $x \in R(p \circ f)$, the last regular set $R(p \circ f) = \{x \in M \mid f \pitchfork_x \mathcal{F}_{f(x)}\}$ will be called *the transversal set of f*.

COROLLARY 2.11. *Let M^n, N^n be compact connected differential manifolds such that $n \geq 2$ and $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$. If $p\colon E \to N$ is a submersion, then $\dim[C(p \circ f)] \geq n - 1$ for every C^1 map $f\colon M \to E$. In fact the transversal set $R(p \circ f)$ of f is either empty or it is is not connected.*

3. On the algebraic φ-category of some pairs of finite groups

In this section we estimate the algebraic φ-category of pairs of finite abelian groups G, H in terms of their orders and some powers of the primes within the prime decomposition of $\gcd(o(H), o(H))$.

Remark 3.1. If G, H are finite groups, then

$$\varphi_{\mathrm{alg}}(G,H) \geq \frac{o(H)}{\gcd(o(G), o(H))}.$$

Indeed, for every group homomorphism $f \in \mathrm{Hom}(G,H)$, the following relations hold

$$\begin{aligned} o(G) &= [G:\ker(f)]o(\ker(f)) = o(G/\ker(f))o(\ker(f)) = o(\mathrm{Im}(f))o\big(\ker(f)\big),\\ o(H) &= [H:\mathrm{Im}(f)]o(\mathrm{Im}(f)). \end{aligned}$$

Consequently $o(\mathrm{Im}(f))$ is a common divisor of both $o(G)$ and $o(H)$, which shows that $o(\mathrm{Im}(f))|\gcd(o(G), o(H))$ and $o(\mathrm{Im}(f)) \leq \gcd(o(G), o(H))$. Thus, for all $f \in \mathrm{Hom}(G,H)$, the following relations

$$[H:\mathrm{Im}(f)] = \frac{o(H)}{o(\mathrm{Im}(f))} \geq \frac{o(H)}{\gcd(o(G), o(H))}$$

hold, which shows that

$$\varphi_{\mathrm{alg}}(G,H) \geq \frac{o(H)}{\gcd(o(G), o(H))}.$$

Lemma 3.2. *If $G = \mathbb{Z}_{p^{\alpha_1}} \times \ldots \times \mathbb{Z}_{p^{\alpha_m}}$ and $H = \mathbb{Z}_{p^{\beta_1}} \times \ldots \times \mathbb{Z}_{p^{\beta_n}}$, $\alpha_1 \geq \ldots \geq \alpha_m$ and $\beta_1 \geq \ldots \geq \beta_n$, for some prime number p, then*

$$\frac{p^{\beta_1+\ldots+\beta_m}}{p^{\min(\alpha_1+\ldots+\alpha_m, \beta_1+\ldots+\beta_n)}} \leq \varphi_{\mathrm{alg}}(G,H) \leq \frac{p^{\beta_1+\ldots+\beta_m}}{p^{\min(\alpha_1,\beta_1)+\ldots+\min(\alpha_k,\beta_k)}},$$

where $k = \min(m,n)$.

Proof. We only have to show that

$$\varphi_{\mathrm{alg}}(G,H) \leq \frac{p^{\beta_1+\ldots+\beta_m}}{p^{\min(\alpha_1,\beta_1)+\ldots+\min(\alpha_k,\beta_k)}},$$

since the other inequality was justified, in a more general context, in Remark 3.1. In this respect we first consider the group homomorphism $f\colon \mathbb{Z}_{p^{\alpha_1}} \times \ldots \times \mathbb{Z}_{p^{\alpha_m}} \to \mathbb{Z}_{p^{\beta_1}} \times \ldots \times \mathbb{Z}_{p^{\beta_n}}$,

$$\begin{aligned} &f(x_1 + p^{\alpha_1}\mathbb{Z}, \ldots, x_m + p^{\alpha_m}\mathbb{Z})\\ &= \begin{cases} \left(\dfrac{p^{\beta_1}x_1}{p^{\min(\alpha_1,\beta_1)}} + p^{\beta_1}\mathbb{Z}, \ldots, \dfrac{p^{\beta_m}x_m}{p^{\min(\alpha_m,\beta_m)}} + p^{\beta_m}\mathbb{Z}, p^{\beta_{m+1}}\mathbb{Z}, \ldots, p^{\beta_n}\mathbb{Z}\right) & \\ & \text{if } m < n,\\ \left(\dfrac{p^{\beta_1}}{p^{\min(\alpha_1,\beta_1)}}x_1 + p^{\beta_1}\mathbb{Z}, \ldots, \dfrac{p^{\beta_n}}{p^{\min(\alpha_n,\beta_n)}}x_n + p^{\beta_n}\mathbb{Z}\right) & \text{if } m \geq n, \end{cases} \end{aligned}$$

and observe that the image of f, $\mathrm{Im}(f)$, is

$$\left\{\left(\frac{p^{\beta_1}x_1}{p^{\min(\alpha_1,\beta_1)}}+p^{\beta_1}\mathbb{Z},\dots,\frac{p^{\beta_m}x_m}{p^{\min(\alpha_m,\beta_m)}}+p^{\beta_m}\mathbb{Z},p^{\beta_{m+1}}\mathbb{Z},\dots,p^{\beta_n}\mathbb{Z}\right)\ \Big|\ 0\le x_i<p^{\min(\alpha_i,\beta_i)},\ i=\overline{1,m}\right\},$$

if $m<n$ and, for $m\ge n$, the image of f is

$$\left\{\left(\frac{p^{\beta_1}x_1}{p^{\min(\alpha_1,\beta_1)}}+p^{\beta_1}\mathbb{Z},\dots,\frac{p^{\beta_n}x_n}{p^{\min(\alpha_n,\beta_n)}}+p^{\beta_n}\mathbb{Z}\right)\ \Big|\ 0\le x_1<p^{\min(\alpha_1,\beta_1)},\dots,0\le x_n<p^{\min(\alpha_n,\beta_n)}\right\}.$$

These show that

$$o(\mathrm{Im}(f))=p^{\min(\alpha_1,\beta_1)}\cdot\ldots\cdot p^{\min(\alpha_k,\beta_k)}=p^{\min(\alpha_1,\beta_1)+\ldots+\min(\alpha_k,\beta_k)},$$

where $k=\min(m,n)$, since

$$\frac{p^{\beta_i}x_i}{p^{\min(\alpha_i,\beta_i)}}+p^{\beta_i}\mathbb{Z}\ne p^{\beta_i}\mathbb{Z}\quad\text{for } 0\le x_i<p^{\min(\alpha_i,\beta_i)}\text{ and } 0\le i\le k.$$

Consequently $o(\mathrm{Im}(f))=p^{\min(\alpha_1,\beta_1)+\ldots+\min(\alpha_k,\beta_k)}$, which shows, by means of Lagrange theorem, that

$$[H:\mathrm{Im}(f)]=o(H/\mathrm{Im}(f))=\frac{p^{\beta_1+\ldots+\beta_m}}{p^{\min(\alpha_1,\beta_1)+\ldots+\min(\alpha_k,\beta_k)}}$$

and

$$\varphi_{\mathrm{alg}}(G,H)\le\frac{p^{\beta_1+\ldots+\beta_m}}{p^{\min(\alpha_1,\beta_1)+\ldots+\min(\alpha_k,\beta_k)}}.\qquad\square$$

THEOREM 3.3. *If G, H are finite abelian groups such that $\gcd(o(G),o(H))=p_1^{r_1}\cdot\ldots\cdot p_k^{r_k}$, then*

$$\frac{o(H)}{\gcd(o(G),o(H))}\le\varphi_{\mathrm{alg}}(G,H)\le\frac{o(H)}{p_1^{\gamma_1}\ldots p_k^{\gamma_k}},$$

where $\gamma_i=\min(\alpha_1,\beta_1)+\ldots+\min(\alpha_z,\beta_z)$, $z=\min(x,y)$, and $S_i=\mathbb{Z}_{p_i^{\alpha_1}}\times\ldots\times\mathbb{Z}_{p_i^{\alpha_x}}$, $\Sigma_i=\mathbb{Z}_{p_i^{\beta_1}}\times\ldots\times\mathbb{Z}_{p_i^{\beta_y}}$ are the p_i-Sylow subgroups of G and H respectively, and the exponents $\alpha_1,\dots,\alpha_x,\beta_1,\dots,\beta_y$ satisfy $\alpha_1\ge\ldots\ge\alpha_x,\beta_1\ge\ldots\ge\beta_y$.

PROOF. If $m_i:=\alpha_1+\ldots+\alpha_x$ and $n_i:=\beta_1+\ldots+\beta_y$, we first recall that

$$o(G)=p_1^{m_1}\cdot\ldots\cdot p_k^{m_k}r,\quad o(H)=p_1^{n_1}\cdot\ldots\cdot p_k^{n_k}l,$$
$$p_i^{m_i+1}\nmid o(G),\quad p_i^{n_i+1}\nmid o(H),\quad i=\overline{1,k}\quad\text{and}\quad(r,l)=1,$$

and that $G = S_1 \times \ldots \times S_k \times R$, $H = \Sigma_1 \times \ldots \times \Sigma_k \times L$ and $o(R) = r$, $o(L) = l$. By using Lemma 3.2, there exists some group homomorphisms $f_i \in \mathrm{Hom}(S_i, \Sigma_i)$ such that

$$o(\mathrm{Im}(f_i)) = p_i^{\gamma_i} \quad \text{and} \quad [\Sigma_i : \mathrm{Im}(f_i)] = \frac{o(\Sigma_i)}{p_i^{\gamma_i}}.$$

Consider the composed group homomorphism

$$G = S_1 \times \ldots \times S_k \times R \xrightarrow{\mathrm{Proj}} S_1 \times \ldots \times S_k \xrightarrow{f_1 \times \ldots \times f_k} \Sigma_1 \times \ldots \times \Sigma_k \overset{\mathrm{Incl}}{\hookrightarrow} \Sigma_1 \times \ldots \times \Sigma_k \times L,$$

namely $f = \mathrm{Incl} \circ (f_1 \times \ldots \times f_k) \circ \mathrm{Proj}$ and show that $[H : \mathrm{Im}(f)] = o(H)/p_i^{\gamma_1} \ldots p_k^{\gamma_k}$. Indeed, we have successively:

$$\begin{aligned} o(\mathrm{Im}(f)) &= o(\mathrm{Im}(f_1 \times \ldots \times f_k)) = o(\mathrm{Im}(f_1) \times \ldots \times \mathrm{Im}(f_k)) \\ &= o(\mathrm{Im}(f_1)) \cdot \ldots \cdot o(\mathrm{Im}(f_k)) = p_1^{\gamma_1} \cdot \ldots \cdot p_k^{\gamma_k}. \end{aligned}$$

This shows that

$$[H : \mathrm{Im}(f)] = o(H/\mathrm{Im}(f)) = \frac{o(H)}{o(\mathrm{Im}(f))} = \frac{o(H)}{p_1^{\gamma_1} \cdot \ldots \cdot p_k^{\gamma_k}}. \qquad \square$$

COROLLARY 3.4. *$\varphi_{\mathrm{alg}}(\mathbb{Z}_m, \mathbb{Z}_n) = n/\gcd(m, n)$.*

PROOF. Indeed, if $o(\mathbb{Z}_m) = m = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$, $o(\mathbb{Z}_n) = n = p_1^{\beta_1} \cdot \ldots \cdot p_k^{\beta_k}$, then the p_i-Sylow subgroups of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are $\mathbb{Z}_{p_i^{\alpha_i}}$ and $\mathbb{Z}_{p_i^{\beta_i}}$, respectively. Thus $\gamma_i = \min(\alpha_i, \beta_i)$, which shows that

$$p_1^{\gamma_1} \cdot \ldots \cdot p_k^{\gamma_k} = p_1^{\min(\alpha_1, \beta_1)} \cdot \ldots \cdot p_k^{\min(\alpha_k, \beta_k)} = \gcd(m, n). \qquad \square$$

A direct proof of Corollary 3.4 appears in Hazar's master thesis [7].

THEOREM 3.5. *If M^n, N^n, $n \geq 3$ are compact orientable smooth manifolds and $f\colon M \to N$ is a smooth mapping such that $\dim[C(f)] \leq n - 3$, then*

$$|\deg(f)| \geq \varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)).$$

PROOF. According to P. T. Church [3], there exists a factorization $f = hg$ such that $g\colon M \to K^n$ is a smooth monotone map onto the smooth manifold K^n, i.e. a map with connected preimages of all points in K^n, and $h\colon K^n \to N^n$ is a smooth k-to-1 diffeo-covering. Note that each preimage of the monotone map g is actually a continuum ([8, p. 411]), as the manifold M is compact. According to P. T. Church [4], $g_*\colon \pi_1(M) \to \pi_1(K)$ is an isomorphism and $k = |\deg(f)|$. By using the theory of covering mappings, it follows that the cardinality of $h^{-1}(y)$ is $[\pi_1(N) : \mathrm{Im}(h_*)]$, for all $y \in N$. Consequently, we have successively:

$$\begin{aligned} |\deg(f)| &= k = \#[h^{-1}(y)] = [\pi_1(N) : \mathrm{Im}(h_*)] = [\pi_1(N) : \mathrm{Im}(h_* \circ g_*)] \\ &= [\pi_1(N) : \mathrm{Im}(hg)_*] = [\pi_1(N) : \mathrm{Im}(f_*)] \geq \varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)). \quad \square \end{aligned}$$

Recall that the ıtlens space $L(m; l_1, \dots, l_n)$ [6, p. 144], where $m > 1$ is an integer and $l_1, \dots, l_n$ are integers relatively prime to m, is defined to be the quotient space $S^{2n-1}/\mathbb{Z}_m$ under the free action of $\mathbb{Z}_m = \{1, e^{2\pi/m}, \dots, e^{2(m-1)\pi/m}\}$ on $S^{2n-1} \subseteq \mathbb{C}^n$ given by

$$e^{2\pi i/m}(z_1, \dots, z_m) := (e^{2\pi i l_1/m} z_1, \dots, e^{2\pi i l_n/m} z_m).$$

COROLLARY 3.6. *If* $f\colon L(r; l_1, \dots, l_n) \to L(s; q_1, \dots, q_n)$ *is a smooth mapping such that* $\dim[C(f)] \leq 2n-4$, *then* $|\deg(f)| \geq s/\gcd(r,s)$. *Equivalently, if* $0 \leq |\deg(g)| < s/\gcd(r,s)$ *for some* $g\colon L(r; l_1, \dots, l_n) \to L(s; q_1, \dots, q_m)$, *then* $\dim[C(g)] \geq 2n-3$.

PROOF. We just need to combine Corollary 3.4 with Theorem 3.5, taking into account that the lens spaces are orientable [6, p. 251] and

$$\pi_1(L(r; p_1, \dots, p_n)) = \mathbb{Z}_r.$$

In fact, if $\deg(g) = 0$, then actually the stronger inequality $\dim[C(g)] \geq 2n$ holds, according to Corollary 1.5(a). □

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