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A BORSUK-TYPE THEOREM FOR SOME CLASSES OF PERTURBED FREDHOLM MAPS

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Abstract. We prove an odd mapping theorem of Borsuk type for locally compact perturbations of Fredholm maps of index zero between Banach spaces. We extend this result to a more general class of perturbations of Fredholm maps, defined in terms of measure of noncompactness.

1. Introduction

In two recent papers (see [1], [2]) the authors and M. Furi defined a concept of topological degree for a special class of noncompact perturbations of nonlinear Fredholm maps of index zero, called *α-Fredholm maps*, between infinite dimensional real Banach spaces. The definition of these maps is based on the following two numbers (see e.g. [11], [12]) associated with a map $f\colon \Omega \to F$ from an open subset Ω of a Banach space E to a Banach space F:

$$\alpha(f) = \sup\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded}, \ \alpha(A) > 0\right\},$$
$$\omega(f) = \inf\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded}, \ \alpha(A) > 0\right\},$$

where α is the Kuratowski measure of noncompactness (see e.g. [14]).

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Roughly speaking, an α-Fredholm map is of the type $f = g - k$, where g is a nonlinear Fredholm map of index zero, k is continuous, and the inequality $\alpha(k) < \omega(g)$ is locally satisfied. The class of α-Fredholm maps includes locally compact perturbations of Fredholm maps (called *quasi-Fredholm maps*) since, when g is Fredholm and k is locally compact, one has $\alpha(k) = 0$ and $\omega(g) > 0$, locally. Moreover, this class contains locally α-contractive perturbations of the identity, where, following Darbo [8], a map k is α-contractive if $\alpha(k) < 1$.

The degree for α-Fredholm maps is a generalization of a notion of degree for quasi-Fredholm maps defined for the first time in [18] by means of the Elworthy–Tromba theory, and recently redefined in [5] by a different approach based on a natural concept of orientation for nonlinear Fredholm maps introduced in [3] and [4]. The degree for α-Fredholm maps, $\deg_{\alpha\mathrm{F}}(g, U, k)$ in symbols, is an integer associated to any *admissible triple* (g, U, k), where $f = g - k$ is an α-Fredholm map, U is an open subset of the domain of f and other conditions are satisfied (we recall the details in Section 4). As shown in [1], [2], the degree verifies classical properties, as normalization, additivity and homotopy invariance.

We point out that the degree of a quasi-Fredholm map f is independent of the representation $f = g - k$ (see Section 3 for more details). On the other hand, as the use of the notation $\deg_{\alpha\mathrm{F}}(g, U, k)$ suggests, the degree for α-Fredholm maps could depend on the representation $f = g - k$ of an α-Fredholm map. As a matter of fact, we do not know if two admissible triples (g, U, k) and $(\widetilde{g}, U, \widetilde{k})$, such that $g - k = \widetilde{g} - \widetilde{k}$, have the same degree. For this reason we will use in the sequel the expression *degree for α-Fredholm triples*.

The purpose of this paper is to prove that an odd mapping theorem of Borsuk type holds true both for quasi-Fredholm maps, and for α-Fredholm triples. A crucial tool to obtain these results is a version of the Borsuk odd mapping theorem for the Brouwer degree of maps between C^1 real manifolds, which is recalled in the next section. This result is probably known to the experts. However, since we did not find a precise reference in the literature, we explicitely give a proof.

In Section 3 we summarize the notions of orientability and degree for the class of quasi-Fredholm maps, and we prove a Borsuk-type theorem for the degree of these maps. In Section 4 we give an extended version of a Borsuk-type theorem for the degree of α-Fredholm triples. In the main result of this section, Theorem 4.8, we prove that if (g, U, k) is an admissible α-Fredholm triple such that U is symmetric with respect to $0 \in U$, and g and k are odd, then the degree of (g, U, k) is odd. Finally, in Section 5 we present a further extension of a Borsuk-type theorem for a more general class of maps, called *weakly α-Fredholm*, introduced in [2] and including the α-Fredholm maps.

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2. Preliminaries: Brouwer degree and Borsuk Theorem in finite dimension

In this section we recall some properties of the Brouwer degree for continuous maps between C^1 real manifolds. In particular, in Theorem 2.4 below we state a version of Borsuk odd mapping theorem for this degree. This result, well known for maps between Euclidean spaces, still holds in the context of manifolds and its proof has been probably published. However, as we are not able to give a reference for that result, we present a proof, following the idea used in the "flat" case (see e.g. [15]).

The version of the Brouwer degree we refer to is a slight extension of that exposed by Nirenberg in [16]. In our approach, the Brouwer degree is an integer assigned to any *admissible triple* (f, U, y), that is, any triple verifying the following conditions: $f\colon M \to N$ is a continuous map, M and N are two oriented C^1 real manifolds of the same finite dimension, U is open in M, $y \in N$ and $f^{-1}(y) \cap U$ is compact. The Brouwer degree of a triple (f, U, y) will be denoted by $\deg_B(f, U, y)$.

The classical properties of the Brouwer degree still hold in this extended version, and they can be easily obtained by a straightforward generalization of the analogous ones given in [16]. To help the reader we recall below the properties that will be explicitly used later on.

- (Additivity) Given an admissible triple (f, U, y) and two disjoint open subsets U_1, U_2 of U, such that $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$, then

$$\deg_B(f, U, y) = \deg_B(f, U_1, y) + \deg_B(f, U_2, y).$$

- (Excision) If (f, U, y) is admissible and V is an open subset of U containing $f^{-1}(y) \cap U$, then

$$\deg_B(f, V, y) = \deg_B(f, U, y).$$

- (Boundary dependence) Let (f, U, y) and (g, U, y) be two admissible triples, where U is bounded, f and g are defined on a manifold M containing $\overline{U}$ and coincide on ∂U. Assume that f and g are $\mathbb{R}^n$-valued, with $n = \dim M$. Then,

$$\deg_B(f, U, y) = \deg_B(g, U, y).$$

Remark 2.1. It is interesting to observe that the above boundary dependence property is generally false if the target space of f and g is not an Euclidean space, as the following simple example shows. Denote by S^1 the unit circle in $\mathbb{R}^2$, by H_+ the open (relatively to the topology of S^1) upper half-circle, and let $g: S^1 \to S^1$ be defined by $g(\alpha, \beta) = (\alpha, -\beta)$. It is immediate to observe that g coincides on ∂H_+ with the identity $I: S^1 \to S^1$. On the other hand, given any element $q \in H_+$, one has $\deg_B(I, H_+, q) = 1$, while $\deg_B(g, H_+, q) = 0$.

Another well known property, which will play a crucial role, is summarized in the next proposition.

Proposition 2.2. *Let M be a C^1 manifold of dimension n, embedded in a real Banach space E. Let D be an open subset of M such that $D \cap (-D) = \emptyset$. Let $f: D \cup (-D) \to \mathbb{R}^n$ be continuous and odd. Assume, in addition, that $f^{-1}(0)$ is compact. Then,*

$$\deg_B(f, D, 0) = (-1)^n \deg_B(f, -D, 0).$$

Before presenting Theorem 2.4, we need the following lemma (see e.g. [15] for the flat case).

Lemma 2.3. *Let X be a compact C^1 manifold of dimension m. Given a compact subset K of X, consider a continuous map $\phi : K \to \mathbb{R}^n$, with $n > m$, such that $\phi(x) \neq 0$ for any $x \in K$. Then, ϕ admits a continuous extension $\eta: X \to \mathbb{R}^n$ such that $\eta(x) \neq 0$ for any $x \in X$.*

Proof. By Tietze's Theorem ϕ admits a continuous extension $\phi_1: X \to \mathbb{R}^n$. Let $c = \inf\{\|\phi(x)\|, x \in K\}$, which is clearly positive. Let $\varepsilon \in (0, c/2)$ be given. Recalling that X is compact, consider a C^1 map $\phi_2: X \to \mathbb{R}^n$ such that $\|\phi_2 - \phi_1\|_\infty = \sup\{\|\phi_2(x) - \phi_1(x)\| : x \in X\} < \varepsilon/2$.

Since $m < n$, the Lebesgue measure of $\phi_2(X)$ in $\mathbb{R}^n$ is zero, so there exists $p \in \mathbb{R}^n$, with $\|p\| < \varepsilon/2$, such that $\psi := \phi_2 - p$ is nowhere zero. Observe that $\|\psi - \phi_1\|_\infty < \varepsilon$. Define $\sigma: \mathbb{R} \to \mathbb{R}$, by

$$\sigma(t) = \begin{cases} \dfrac{2t}{c} & \text{if } t < \dfrac{c}{2}, \\ 1 & \text{if } t \geq \dfrac{c}{2}. \end{cases}$$

Then, define $\beta: X \to \mathbb{R}^n$ as $\beta(x) = \psi(x)/\sigma(\|\psi(x)\|)$. It is easily seen that β coincides with ψ on K and $\|\beta(x)\| \geq c/2$ for every $x \in X$. Again by Tietze's Theorem there exists a continuous map $\alpha: X \to \mathbb{R}^n$ with $\alpha(x) = \beta(x) - \phi(x)$ for all $x \in K$ and with $\|\alpha\|_\infty \leq \varepsilon$ (since $\|\beta(x) - \phi(x)\| \leq \varepsilon$ for all $x \in K$).

Finally, define $\eta = \beta - \alpha$. The map η coincides with ϕ on K. Moreover, $\|\eta(x)\| \geq c/2 - \varepsilon > 0$, for all $x \in X$, that is, η is nowhere zero, and the proof is complete. □

Theorem 2.4 (Finite dimensional Borsuk odd mapping theorem). *Let M be an oriented C^1 manifold of dimension n, embedded in a real Banach space E. Let D be an open subset of M, symmetric with respect to the origin of E. Suppose* *that* 0 *belongs to* D. *Let* $f\colon D \to \mathbb{R}^n$ *be continuous and odd. Assume, in addition, that* $f^{-1}(0)$ *is compact. Then* $\deg_B(f, D, 0)$ *is odd.*

Proof. It is easily seen that the compact set $f^{-1}(0)$ can be covered by a finite family $\{U, A_1, \dots, A_k, B_1, \dots, B_k\}$ of open subsets of M, satisfying the following properties:

(1) The closure in M of every set of the covering is contained in D.
(2) U is a symmetric bounded open neighbourhood of the origin of E. In addition, there exists an odd diffeomorphism g between U and an open subset of $\mathbb{R}^n$.
(3) For any $i = 1, \dots, k$, $\overline{A}_i$ and $\overline{B}_i$ are diffeomorphic to a closed ball in $\mathbb{R}^n$ and do not contain 0.
(4) For each $i = 1, \dots, k$, $B_i = -A_i$ and $\overline{A}_i \cap \overline{B}_i = \emptyset$.
(5) Called $\mathcal{O}$ the union of the sets of the covering, then $\partial U \cap \partial\mathcal{O} = \emptyset$.
(6) For each $i = 1, \dots, k$, $\mathcal{O} \setminus (A_i \cup B_i) \neq \mathcal{O}$.

The existence of a covering verifying property (5) follows from of the compactness of ∂U. In addition, the same property and the fact that $\partial\mathcal{O} \neq \emptyset$ ensure that there exists at least one set A_i such that $\partial A_i \cap \partial\mathcal{O} \neq \emptyset$. Without loss of generality, let A_1 verify this condition.

Define $\phi_1\colon (\partial A_1 \cap \partial\mathcal{O}) \cup (\partial A_1 \cap \overline{U}) \to \mathbb{R}^n$ by

$$\phi_1(x) = \begin{cases} f(x) & \text{if } x \in \partial A_1 \cap \partial\mathcal{O}, \\ g(x) & \text{if } x \in \partial A_1 \cap \overline{U}. \end{cases}$$

By the above property (3), ∂A_1 is a compact boundaryless manifold of dimension $n-1$. Thus, by Lemma 2.3, ϕ_1 admits a continuous and nowhere zero extension $\widehat{\phi}_1\colon \partial A_1 \to \mathbb{R}^n$. In addition, by Tietze's Theorem and the property (4), we can extend $\widehat{\phi}_1$ to a continuous and odd $\widetilde{\phi}_1\colon \overline{A}_1 \cup \overline{B}_1 \to \mathbb{R}^n$. Then, call $V_1 = A_1 \cup U \cup B_1$ and define $\psi_1\colon \overline{V}_1 \to \mathbb{R}^n$ by

$$\psi_1(x) = \begin{cases} \widetilde{\phi}_1(x) & \text{if } x \in \overline{A}_1 \cup \overline{B}_1, \\ g(x) & \text{if } x \in \overline{U} \setminus (\overline{A}_1 \cup \overline{B}_1). \end{cases}$$

Observe that ψ_1 coincides with f on $\partial V_1 \cap \partial\mathcal{O}$. The triple $(\psi_1, V_1, 0)$ is admissible for the Brouwer degree and

$$\deg_B(\psi_1, V_1, 0) = \deg_B(\widetilde{\phi}_1, A_1, 0) + \deg_B(g, U \setminus (\overline{A}_1 \cup \overline{B}_1), 0) + \deg_B(\widetilde{\phi}_1, B_1, 0).$$

By Proposition 2.2, $\deg_B(\widetilde{\phi}_1, A_1, 0) = \pm\deg_B(\widetilde{\phi}_1, B_1, 0)$. In addition, one has that $\deg_B(g, U \setminus (\overline{A}_1 \cup \overline{B}_1), 0) = \pm 1$ since g is a diffeomorphism and $0 \notin \overline{A}_1 \cup \overline{B}_1$. Therefore, $\deg_B(\psi_1, V_1, 0)$ is odd.

Consider, as a second step, the set A_2. By the above property (6), A_2 is not contained in V_1. Suppose first that $(\partial A_2 \cap \partial\mathcal{O}) \cup (\overline{A}_2 \cap \overline{V}_1) \neq \emptyset$. In particular, if $\overline{A}_2 \cap \overline{V}_1 \neq \emptyset$, from the property (6) it follows that $\partial V_1 \cap \overline{A}_2$ is nonempty. Hence,

$(\partial A_2 \cap \partial \mathcal{O}) \cup (\partial V_1 \cap \overline{A}_2)$ is nonempty. Let $\phi_2\colon (\partial A_2 \cap \partial \mathcal{O}) \cup (\partial V_1 \cap \overline{A}_2) \to \mathbb{R}^n$ be given by

$$\phi_2(x) = \begin{cases} f(x) & \text{if } x \in \partial A_2 \cap \partial \mathcal{O}, \\ \psi_1(x) & \text{if } x \in \partial V_1 \cap \overline{A}_2. \end{cases}$$

The definition of ϕ_2 is well posed since $\psi_1(x) = f(x)$ for any $x \in \partial A_2 \cap \partial \mathcal{O} \cap \partial V_1$. It is immediate to see that ϕ_2 is nowhere zero on the compact set $C := \partial A_2 \cap (\partial \mathcal{O} \cup \partial V_1)$. Therefore, by Lemma 2.3, we can extend $\phi_2|_C$ to a continuous and nowhere zero $\widehat{\phi}_2 : \partial A_2 \to \mathbb{R}^n$. Thus, let $\overline{\phi}_2\colon \partial A_2 \cup (\partial V_1 \cap \overline{A}_2) \to \mathbb{R}^n$ be given by

$$\overline{\phi}_2 = \begin{cases} \widehat{\phi}_2(x) & \text{if } x \in \partial A_2, \\ \psi_1(x) & \text{if } x \in \partial V_1 \cap \overline{A}_2. \end{cases}$$

Clearly, $\overline{\phi}_2$ is well defined and continuous. Then, we extend it to an odd continuous map $\widetilde{\phi}_2\colon \overline{A}_2 \cup \overline{B}_2 \to \mathbb{R}^n$. Furthermore, call $V_2 = A_2 \cup V_1 \cup B_2$ and define $\psi_2\colon \overline{V}_2 \to \mathbb{R}^n$ by

$$\psi_2(x) = \begin{cases} \widetilde{\phi}_2(x) & \text{if } x \in (\overline{A}_2 \cup \overline{B}_2) \setminus \overline{V}_1, \\ \psi_1(x) & \text{if } x \in \overline{V}_1. \end{cases}$$

The definition of ϕ_2 obviously ensures the continuity of ψ_2.

Consider now the case when $(\overline{A}_2 \cap \overline{V}_1) \cup (\partial A_2 \cap \partial \mathcal{O}) = \emptyset$. In this case, let $\phi_2\colon \overline{A}_2 \to \mathbb{R}^n$ be a diffeomorphism between $\overline{A}_2$ and a closed ball of $\mathbb{R}^n$ centered at zero (recall the property (3) of the covering). Then, call $\widetilde{\phi}_2 : \overline{A}_2 \cup \overline{B}_2 \to \mathbb{R}^n$ the odd extension of ϕ_2 and define $\psi_2\colon \overline{V}_2 \to \mathbb{R}^n$ by

$$\psi_2(x) = \begin{cases} \widetilde{\phi}_2(x) & \text{if } x \in \overline{A}_2 \cup \overline{B}_2, \\ \psi_1(x) & \text{if } x \in \overline{V}_1. \end{cases}$$

In both cases one obtains that ψ_2 is continuous, odd and nowhere zero on ∂V_2. Therefore, by the additivity property of the Brouwer degree and Proposition 2.2, the triple $(\psi_2, V_2, 0)$ is admissible for the Brouwer degree and

$$\deg_B(\psi_2, V_2, 0) = \deg_B(\widetilde{\phi}_2, A_2 \setminus \overline{V}_1, 0) + \deg_B(\psi_1, V_1, 0) + \deg_B(\widetilde{\phi}_2, B_2 \setminus \overline{V}_1, 0)$$

is odd.

Iterating the process, we define an odd continuous map $\psi_k\colon \overline{\mathcal{O}} \to \mathbb{R}^n$ verifying the following conditions:

(i) ψ_k coincides with f on $\partial \mathcal{O}$;
(ii) $\deg_B(\psi_k, \mathcal{O}, 0)$ is well defined and odd.

By the boundary dependence property of the Brouwer degree we obtain that

$$\deg_B(f, \mathcal{O}, 0) = \deg_B(\psi_k, \mathcal{O}, 0) \text{ is odd}$$

and this implies that $\deg_B(f, D, 0)$ is odd applying the excision property. □

3. Borsuk Theorem for quasi-Fredholm maps

In this section we present an extension of the odd mapping theorem to locally compact perturbations of Fredholm maps of index zero between Banach spaces, called quasi-Fredholm maps. In the first part of the section we summarize the notions of orientability and degree for the quasi-Fredholm maps, introduced in [5], and we sketch the construction of the degree. Then, we prove the Borsuk-type theorem.

From now on E and F will denote two real Banach spaces. The space of bounded linear operators from E to F will be denoted by $L(E,F)$ and $\Phi_0(E,F)$ will be the open subset of Fredholm operators of index zero.

Consider an operator $L \in \Phi_0(E,F)$. A bounded linear operator $A\colon E \to F$ with finite dimensional image is called a *corrector* of L if $L+A$ is an isomorphism. On the (nonempty) set $\mathcal{C}(L)$ of correctors of L we define an equivalence relation as follows. Let $A, B \in \mathcal{C}(L)$ be given and consider the following automorphism of E:

$$T = (L+B)^{-1}(L+A) = I - (L+B)^{-1}(B-A).$$

Clearly, the image of $I-T$ has finite dimension. Hence, given any nontrivial finite dimensional subspace E_0 of E containing $\operatorname{Im}(I-T)$, the restriction of T to E_0 is an automorphism. Therefore, its determinant is nonzero and independent of the choice of E_0. This value can be defined as the determinant of T, $\det T$ in symbols (see e.g. [13]). We say that A is *equivalent* to B if $\det T > 0$.

As shown in [3], this is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes.

Definition 3.1. Let $L \in \Phi_0(E,F)$ be given. An *orientation* of L is the choice of one of the two classes of $\mathcal{C}(L)$, and L is *oriented* when an orientation is chosen. If L is oriented, the elements of its orientation are called the *positive correctors* of L.

Since the set of the isomorphisms of E into F is open in $L(E,F)$, a corrector of $L \in \Phi_0(E,F)$ is a corrector of every operator in $\Phi_0(E,F)$ close enough of L. This allows us to give the following definition.

Definition 3.2. Let X be a topological space and $h\colon X \to \Phi_0(E,F)$ a continuous map. An *orientation* of h is a choice of an orientation $\beta(x)$ of $h(x)$ for each $x \in X$, such that for any $x \in X$ there exists $A \in \beta(x)$ which is a positive corrector of $h(x')$ for any x' in a neighbourhood of x. A map is *orientable* if it admits an orientation and *oriented* when an orientation is chosen.

Remark 3.3. With an abuse of terminology we can say that, if a map h is oriented, the orientation $\beta(x)$ of $h(x)$ depends continuously on x.

By Definition 3.2 we can give a notion of orientability for Fredholm maps of index zero between Banach spaces. Recall that, given an open subset Ω of E, a C^1 map $g\colon \Omega \to F$ is *Fredholm of index n* if its Fréchet derivative, $g'(x)$, is a Fredholm operator of index n for all $x \in \Omega$.

Definition 3.4. An *orientation* of a Fredholm map of index zero $g\colon \Omega \to F$ is an orientation of the continuous map $g'\colon x \mapsto g'(x)$, and g is *orientable*, or *oriented*, if so is g' according to Definition 3.2.

The notion of orientability of Fredholm maps of index zero is accurately discussed in [3] and [4]. Here we only recall the properties which will be used in this paper. Theorem 3.6 below deals with a sort of continuous transport of an orientation along a homotopy of Fredholm maps. We need first the following definition.

Definition 3.5. Let $H\colon \Omega \times [0,1] \to F$ be a C^1 Fredholm map of index 1. We call H a *Fredholm homotopy*. For any $\lambda \in [0,1]$, denote by H_λ the partial map $x \mapsto H(x,\lambda)$, defined on Ω. An *orientation* of H is an orientation of the derivative with respect to the first variable

$$\partial_1 H\colon \Omega \times [0,1] \to \Phi_0(E,F), \quad \partial_1 H(x,\lambda) = (H_\lambda)'(x);$$

H is *orientable*, or *oriented*, if so is $\partial_1 H$ according to Definition 3.2.

Theorem 3.6. *Let $H\colon \Omega \times [0,1] \to F$ be a Fredholm homotopy. Suppose that H_λ is orientable for a given $\lambda \in [0,1]$. Then H is orientable. In addition, assume that, for some $\lambda \in [0,1]$, the partial map H_λ is oriented and call β its orientation. Then there exists a unique orientation of H, say Γ, such that $\Gamma(x,\lambda) = \beta(x)$ for any $x \in \Omega$.*

Remark 3.7. Given an oriented map $g\colon \Omega \to F$, let us show how its orientation is related to the orientations of domain and codomain of suitable restrictions of g. Call Z a finite dimensional subspace of F, transverse to g. By classical transversality results, $M := g^{-1}(Z)$ is a C^1 manifold of the same dimension as Z. Let Z be oriented. Consider $x \in M$ and a positive corrector A of $g'(x)$ with image contained in Z. Then, orient the tangent space T_xM in such a way that the isomorphism

$$(g'(x) + A)|_{T_xM}\colon T_xM \to Z$$

is orientation preserving. As proved in [3] (see in particular Remark 2.5 and Lemma 3.1 of that paper), the orientation of T_xM does not depend on the choice of the corrector A, but only on the orientations of Z and $g'(x)$. Moreover, the orientation of T_xM depends continuously on x, that is, defines an orientation on M. We will call M the *oriented g-preimage* of Z.

We are now ready to recall the concepts of orientability and degree for quasi-Fredholm maps.

Definition 3.8. Let Ω be an open subset of E, $g\colon \Omega \to F$ a Fredholm map of index zero and $k\colon \Omega \to F$ a locally compact map. The map $f : \Omega \to F$, defined by $f = g - k$, is called a *quasi-Fredholm map* and g is a *smoothing map* of f.

Definition 3.9. A quasi-Fredholm map $f\colon \Omega \to F$ is *orientable* if it has an orientable smoothing map. If f is orientable, an *orientation* of f is the choice of an orientation of any of its smoothing maps.

The above definition is well posed because, as shown in [5], if f is an orientable quasi-Fredholm map, the following facts hold:

(i) any smoothing map of f is orientable;
(ii) an orientation of a smoothing map f determines uniquely an orientation of any other smoothing map.

In the sequel, if f is oriented and g is an oriented smoothing map that determines the orientation of f, we will refer to g as a *positively oriented smoothing map* of f.

Definition 3.10. Let $f\colon \Omega \to F$ be an oriented quasi-Fredholm map and U an open subset of Ω. The triple $(f, U, 0)$ is said to be *qF-admissible* provided that $f^{-1}(0) \cap U$ is compact.

The degree is an integer valued map defined in the set of the qF-admissible triples. Let us sketch its construction, proceeding in two steps.

In the first one we consider triples $(f, U, 0)$ such that f has a smoothing map g with $(f - g)(U)$ contained in a finite dimensional subspace of F. Then we remove this assumption, defining the degree for all the qF-admissible triples.

Let $(f, U, 0)$ be a qF-admissible triple and let g be a positively oriented smoothing map of f such that $(f - g)(U)$ is contained in a finite dimensional subspace of F. As $f^{-1}(0) \cap U$ is compact, there exist a finite dimensional subspace Z of F and an open neighbourhood W of $f^{-1}(0) \cap U$ in U, such that g is transverse to Z in W. Assume that Z is oriented and contains $(f - g)(U)$. Let $M = g^{-1}(Z) \cap W$ be the oriented $g|_W$-preimage of Z (recall Remark 3.7).

One can easily verify that $(f|_M)^{-1}(0) = f^{-1}(0) \cap U$. Thus $(f|_M)^{-1}(0)$ is compact and the Brouwer degree of the triple $(f|_M, M, 0)$ is well defined. Then, the degree of $(f, U, 0)$ is given by

$$\deg_{\mathrm{qF}}(f, U, 0) = \deg_B(f|_M, M, 0). \tag{3.1}$$

As proved in [5], this definition is well posed since the right hand side of (3.1) is independent of the choice of the smoothing map g, the open set W and the subspace Z.

To define the degree of a general qF-admissible triple $(f,U,0)$, let us consider:

(1) a positively oriented smoothing map g of f;
(2) an open neighbourhood V of $f^{-1}(0)\cap U$ such that $\overline{V}\subseteq U$, g is proper on $\overline{V}$ and $(f-g)|_{\overline{V}}$ is compact;
(3) a continuous map $\xi:\overline{V}\to F$ having bounded finite dimensional image and such that

$$\|g(x)-f(x)-\xi(x)\|<\rho,\quad \text{for all } x\in\partial V,$$

where ρ is the distance in F between 0 and $f(\partial V)$.

Then, the degree of $(f,U,0)$ is

$$\deg_{\mathrm{qF}}(f,U,0)=\deg_{\mathrm{qF}}(g-\xi,V,0). \tag{3.2}$$

In [5] it is shown that formula (3.2) is well posed since the right hand side does not depend on g, ξ and V.

Finally, the degree for quasi-Fredholm maps verifies classical properties in degree theory, as additivity and homotopy invariance. The reader can find details in [5].

We are now in the position to prove a Borsuk-type theorem for the class of quasi-Fredholm maps.

Theorem 3.11. *Let $(f,U,0)$ be a qF-admissible triple. Suppose that $0\in U$ and that U is symmetric with respect to 0. In addition, assume that f is odd on U. Then $\deg_{\mathrm{qF}}(f,U,0)$ is odd.*

Proof. By the definition of degree for quasi-Fredholm maps, one has

$$\deg_{\mathrm{qF}}(f,U,0)=\deg_{\mathrm{qF}}(g-\xi,V,0),$$

where g, ξ and V verify conditions (1)–(3) above. Let us observe that V can be chosen symmetric with respect to the origin because so is $f^{-1}(0)$. In addition, without loss of generality, we can assume g and ξ to be odd, since they can be replaced, if necessary, by $(g(x)-g(-x))/2$ and $(\xi(x)-\xi(-x))/2$, respectively.

Let now Z be a finite dimensional subspace of F and W an open neighbourhood of $(g-\xi)^{-1}(0)\cap V$, symmetric with respect to the origin and such that g is transverse to Z in W. We may assume that Z contains $\xi(U)$. Let $M=g^{-1}(Z)\cap W$. We orient Z and consequently M in such a way that this latter is the oriented $g|_W$-preimage of Z. Thus,

$$\deg_{\mathrm{qF}}(g-\xi,V,0)=\deg_B(f|_M,M,0)$$

(recall formula (3.1)). Moreover, as it is not difficult to prove, M turns out to be symmetric with respect to the origin. Hence, Theorem 2.4 can be applied and this concludes the proof. □

4. Borsuk theorem for α-Fredholm triples

We present here another infinite dimensional version of the odd mapping theorem of Borsuk type. Precisely, we extend the result of the above section to the more general class of α-Fredholm maps. After recalling the definition of α-Fredholm maps, we summarize the construction of the degree for the α-Fredholm triples. Finally we present the Borsuk-type theorem for this degree.

Let us begin by recalling the definition of the Kuratowski measure of noncompactness together with some related concepts. For general references see e.g. [9] or [14]. From now on the Banach spaces E and F are assumed to be infinite dimensional.

The *Kuratowski measure of noncompactness* $\alpha(A)$ of a bounded subset A of E is defined as the infimum of real numbers $d > 0$ such that A admits a finite covering by sets of diameter less than d. If A is unbounded, we set $\alpha(A) = \infty$.

Given an open subset Ω of E and a continuous map $f\colon \Omega \to F$, we recall the definition of the following two extended real numbers (see e.g. [12]) associated with f:

$$\begin{aligned} \alpha(f) &= \sup\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded, } \alpha(A) > 0\right\}, \\ \omega(f) &= \inf\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded, } \alpha(A) > 0\right\}. \end{aligned}$$

We point out that $\alpha(f) = 0$ if and only if f is completely continuous and $\omega(f) > 0$ only if f is proper on bounded closed sets. For a comprehensive list of properties of $\alpha(f)$ and $\omega(f)$ we refer to [12].

Proposition 4.1. *Let $L\colon E \to F$ be a bounded linear operator. Then, $\omega(L) > 0$ if and only if $\mathrm{Im}L$ is closed and $\dim \mathrm{Ker}L < \infty$.*

As a consequence of Proposition 4.1 one gets that a bounded linear operator L is Fredholm if and only if $\omega(L) > 0$ and $\omega(L^*) > 0$, where L^* is the adjoint of L.

Let $p \in \Omega$ be fixed. We recall the definitions of $\alpha_p(f)$ and $\omega_p(f)$ given in [6] (see also [7]). Roughly speaking, these numbers are the local analogues of $\alpha(f)$ and $\omega(f)$. Let $B(p,s)$ denote the open ball in E centered at p with radius $s > 0$. Define

$$\alpha_p(f) = \lim_{s\to 0} \alpha(f|_{B(p,s)}) \quad \text{and} \quad \omega_p(f) = \lim_{s\to 0} \omega(f|_{B(p,s)}).$$

Clearly, $\alpha_p(f) \le \alpha(f)$ and $\omega_p(f) \ge \omega(f)$. With only minor changes, it is easy to show that the main properties of α and ω hold for α_p and ω_p (see [6] for details).

In the case of a bounded linear operator $L\colon E \to F$, the numbers $\alpha_p(L)$ and $\omega_p(L)$ do not depend on the point p and coincide with $\alpha(L)$ and $\omega(L)$, respectively.

Proposition 4.2 ([6]). *Let $f:\Omega \to F$ be of class C^1. Then, for any $p \in \Omega$ we have $\alpha_p(f) = \alpha(f'(p))$ and $\omega_p(f) = \omega(f'(p))$.*

If $f:\Omega \to F$ is a Fredholm map, as a straightforward consequence of Propositions 4.1 and 4.2, we obtain $\omega_p(f) > 0$ for any $p \in \Omega$.

Based on the two numbers α_p and ω_p, we can now recall the definition of α-Fredholm map.

Definition 4.3. an *α-Fredholm map* $f:\Omega \to F$ is of the form $f = g - k$, where g is a Fredholm map of index zero, k is continuous and $\alpha_p(k) < \omega_p(g)$ for every $p \in \Omega$.

As already pointed out, a quasi-Fredholm map $f = g - k$ is also α-Fredholm since $\omega_p(g) > 0$ and $\alpha_p(k) = 0$ for any p in the domain of f.

Now we sketch the construction of the degree for α-Fredholm triples, reporting, in particular, some details which will be useful in the proof of Theorem 4.8 below. We recall first the definition of *admissible α-Fredholm triple*.

Definition 4.4. Let $g:\Omega \to F$ be an oriented Fredholm map of index zero, $k:\Omega \to F$ a continuous map, and U an open subset of Ω such that $\alpha_p(k) < \omega_p(g)$ for any $p \in U$. The triple (g, U, k) is said to be an *admissible α-Fredholm triple* if the *solution set* $S = \{x \in U : g(x) = k(x)\}$ is compact.

Definition 4.5. Let (g, U, k) be an admissible α-Fredholm triple and consider a finite covering $\mathcal{V} = \{V_1, \dots, V_N\}$ of open balls of its solution set S. We say that $\mathcal{V}$ is an *α-covering* of S, relative to (g, U, k), if, for any $i \in \{1, \dots, N\}$, the following conditions hold:

(a) the ball $\widetilde{V}_i$ of double radius and same center as V_i is contained in U;
(b) $\alpha(k|_{\widetilde{V}_i}) < \omega(g|_{\widetilde{V}_i})$.

Let (g, U, k) be an admissible α-Fredholm triple and $\mathcal{V} = \{V_1, \dots, V_N\}$ an α-covering of the solution set S. We define the following sequence $\{C_n\}$ of convex closed subsets of E:

$$C_1 = \overline{\mathrm{co}}\left(\bigcup_{i=1}^{N}\{x \in V_i : g(x) \in k(\widetilde{V}_i)\}\right) \tag{4.1}$$

and, inductively,

$$C_n = \overline{\mathrm{co}}\left(\bigcup_{i=1}^{N}\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_{n-1})\}\right), \quad n \geq 2.$$

One can prove, by induction, that $C_{n+1} \subseteq C_n$ and $S \subseteq C_n$ for any $n \geq 1$. Then, the set

$$C_\infty = \bigcap_{n \geq 1} C_n$$

turns out to be closed, convex, and containing S. Consequently, if S is nonempty, so is C_∞. In addition C_∞ verifies the following two properties (see [1] for the proof):

(1) $\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_\infty)\} \subseteq C_\infty$, for any $i = 1, \dots, N$;
(2) C_∞ is compact.

Definition 4.6. Let (g, U, k) be an admissible α-Fredholm triple, $\mathcal{V} = \{V_1, \dots, V_N\}$ an α-covering of the solution set S, and C a compact convex set. We say that $(\mathcal{V}, C)$ is an *α-pair* (relative to (g, U, k)) if the following properties hold:

(a) $U \cap C \neq \emptyset$;
(b) $C_\infty \subseteq C$;
(c) $\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C)\} \subseteq C$ for any $i = 1, \dots, N$.

Let (g, U, k) be an admissible α-Fredholm triple and let $(\mathcal{V}, C)$ be an α-pair. Consider a retraction $r \colon E \to C$, whose existence is ensured by Dugundji's Extension Theorem (see e.g. [10]). Denote $V = \bigcup_{i=1}^N V_i$, where $\{V_1, \dots, V_N\} = \mathcal{V}$, and let W be a (possibly empty) open subset of V containing S such that, for any i, $r(x) \in \widetilde{V}_i$ if $x \in W \cap V_i$. Observe that the triple $(g - kr, W, 0)$ is qF-admissible (recall Definition 3.10). Hence, we define the degree of the triple (g, U, k) as

$$\deg_{\alpha\mathrm{F}}(g, U, k) = \deg_{\mathrm{qF}}(g - kr, W, 0), \tag{4.2}$$

where the right hand side is the degree for quasi-Fredolm maps, recalled in Section 3.

In [1] it is proved that this definition is well posed since the right hand side of equality (4.2) is independent of the choice of the α-pair $(\mathcal{V}, C)$, of the retraction r and of the open set W.

The degree for α-Fredholm triples verifies classical properties in degree theory, such as additivity and homotopy invariance. The reader can find detailed proofs in [1].

We conclude this section proving a Borsuk-type theorem for the degree for α-Fredholm triples. We start with the following result whose proof can be found e.g. in [17].

Lemma 4.7. *Let C be a closed, convex set in a Banach space E. Assume that C is symmetric with respect to the origin of E. Then there is an odd retraction $r \colon E \to C$.*

Let us now state the Borsuk-type theorem.

Theorem 4.8. *Let (g, U, k) be an admissible α-Fredholm triple. Suppose that $0 \in U$ and that U is symmetric with respect to 0. In addition, assume that g and k are odd mappings. Then, $\deg_{\alpha\mathrm{F}}(g, U, k)$ is odd.*

Proof. Observe first that the set $S = \{x \in U : g(x) = k(x)\}$ is symmetric with respect to $0 \in S$ since g and k are odd mappings. Let $\mathcal{V} = \{V_1, \ldots, V_N\}$ be an α-covering of S. Without loss of generality, assume that if $V_i \in \mathcal{V}$, then $-V_i \in \mathcal{V}$.

Consider now the set C_1, defined as in (4.1). It is easy to see that it is symmetric with respect to 0, because g and k are odd and the inclusion of V_i in $\mathcal{V}$ implies that $-V_i \in \mathcal{V}$. By induction, one has that any C_n is symmetric with respect to 0 as well. Consequently, C_∞ turns out to be symmetric with respect to 0, being the intersection of symmetric sets.

Let $r: E \to C_\infty$ be an odd retraction, whose existence is ensured by Lemma 4.7. Denote $V = \bigcup_{i=1}^N V_i$, where $\{V_1, \ldots, V_N\} = \mathcal{V}$. Observe that V is symmetric with respect to the origin of E, which belongs to V. Let W be an open subset of V containing S such that, for any i, $r(x) \in \widetilde{V}_i$ if $x \in W \cap V_i$. Moreover, we can assume that W is symmetric with respect to the origin of E, which belongs to W. Then, by definition of the degree,

$$\deg_{\alpha\mathrm{F}}(g, U, k) = \deg_{\mathrm{qF}}(g - kr, W, 0).$$

Therefore, the assertion follows from the Borsuk-type theorem for quasi-Fredholm maps (Theorem 3.11). □

5. An extension of Borsuk theorem to weakly α-Fredholm triples

We conclude the paper presenting an extension of the odd mapping theorem to a more general class of maps, called *weakly α-Fredholm*. In [2] these maps have been introduced and a concept of degree for them has been defined. Let us sketch the construction of that degree.

A weakly α-Fredholm map $f: \Omega \to F$ is of the form $f = g - k$, where g is Fredholm of index zero, k is continuous and the following condition is verified: for any $p \in \Omega$ there exists $s > 0$ such that for any $A \subseteq B(p, s)$ with $\alpha(A) > 0$ we have $\alpha(k(A)) < \omega_p(g)\alpha(A)$. Notice that α-Fredholm maps are also weakly α-Fredholm.

Definition 5.1. Let $g: \Omega \to F$ be a Fredholm map of index zero, $k: \Omega \to F$ a continuous map and U an open subset of Ω. The triple (g, U, k) is said to be *weakly α-Fredholm* if for any $p \in U$ there exists $s > 0$ such that for any $A \subseteq B(p, s)$ with $\alpha(A) > 0$ we have

$$\alpha(k(A)) < \omega_p(g)\alpha(A).$$

Let (g, U, k) be a weakly α-Fredholm triple. As a consequence of Definition 5.1, given $p \in U$, there exists $s > 0$ such that $\alpha(k(A)) < \alpha(g(A))$, for any $A \subseteq B(p, s)$ with $\alpha(A) > 0$. Thus, any compact subset of U admits a neighbourhood as in the following definition.

Definition 5.2. Let (g, U, k) be a weakly α-Fredholm triple, and Q a compact subset of U. An open neighbourhood V of Q is said to be an *α-neighbourhood* of Q (relative to (g, U, k)) if the following conditions hold:

(a) $\overline{V} \subseteq U$ and $k(\overline{V})$ is bounded;
(b) $\alpha(k(A)) < \alpha(g(A))$, for any $A \subseteq \overline{V}$ with $\alpha(A) > 0$.

We recall now the concept of *admissible weakly α-Fredholm triple*.

Definition 5.3. A weakly α-Fredholm triple (g, U, k) is said to be *admissible* if

(a) g is oriented;
(b) the *solution set* $S = \{x \in U : g(x) = k(x)\}$ is compact.

In [2] it is proved that, given an admissible weakly α-Fredholm triple (g, U, k) and an α-neighbourhood V of S, the triple $(g, V, (1-\varepsilon)k)$ is an admissible α-Fredholm triple, for $\varepsilon > 0$ sufficiently small. Moreover, for ε small, the degree $\deg_{\alpha\mathrm{F}}(g, V, (1-\varepsilon)k)$ is constant. Thus, the following definition makes sense.

Definition 5.4. Let (g, U, k) be an admissible weakly α-Fredholm triple, and V an α-neighbourhood of the solution set S. Let $\varepsilon_0 > 0$ be such that $\deg_{\alpha\mathrm{F}}(g, V, (1-\varepsilon)k)$ is constant, for $0 < \varepsilon < \varepsilon_0$. Then, define

$$\deg_{\mathrm{wF}}(g, U, k) = \deg_{\alpha\mathrm{F}}(g, V, (1-\varepsilon)k), \quad 0 < \varepsilon < \varepsilon_0.$$

As shown in [2], the above definition is well posed and the degree for weakly α-Fredholm maps verifies the classical properties in degree theory.

Let us now state a Borsuk-type theorem for weakly α-Fredholm maps.

Theorem 5.5. *Let (g, U, k) be an admissible weakly α-Fredholm triple. Suppose that $0 \in U$ and that U is symmetric with respect to 0. In addition, assume that g and k are odd. Then, $\deg_{\mathrm{wF}}(g, U, k)$ is odd.*

Proof. It is a consequence of Definition 5.4 and of Theorem 4.8. □

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