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NONCONVEX PERTURBATIONS OF SECOND ORDER MAXIMAL MONOTONE DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove the existence of solutions for a two point boundary value problem for a second order differential inclusion governed by a maximal monotone operator with a mixed semicontinuous perturbation.

1. Introduction

Existence of solutions for second order differential inclusions of the form $-\ddot{u}(t) \in A(t)u(t) + F(t, u(t), \dot{u}(t))$ with three point boundary conditions has been studied in [2], where $A(t): E \rightrightarrows E$, $(t \in [0, 1])$ is a maximal monotone operator and $F: [0, 1] \times E \times E \rightrightarrows E$ is a nonempty convex compact valued multifunction, Lebesgue-measurable on $[0, 1]$ and upper semicontinuous on $E \times E$. There are several results concerning the first order differential inclusions governed by maximal monotone operators with several classes of perturbations (see [8]–[11]).

The existence of solutions of a number of differential inclusions with the boundary conditions

$$\begin{cases} a_1 u(t_0) - a_2 \dot{u}(t_0) = c_1, \\ b_1 u(T) + b_2 \dot{u}(T) = c_2, \end{cases} \tag{1.1}$$

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have been discussed in the literature, see for example [6], [12] and the references therein, with $a_1, a_2, b_1, b_2 \geq 0$, $a_1 + b_1 > 0$ and $a_2 + b_2 > 0$, which is a sufficient condition to be able to construct a Green's function for the boundary value problem in consideration.

We will be concerned, in this work, with the existence of solutions of the perturbed second order differential inclusion governed by a maximal monotone operator of the form

$$-\ddot{u}(t) \in A(t)u(t) + F(t, u(t), \dot{u}(t)), \quad \text{for a.e. } t \in [0,1],$$

satisfying the boundary conditions (1.1) where $a_2 = b_2 = c_1 = c_2 = 0$ and $a_1 = b_1 = 1$, $t_0 = 0$ and $T = 1$ and where F is a measurable multifunction with nonempty closed values satisfying the following mixed semicontinuity condition: for every $t \in [0,1]$, at each $(x,y) \in E \times E$ such that $F(t,x,y)$ is convex the multifunction $F(t,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous on $E \times E$ and whenever $F(t,x,y)$ is not convex the multifunction $F(t,\,\cdot\,,\,\cdot\,)$ is lower semicontinuous on some neighbourhood of (x,y).

Many existence results for problems with mixed semicontinuous perturbations have been studied in the literature see for example [1], [3], [4], [13], [15] and [16].

2. Notation and preliminaries

Throughout $(E, \|\cdot\|)$ is a finite dimensional space, $\overline{\mathrm{B}}_E(0,r)$ is the closed ball of E of center 0 and radius $r > 0$, $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue-measurable sets of $[0,1]$ and $\mathcal{B}(E)$ is the σ-algebra of Borel subsets of E. By $\mathrm{L}^1_E([0,1])$ we denote the space of all Lebesgue–Bochner integrable E-valued mappings defined on $[0,1]$.

Let $\mathrm{C}_E([0,1])$ be the Banach space of all continuous mappings $u\colon [0,1] \to E$, endowed with the sup norm, and $\mathrm{C}^1_E([0,1])$ be the Banach space of all continuous mappings $u\colon [0,1] \to E$ with continuous derivative, equipped with the norm

$$\|u\|_{\mathrm{C}^1} = \max\Big\{ \max_{t\in[0,1]} \|u(t)\|, \max_{t\in[0,1]} \|\dot{u}(t)\| \Big\}.$$

Recall that a mapping $v\colon [0,1] \to E$ is said to be scalarly derivable when there exists some mapping $\dot{v}\colon [0,1] \to E$ (called the weak derivative of v) such that, for every $x' \in E'$, the scalar function $\langle x', v(\,\cdot\,)\rangle$ is derivable and its derivative is equal to $\langle x', \dot{v}(\,\cdot\,)\rangle$. The weak derivative $\ddot{v}$ of $\dot{v}$ when it exists is the weak second derivative.

By $\mathrm{W}^{2,1}_E([0,1])$ we denote the space of all continuous mappings $u \in \mathrm{C}_E([0,1])$ such that their first usual derivatives are continuous and scalarly derivable and $\ddot{u} \in \mathrm{L}^1_E([0,1])$.

Recall that a multivalued operator $A\colon E \rightrightarrows E$ is monotone if, for each $\lambda > 0$, and for each x_1, $x_2 \in D(A)$, $y_1 \in Ax_1$, $y_2 \in Ax_2$, we have

$$\|x_1 - x_2\| \leq \|(x_1 - x_2) + \lambda(y_1 - y_2)\|. \tag{2.1}$$

Furthermore, if $\mathcal{R}(I_E + \lambda A) = E$ we said that A is a maximal monotone operator, where $D(A) = \{x \in E : Ax \neq \emptyset\}$ and $\mathcal{R}(A) = \bigcup_{x\in E} Ax$.

Proposition 2.1. *If $A\colon E \rightrightarrows E$ is monotone and $\lambda > 0$, then*

(a) *$J_\lambda A$ is a single-valued mapping and, for each $x, y \in \mathcal{R}(I_E + \lambda A)$,*

$$\|J_\lambda Ax - J_\lambda Ay\| \leq \|x - y\|; \tag{2.2}$$

(b) *A_λ is single-valued, monotone and Lipschitz continuous on $\mathcal{R}(I_E + \lambda A)$ with Lipschitz constant $2/\lambda$;*

(c) *$A_\lambda x \in AJ_\lambda Ax$ for each $x \in \mathcal{R}(I_E + \lambda A)$;*

(d)

$$\frac{1}{\lambda}\|J_\lambda Ax - x\| = \|A_\lambda x\| \leq |Ax|_0 = \inf\{\|y\|,\ y \in Ax\}, \tag{2.3}$$

for all $x \in \mathcal{R}(I_E + \lambda A) \cap D(A)$ where I_E is the identity operator in E, $J_\lambda A = (I_E + \lambda A)^{-1}$ is the resolvent of A, and $A_\lambda = (I_E - J_\lambda A)/\lambda$ is the Yosida approximation of A.

Theorem 2.2. *Let E be a Banach space which has his topological dual uniformly convex. Then the graph of all maximal monotone operator $A\colon E \rightrightarrows E$ is strongly-weakly sequentially closed.*

Lemma 2.3. *Suppose that H is a separable Hilbert space and $A(t)\colon H \rightrightarrows H$, ($t \in [0,1]$) is a maximal monotone operator satisfying the assumption:*

(H) *For every $x \in H$ and for every $\lambda > 0$, the mapping $t \mapsto (I_H + \lambda A(t))^{-1}x$ is Lebesgue-measurable and there exists $\overline{g} \in \mathrm{L}^2_H([0,1])$ such that $t \mapsto (I_H + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $\mathrm{L}^2_H([0,1])$.*

Let (u_n) and (v_n) be sequences in $\mathrm{L}^2_H([0,1])$ satisfying:

(a) *(u_n) converges strongly to $u \in \mathrm{L}^2_H([0,1])$ and (v_n) converges to $v \in \mathrm{L}^2_H([0,1])$ with respect to the weak topology $\sigma(\mathrm{L}^2_H, \mathrm{L}^2_H)$;*

(b) *$v_n(t) \in A(t)u_n(t)$ for all n and all $t \in [0,1]$.*

Then we have $v(t) \in A(t)u(t)$ for almost every $t \in [0,1]$.

Proof. We include the proof of this lemma for the convenience of the reader. Let $\mathcal{A}\colon \mathrm{L}^2_H([0,1]) \rightrightarrows \mathrm{L}^2_H([0,1])$ be the operator defined by

$$v \in \mathcal{A}u \Leftrightarrow v(t) \in A(t)u(t) \quad \text{for a.e. } t \in [0,1].$$

$\mathcal{A}$ is a monotone operator. Indeed, let $u_1, u_2 \in D(\mathcal{A})$, $v_1 \in \mathcal{A}u_1, v_2 \in \mathcal{A}u_2$, $t \in [0,1]$ and $\lambda > 0$, we have $u_1(t), u_2(t) \in D(A(t))$ for all $t \in [0,1]$ and

$$\begin{aligned}\|u_1 - u_2\|^2_{\mathrm{L}^2_H([0,1])} &= \int_0^1 \|(u_1(t) - u_2(t))\|^2\,dt \\ &\le \int_0^1 \|u_1(t) - u_2(t) + \lambda(v_1(t) - v_2(t)\|^2\,dt \\ &= \|u_1 - u_2 + \lambda(v_1 - v_2)\|^2_{\mathrm{L}^2_H([0,1])},\end{aligned}$$

using (2.1). Let us prove now that $\mathcal{A}$ is a maximal monotone operator, that is, for all $\lambda > 0$

$$\mathcal{R}(I_{\mathrm{L}^2_H} + \lambda\mathcal{A}) = \mathrm{L}^2_H([0,1]).$$

Let $\lambda > 0$ and let $g \in \mathrm{L}^2_H([0,1])$. By the asumption (H), there exists $\overline{g} \in \mathrm{L}^2_H([0,1])$ such that the mapping $\overline{h}\colon t \mapsto (I_H + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $\mathrm{L}^2_H([0,1])$.

Consider the mapping $h\colon t \mapsto (I_H + \lambda A(t))^{-1}g(t)$. Using the fact that $(I_H + \lambda A(t))^{-1}$ is nonexpansive (see the relation (2.2)), we obtain

$$\|h\|_{\mathrm{L}^2_H([0,1])} \le \|g - \overline{g}\|_{\mathrm{L}^2_H([0,1])} + \|\overline{h}\|_{\mathrm{L}^2_H([0,1])}.$$

Since g, $\overline{g}$ and $\overline{h}$ belong to $\mathrm{L}^2_H([0,1])$, we conclude that h is Lebesgue-measurable and belongs to $\mathrm{L}^2_H([0,1])$, and furthermore,

$$\begin{aligned} & h(t) = (I_H + \lambda A(t))^{-1}g(t) && \text{for all } t \in [0,1] \\ \Leftrightarrow\ & g(t) \in (I_H + \lambda A(t))h(t) && \text{for all } t \in [0,1] \\ \Leftrightarrow\ & g \in (h + \lambda\mathcal{A}h) \\ \Leftrightarrow\ & g \in (I_{\mathrm{L}^2_H} + \lambda\mathcal{A})h \\ \Rightarrow\ & \mathcal{R}(I_{\mathrm{L}^2_H} + \lambda\mathcal{A}) = \mathrm{L}^2_H([0,1]).\end{aligned}$$

Thus $\mathcal{A}$ is a maximal monotone operator in the Hilbert space $\mathrm{L}^2_H([0,1])$, by Theorem 2.2, its graph is strongly-weakly sequentially closed. As $u_n \to u$ strongly and $v_n \to v$ weakly in $\mathrm{L}^2_H([0,1])$, we conclude that $v \in \mathcal{A}u$ that is, $v(t) \in A(t)u(t)$ almost everywhere. □

We refer the reader to [5], [7] and [17] for the theory of maximal monotone operators.

3. Main results

We begin this section by a useful lemma which summarizes some properties of some Green type function. See [2], [6] and [14].

LEMMA 3.1. *Let E be a separable Banach space, E' its topological dual and let $G\colon [0,1]\times[0,1]\to\mathbb{R}$ be the function defined by*

$$G(t,s)=\begin{cases}(t-1)s & \text{if } 0\le s\le t,\\ t(s-1) & \text{if } t\le s\le 1.\end{cases}$$

Then the following assertions hold:

(a) *If $u\in \mathrm{W}^{2,1}_E([0,1])$ with $u(0)=u(1)=0$, then*

$$u(t)=\int_0^1 G(t,s)\ddot{u}(s)\,ds \quad \text{for all } t\in[0,1]. \tag{3.1}$$

(b) *$G(\,\cdot\,,s)$ is derivable on $[0,1]$, for every $s\in[0,1]$, and its derivative is given by*

$$\frac{\partial G}{\partial t}(t,s)=\begin{cases}s & \text{if } 0\le s\le t,\\ (s-1) & \text{if } t< s\le 1.\end{cases}$$

(c) *$G(\,\cdot\,,\,\cdot\,)$ and $\frac{\partial G}{\partial t}(\,\cdot\,,\,\cdot\,)$ satisfy*

$$\sup_{t,s\in[0,1]}|G(t,s)|\le 1,\qquad \sup_{t,s\in[0,1]}\left|\frac{\partial G}{\partial t}(t,s)\right|\le 1. \tag{3.2}$$

(d) *For $f\in \mathrm{L}^1_E([0,1])$ and for the mapping $u_f\colon[0,1]\to E$ defined by*

$$u_f(t)=\int_0^1 G(t,s)f(s)\,ds \quad \text{for all } t\in[0,1], \tag{3.3}$$

one has $u_f(0)=u_f(1)=0$. Furthermore, the mapping u_f is derivable, and its derivative $\dot{u}_f$ satisfies

$$\lim_{h\to 0}\frac{u_f(t+h)-u_f(t)}{h}=\dot{u}_f(t)=\int_0^1\frac{\partial G}{\partial t}(t,s)f(s)\,ds, \tag{3.4}$$

for all $t\in[0,1]$. Consequently, $\dot{u}_f$ is a continuous mapping from $[0,1]$ into E.

(e) *The mapping $\dot{u}_f$ is scalarly derivable, that is, there exists a mapping $\ddot{u}_f\colon[0,1]\to E$ such that, for every $x'\in E'$, the scalar function $\langle x',\dot{u}_f(\,\cdot\,)\rangle$ is derivable, with $\frac{d}{dt}\langle x',\dot{u}_f(t)\rangle=\langle x',\ddot{u}_f(t)\rangle$. Furthermore*

$$\ddot{u}_f=f \quad \text{a.e. on } [0,1]. \tag{3.5}$$

Let us mention a useful consequence of Lemma 3.1.

PROPOSITION 3.2. *Let E be a separable Banach space and let $f\colon[0,1]\to E$ be a continuous mapping (respectively, a mapping in $\mathrm{L}^1_E([0,1])$). Then the mapping*

$$u_f(t)=\int_0^1 G(t,s)f(s)\,ds \quad \text{for all } t\in[0,1],$$

is the unique $\mathrm{C}^2_E([0,1])$*-solution* (*respectively,* $\mathrm{W}^{2,1}_E([0,1])$*-solution*) *to the differential equation*

$$\begin{cases} \ddot{u}(t) = f(t) & \text{for all } t \in [0,1], \\ u(0) = u(1) = 0. \end{cases}$$

Now we are able to give our first main result.

Theorem 3.3. *Let E be a finite dimensional space, $A(t)\colon E \rightrightarrows E$, $(t \in [0,1])$, be a maximal monotone operator and $F\colon [0,1] \times E \times E \rightrightarrows E$ be a closed valued multifunction, satisfying the following assumptions:*

(a) *F is $\mathcal{L}([0,1]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable;*
(b) *for every $t \in [0,1]$, at each $(x,y) \in E \times E$ such that $F(t,x,y)$ is convex $F(t,\cdot\,,\cdot\,)$ is upper semicontinuous, and whenever $F(t,x,y)$ is not convex $F(t,\cdot\,,\cdot\,)$ is lower semicontinuous on some neighbourhood of (x,y);*
(c) *$F(t,x,y) \subset \rho_1(t)\overline{\mathrm{B}}_E(0,1)$ for all $(t,x,y) \in [0,1] \times E \times E$, for some nonnegative function $\rho_1 \in \mathrm{L}^2_{\mathbb{R}}([0,1])$.*

Suppose that the following assumptions are also satisfied:

(H1) *For every $x \in E$ and for every $\lambda > 0$, the mapping $t \mapsto (I_E + \lambda A(t))^{-1}x$ is Lebesgue-measurable and there exists $\overline{g} \in \mathrm{L}^2_E([0,1])$ such that $t \mapsto (I_E + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $\mathrm{L}^2_E([0,1])$;*
(H2) *there is a nonnegative function $m_2 \in \mathrm{L}^2_{\mathbb{R}}([0,1])$ such that*

$$|A(t)x|_0 \le m_2(t) \quad \text{for all } (t,x) \in [0,1] \times E.$$

Then, there is a $\mathrm{W}^{2,1}_E([0,1])$-solution to the problem:

$$(\mathrm{P}_F) \qquad \begin{cases} -\ddot{u}(t) \in A(t)u(t) + F(t,u(t),\dot{u}(t)) & \text{for a.e. } t \in [0,1], \\ u(0) = u(1) = 0. \end{cases}$$

For the proof of our theorem we will need the following result which is a direct consequence of Theorem 2.1 in [16].

Theorem 3.4. *Let $M\colon [0,1] \times E \times E \rightrightarrows E$ be a closed valued multifunction satisfying hypotheses* (a), (b) *of Theorem* 3.3 *and the following one:*

(d) *there exits a Carathéodory function $\zeta\colon [0,1] \times E \times E \to \mathbb{R}^+$ which is integrably bounded and such that $M(t,x,y) \bigcap \overline{\mathrm{B}}_E(0,\zeta(t,x,y)) \neq \emptyset$ for all $(t,x,y) \in [0,1] \times E \times E$.*

Then for any $\varepsilon > 0$ and any compact set $K \subset \mathrm{C}^1_E([0,1])$ there is a nonempty closed convex valued multifunction $\Phi\colon K \rightrightarrows \mathrm{L}^1_E([0,1])$ which has a strongly-weakly sequentially closed graph such that, for any $u \in K$ and $\varphi \in \Phi(u)$, one has

$$\varphi(t) \in M(t,u(t),\dot{u}(t)), \tag{3.6}$$

$$\|\varphi(t)\| \le \zeta(t,u(t),\dot{u}(t)) + \varepsilon, \tag{3.7}$$

for almost every $t \in [0,1]$.

PROOF OF THEOREM 3.3. *Step 1.* Let $m_1 = \rho_1 + 1/2$,

$$\mathrm{S} = \{f \in \mathrm{L}^2_E([0,1]) : \|f(t)\| \leq m(t), \text{ a.e. } t \in [0,1]\},$$

and

$$\mathrm{X} = \left\{u_f\colon [0,1] \to E : u_f(t) = \int_0^1 G(t,s)f(s)\,ds, \text{ for all } t \in [0,1],\ f \in \mathrm{S}\right\}.$$

It is clear that S is a convex $\sigma(\mathrm{L}^2,\mathrm{L}^2)$-compact subset of $\mathrm{L}^2_E([0,1])$ and that X is a convex compact subset of $\mathrm{C}^1_E([0,1])$ equipped with norm $\|\cdot\|_{\mathrm{C}^1}$. Indeed, for any $u_f \in \mathrm{X}$ and for all $t,\tau \in [0,1]$ we have

$$\begin{aligned}\|u_f(t) - u_f(\tau)\| &= \left\|\int_0^1 G(t,s)f(s)ds - \int_0^1 G(\tau,s)f(s)(s)\,ds\right\| \\ &\leq \int_0^1 |G(t,s) - G(\tau,s)| m(s)\,ds.\end{aligned}$$

and, by the relation (3.4) in Lemma 3.1,

$$\begin{aligned}\|\dot u_f(t) - \dot u_f(\tau)\| &= \left\|\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds - \int_0^1 \frac{\partial G}{\partial t}(\tau,s)f(s)\,ds\right\| \\ &\leq \int_0^1 \left|\frac{\partial G}{\partial t}(t,s) - \frac{\partial G}{\partial t}(\tau,s)\right| m(s)\,ds.\end{aligned}$$

Since $m \in \mathrm{L}^2_E([0,1])$ and G is uniformly continuous, we get the equicontinuity of the sets X and $\{\dot u_f : u_f \in \mathrm{X}\}$. On the other hand, for any $u_f \in \mathrm{X}$ and for all $t \in [0,1]$

$$\|u_f(t)\| = \left\|\int_0^1 G(t,s)f(s)\,ds\right\| \leq \int_0^1 \|f(s)\|\,ds \leq \int_0^1 m(s)\,ds = \|m\|_{\mathrm{L}^1_{\mathbb{R}}}$$

and

$$\|\dot u_f(t)\| = \left\|\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds\right\| \leq \int_0^1 \|f(s)\|\,ds \leq \int_0^1 m(s)\,ds = \|m\|_{\mathrm{L}^1_{\mathbb{R}}}.$$

Hence the sets $\mathrm{X}(t) = \{u_f(t) : u_f \in \mathrm{X}\}$ and $\{\dot u_f(t) : u_f \in \mathrm{X}\}$ are relatively compact in the finite dimensional space E. The Ascoli–Arzelà theorem yields that they are relatively compact in $\mathrm{C}_E([0,1])$ and consequently X is relatively compact in $(\mathrm{C}^1_E([0,1]), \|\cdot\|_{\mathrm{C}^1})$. We claim that X is closed in $(\mathrm{C}^1_E([0,1]), \|\cdot\|_{\mathrm{C}^1})$. Let (u_{f_n}) be a sequence in X converging uniformly to $\zeta \in \mathrm{C}^1_E([0,1]$ with respect to $\|\cdot\|_{\mathrm{C}^1}$. As S is weakly compact in $\mathrm{L}^2_E([0,1])$ and then in $\mathrm{L}^1_E([0,1])$, we extract from (f_n) a subsequence that we do not relabel and which converges in $\mathrm{L}^1_E([0,1])$ with respect to the weak topology $\sigma(\mathrm{L}^1_E([0,1]),\mathrm{L}^\infty_E([0,1]))$ to some mapping $f\in \mathrm{S}$. In particular, for every $t\in[0,1]$

$$\lim_{n\to\infty} u_{f_n}(t)=\lim_{n\to\infty}\int_0^1 G(t,s)f_n(s)\,ds=\int_0^1 G(t,s)f(s)\,ds.$$

Thus we get $\zeta=u_f$. This shows the compactness of X in $\mathrm{C}^1_E([0,1])$.

Step 2. By Theorem 3.6, there is a nonempty closed convex valued multifunction $\Phi\colon \mathrm{X}\rightrightarrows \mathrm{L}^2_E([0,1])$ such that for any $u_f\in \mathrm{X}$ and $\varphi\in\Phi(u_f)$ one has

$$\varphi(t)\in F(t,u_f(t),\dot u_f(t))\quad\text{and}\quad \|\varphi(t)\|\le m_1(t)$$

for almost every $t\in[0,1]$.

Let us define the multifunction $\Psi\colon \mathrm{X}\rightrightarrows \mathrm{C}^1_E([0,1])$ by

$$\begin{aligned}\Psi(v)=\bigg\{u\colon[0,1]\to E\ \bigg|\ &\text{there exists } f\in \mathrm{S} \text{ such that}\\ &u(t)=u_f(t)=\int_0^1 G(t,s)f(s)\,ds,\ \text{for all } t\in[0,1],\\ &f(t)\in -A(t)v(t)-g(t)\ \text{a.e. and } g\in\Phi(v)\bigg\}.\end{aligned}$$

We claim that, for any $v\in \mathrm{X}$, $\Psi(v)$ is a nonempty subset of X. Let (λ_n) be a decreasing sequence in $]0,1[$, such that $\lambda_n\to 0$. For each $n\in\mathbb{N}$ and any $g\in\Phi(v)$, let us consider the mapping f_n defined by

$$f_n(t)=-A_{\lambda_n}(t)v(t)-g(t),\quad \text{for all } t\in[0,1].$$

The mapping f_n is Lebesgue-measurable and in view of (H2) and the relation (2.3) we have

$$\|f_n(t)\|\le m_1(t)+m_2(t)=m(t),\quad \text{a.e. } t\in[0,1],$$

that is, $(f_n)\subset \mathrm{S}$. Hence by extracting a subsequence (that we do not relabel) we may suppose that (f_n) converges $\sigma(\mathrm{L}^2_E,\mathrm{L}^2_E)$ to some mapping $f\in\mathrm{S}$. On the other hand, we have for all $t\in[0,1]$ (see Proposition 2.1(c))

$$-f_n(t)-g(t)=A_{\lambda_n}(t)v(t)\in A(t)J_{\lambda_n}A(t)v(t). \tag{3.8}$$

But, by the relation (2.3) and (H2)

$$\|J_{\lambda_n}A(t)v(t)-v(t)\|=\lambda_n\|A_{\lambda_n}(t)v(t)\|\le\lambda_n m_2(t).$$

As $\lambda_n\to 0$, we conclude that $\|J_{\lambda_n}A(t)v(t)-v(t)\|\to 0$. On the other hand, since $\lambda_n<1$ and $v\in \mathrm{X}$ we get

$$\|J_{\lambda_n}A(t)v(t)\|\le\lambda_n m_2(t)+\|v(t)\|\le m_2(t)+\int_0^1 m(s)\,ds$$

for all $n \in \mathbb{N}$ and all $t \in [0,1]$ using the definition of X and the inequalities of the relation (3.2). Consequently $J_{\lambda_n} A(\cdot)v(\cdot) \to v(\cdot)$ in $\mathrm{L}^2_E([0,1])$ by Lebesgue's theorem. As $(f_n + g)$ converges $\sigma(\mathrm{L}^2_E, \mathrm{L}^2_E)$ to $f+g$, the relation (3.8) and Lemma 2.3 ensure that

$$f(t) + g(t) \in -A(t)v(t) \quad \text{almost everywhere},$$

that is, the mapping u_f defined by

$$u_f(t) = \int_0^1 G(t,s) f(s)\, ds \quad \text{for all } t \in [0,1]$$

belongs to $\Psi(v)$, since $f \in \mathrm{S}$. This shows that $\Psi(v)$ is a nonempty subset of X. Furthermore, $\Psi(v)$ is convex for any $v \in \mathrm{X}$ since $\Phi(v)$ and $A(t)v(t)$ are convex sets. Let us prove now, that $\Psi(v)$ is a compact subset of X. As X is compact it is sufficient to prove that $\Psi(v)$ is closed. Let (u_{f_n}) be a sequence in $\Psi(v)$ converging to $w(\cdot)$ in $(\mathrm{C}^1_E([0,1]), \|\cdot\|_{\mathrm{C}^1})$, that is, for each $n \in \mathbb{N}$

$$u_{f_n}(t) = \int_0^1 G(t,s) f_n(s)\, ds \quad \text{for all } t \in [0,1],\ f_n \in \mathrm{S},$$
$$f_n(t) \in -A(t)v(t) - g_n(t) \quad \text{a.e.} \quad \text{and} \quad g_n \in \Phi(v).$$

Since S and $\Phi(v)$ are $\sigma(\mathrm{L}^2_E, \mathrm{L}^2_E)$-compact, extracting subsequences we may suppose that (f_n) $\sigma(\mathrm{L}^2_E, \mathrm{L}^2_E)$-converges to some mapping $f \in \mathrm{S}$ and (g_n) $\sigma(\mathrm{L}^2_E, \mathrm{L}^2_E)$-converges to some mapping $g \in \Phi(v)$. Hence, by Lemma 2.3 we get

$$f(t) \in -A(t)v(t) - g(t) \quad \text{almost everywhere}.$$

Using the compactness of X and the fact that (f_n) converges $\sigma(\mathrm{L}^2_E, \mathrm{L}^2_E)$ to f we conclude that (u_{f_n}) converges to u_f in $(\mathrm{C}^1_E([0,1]), \|\cdot\|_{\mathrm{C}^1})$. Thus we get $w = u_f$ and consequently $\Psi(v)$ is closed.

Finally, we need to check that Ψ is upper semicontinuous on the convex compact set X or equivalently, the graph of Ψ

$$\mathrm{gph}(\Psi) = \{(v,u) \in \mathrm{X} \times \mathrm{X} : u \in \Psi(v)\}$$

is closed in $\mathrm{X} \times \mathrm{X}$. Let (v_n, u_n) be a sequence in $\mathrm{gph}(\Psi)$ converging to $(v,u) \in \mathrm{X} \times \mathrm{X}$, that is, $(v_n, u_n) \in \mathrm{X} \times \mathrm{X}$ and $u_n \in \Psi(v_n)$. $(u_n) \subset \mathrm{X}$ implies that there is a sequence $(f_n) \subset \mathrm{S}$ such that

$$u_n(t) = u_{f_n}(t) = \int_0^1 G(t,s) f_n(s)\, ds \quad \text{for all } t \in [0,1].$$

Since $(f_n) \subset \mathrm{S}$, extracting a subsequence we may suppose that (f_n) $\sigma(\mathrm{L}^2_E, \mathrm{L}^2_E)$-converges to some mapping $f \in \mathrm{S}$. Hence (u_{f_n}) converges in $(\mathrm{C}^1_E([0,1]), \|\cdot\|_{\mathrm{C}^1})$ to u_f. Thus we get $u = u_f$. On the other hand, $u_n \in \Psi(v_n)$ implies that

$$f_n(t) \in -A(t)v_n(t) - g_n(t) \quad \text{almost everywhere}, \tag{3.9}$$

and $(g_n) \subset \Phi(v_n) \subset m_1(t)\overline{\mathrm{B}}_E(0,1)$ (see the relation (3.7)). Then by extracting a subsequence we may suppose that (g_n) $\sigma(\mathrm{L}^2_E, \mathrm{L}^2_E)$-converges to some mapping $g \in m_1(t)\overline{\mathrm{B}}_E(0,1)$. As (v_n) converges uniformly to v and as the graph of Φ is strongly-weakly sequentially closed we conclude that $g \in \Phi(v)$. Hence, the relation (3.9) and Lemma 2.3 ensure that

$$f(t) \in -A(t)v(t) - g(t) \quad \text{almost everywhere.}$$

This shows that $\mathrm{gph}(\Psi)$ is closed in $\mathrm{X} \times \mathrm{X}$ and hence we get the upper semicontinuity of Ψ. An application of the Kakutani fixed point theorem gives some $u_f \in \Psi(u_f)$. This means $f(t) \in -A(t)u_f(t) - g(t)$ almost everywhere and $g \in \Phi(u_f)$ or equivalently (see the relation (3.6)) $g(t) \in F(t, u_f(t), \dot{u}_f(t))$ almost everywhere. By (3.3) and (3.5) we get

$$\begin{cases} -\ddot{u}_f(t) \in A(t)u_f(t) + F(t, u_f(t), \dot{u}_f(t)) & \text{for almost every } t \in [0,1], \\ u_f(0) = u_f(1) = 0. \end{cases}$$

This completes the proof of our theorem. □

It is worth to mention that if u is a solution of (P_F), then $u \in \mathrm{X}$ and hence $\|u(\,\cdot\,)\|_{\mathrm{C}^1} \le \|m\|_{\mathrm{L}^1_{\mathbb{R}}}$.

Now we present an other existence result of solutions of the problem (P_F) if we replace the hypotheses (c) and (H2) in Theorem 3.4 by the following ones:

(e) there exists a nonnegative function $\rho_1 \in \mathrm{L}^2_{\mathbb{R}}([0,1])$ and two nonnegative functions $p, q \in \mathrm{L}^2_{\mathbb{R}}([0,1])$ satisfying $\|p+q\|_{\mathrm{L}^1_{\mathbb{R}}} < 1$, such that

$$F(t,x,y) \subset (\rho_1(t) + p(t)\|x\| + q(t)\|y\|)\overline{\mathrm{B}}_E(0,1),$$

for all $(t,x,y) \in [0,1] \times E \times E$.

(H3) There is a nonnegative function $m_2 \in \mathrm{L}^2_{\mathbb{R}}([0,1])$ such that

$$\sup\{\|y\| : y \in A(t)x\} \le m_2(t), \quad \text{for all } (t,x) \in [0,1] \times E.$$

For this purpose we need the following fundamental lemma.

Lemma 3.5. *Suppose that the assumptions* (a), (b), (e) (H1) *and* (H3) *are satisfied. If u is a $\mathrm{W}^{2,1}_E([0,1])$-solution of the problem (P_F), then for all $t \in [0,1]$ we have*

$$\|u(t)\| \le \alpha, \quad \|\dot{u}(t)\| \le \alpha \tag{3.10}$$

where $\alpha = \|m\|_{\mathrm{L}^1_{\mathbb{R}}}/(1 - \|p+q\|_{\mathrm{L}^1_{\mathbb{R}}})$, and $m_1 = \rho_1 + 1/2$ and $m = m_1 + m_2$.

Proof. Suppose that u is a solution of the differential inclusion (P_F). By the hypothesis (e) and (H3) we have

$$\begin{aligned}\|\ddot{u}(t)\| &\leq m_2(t) + \|F(t,u(t),\dot{u}(t))\| \\ &\leq m_2(t) + \rho_1(t) + p(t)\|u(t)\| + q(t)\|\dot{u}(t)\| \\ &= m(t) + p(t)\|u(t)\| + q(t)\|\dot{u}(t)\|.\end{aligned}$$

But, by the relation (3.1) and (3.2) in Lemma 3.1 we have

$$\begin{aligned}\|u(t)\| &= \bigg\|\int_0^1 G(t,s)\ddot{u}(s)\,ds\bigg\| \leq \int_0^1 |G(t,s)|\|\ddot{u}(s)\|\,ds \\ &\leq \int_0^1 (m(s) + p(s)\|u(s)\| + q(s)\|\dot{u}(s)\|)\,ds \\ &\leq \int_0^1 m(s)\,ds + \int_0^1 (p(s)\|u\|_{\mathrm{C}^1} + q(s)\|u\|_{\mathrm{C}^1})\,ds \\ &\leq \|m\|_{\mathrm{L}^1_{\mathbb{R}}} + \|u\|_{\mathrm{C}^1}(\|p+q\|_{\mathrm{L}^1_{\mathbb{R}}}),\end{aligned}$$

and by (3.2) and (3.4)

$$\begin{aligned}\|\dot{u}(t)\| &= \bigg\|\int_0^1 \frac{\partial G}{\partial t}(t,s)\ddot{u}(s)\,ds\bigg\| \leq \int_0^1 \bigg|\frac{\partial G}{\partial t}(t,s)\bigg|\|\ddot{u}(s)\|\,ds \\ &\leq \int_0^1 (m(s) + p(s)\|u(s)\| + q(s)\|\dot{u}(s)\|)\,ds \\ &\leq \|m\|_{\mathrm{L}^1_{\mathbb{R}}} + \|u\|_{\mathrm{C}^1}(\|p+q\|_{\mathrm{L}^1_{\mathbb{R}}}).\end{aligned}$$

Then

$$\|u\|_{\mathrm{C}^1} \leq \|m\|_{\mathrm{L}^1_{\mathbb{R}}} + (\|p+q\|_{\mathrm{L}^1_{\mathbb{R}}})\|u\|_{\mathrm{C}^1},$$

or equivalently

$$\|u\|_{\mathrm{C}^1} \leq \frac{\|m\|_{\mathrm{L}^1_{\mathbb{R}}}}{1 - \|p+q\|_{\mathrm{L}^1_{\mathbb{R}}}},$$

this shows the estimates (3.10). □

We mention now our second existence result of solutions of (P_F).

Theorem 3.6. *Let E be a finite dimensional space, $A(t)\colon E \rightrightarrows E$, ($t \in [0,1]$), be a maximal monotone operator and $F\colon [0,1]\times E\times E \rightrightarrows E$ be a closed valued multifunction. Assume that the hypotheses* (a), (b), (e), (H1) *and* (H3) *are satisfied. Then, the differential inclusion* (P_F) *has at least a* $\mathrm{W}^{2,1}_E([0,1]$*-solution.*

Proof. Let us consider the mapping $\pi_\kappa\colon [0,1]\times E \to E$ given by

$$\pi_\kappa(t,x) = \begin{cases} x & \text{if } \|x\| \leq \kappa, \\ \kappa x/\|x\| & \text{if } \|x\| > \kappa, \end{cases}$$

and consider the multifunction $F_0\colon [0,1]\times E\times E \rightrightarrows E$ defined by

$$F_0(t,x,y) = F(t,\pi_\alpha(t,x),\pi_\alpha(t,y)).$$

Then F_0 inherits the properties (a) and (b) on F, and furthermore

$$\begin{aligned}\|F_0(t,x,y)\| &= \|F(t,\pi_\alpha(t,x),\pi_\alpha(t,y))\| \\ &\le \rho_1(t)+p(t)\|\pi_\alpha(t,x)\|+q(t)\|\pi_\alpha(t,y)\| \\ &\le \rho_1(t)+p(t)\alpha+q(t)\alpha=\rho_1(t)+\alpha(p(t)+q(t)):=\beta_1(t),\end{aligned}$$

for all $(t,x,y)\in[0,1]\times E\times E$. Consequently F_0 satisfies all the hypotheses of Theorem 3.4. Hence, we conclude the existence of a $\mathrm{W}^{2,1}_E([0,1])$-solution u of the problem (P_{F_0}). Furthermore, u satisfy the estimates

$$\|u(t)\|\le\|\beta+m_2\|_{\mathrm{L}^1_{\mathbb{R}}},\qquad \|\dot u(t)\|\le\|\beta+m_2\|_{\mathrm{L}^1_{\mathbb{R}}}, \tag{3.11}$$

where $\beta=\beta_1+1/2$.

Now, let us observe that u is a solution of

$$(\mathrm{P}_F)\qquad \begin{cases}-\ddot u(t)\in A(t)u(t)+F(t,u(t),\dot u(t)), & \text{for a.e. } t\in[0,1],\\ u(0)=u(1)=0,\end{cases}$$

if and only if u is a solution of

$$(\mathrm{P}_{F_0})\qquad \begin{cases}-\ddot u(t)\in A(t)u(t)+F_0(t,u(t),\dot u(t)), & \text{for a.e. } t\in[0,1],\\ u(0)=u(1)=0.\end{cases}$$

Indeed, let u be a solution of (P_F). By Lemma 3.5 we have

$$\|u(t)\|\le\alpha,\qquad \|\dot u(t)\|\le\alpha,$$

for all $t\in[0,1]$. Hence $\pi_\alpha(t,u(t))=u(t)$ and $\pi_\alpha(t,\dot u(t))=\dot u(t)$ and consequently

$$\begin{cases}-\ddot u(t)\in A(t)u(t)+F_0(t,u(t),\dot u(t)) & \text{for a.e. } t\in[0,1],\\ u(0)=u(1)=0,\end{cases}$$

that is, u is a solution of (P_{F_0}). Suppose now that u is a solution of (P_{F_0}). Then

$$\begin{aligned}\|\ddot u(t)\| &\le m_2(t)+\|F_0(t,u(t),\dot u(t))\|\le m_2(t)+\beta(t) \\ &= m_2(t)+m_1(t)+\alpha(p(t)+q(t))=m(t)+\alpha(p(t)+q(t)),\end{aligned}$$

and by (3.11) we have

$$\|u(t)\|\le\|m_1+\alpha(p+q)+m_2\|_{\mathrm{L}^1_{\mathbb{R}}}\le\|m\|_{\mathrm{L}^1_{\mathbb{R}}}+\alpha\|p+q\|_{\mathrm{L}^1_{\mathbb{R}}}, \tag{3.12}$$

$$\|\dot u(t)\|\le\|m_1+\alpha(p+q)+m_2\|_{\mathrm{L}^1_{\mathbb{R}}}\le\|m\|_{\mathrm{L}^1_{\mathbb{R}}}+\alpha\|p+q\|_{\mathrm{L}^1_{\mathbb{R}}}. \tag{3.13}$$

But, if we replace $\alpha=\|m\|_{\mathrm{L}^1_{\mathbb{R}}}/(1-\|p+q\|_{\mathrm{L}^1_{\mathbb{R}}})$ in (3.12) and (3.13) we obtain $\|u(t)\|\le\alpha$ and $\|\dot u(t)\|\le\alpha$ for all $t\in[0,1]$, that is, $\pi_\alpha(t,u(t))=u(t)$ and $\pi_\alpha(t,\dot u(t))=\dot u(t)$. Consequently,

$$\begin{aligned}&-\ddot u(t)\in A(t)u(t)+F_0(t,u(t),\dot u(t)) \quad \text{for a.e. } t\in[0,1]\\ \Rightarrow&-\ddot u(t)\in A(t)u(t)+F(t,u(t),\dot u(t)) \quad \text{for a.e. } t\in[0,1],\end{aligned}$$

with $u(0) = u(1) = 0$. We conclude that u is a solution of (P_F). This finished the proof of the theorem. □

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