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DYNAMICS OF THE MODIFIED VISCOUS CAHN–HILLIARD EQUATION IN $\mathbb{R}^N$

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Abstract. Global solvability and dynamical behaviour of the modified viscous Cahn–Hilliard equation is studied in the Sobolev space $H^1(\mathbb{R}^N)$. For $\nu \in [0,1]$ we construct $H^1(\mathbb{R}^N)$ global attractors and show their upper semicontinuity at $\nu = 0$.

1. Introduction

Viscous Cahn–Hilliard equation, a generalization of the classical Cahn–Hilliard model describing decomposition of binary alloys, was introduced by A. Novick–Cohen in [13] to analyze the dynamics of viscous first order phase transitions. The viscous Cahn–Hilliard equation

$$(1.1) \qquad (1-\nu)u_t = -\Delta(\Delta u + f(u) - \nu u_t), \quad \text{in } \Omega,$$

where $\nu \in [0,1]$ and Ω is a bounded smooth domain in $\mathbb{R}^N$, includes as limiting cases the Cahn–Hilliard equation ($\nu = 0$) and semilinear heat equation ($\nu = 1$). The transition of the asymptotic behaviour, as parameter ν varies from 0 to 1, was studied in [7]. This result was extended recently in [4].

The dynamics of the viscous Cahn–Hilliard equation considered in bounded domains with both 'bi-Neumann' or 'bi-Dirichlet' boundary conditions is quite well understood (see e.g. [13], [4]–[8], [17], [19], [23]). A rather natural extension

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is to study that problem in unbounded domains Ω. The case of an unbounded domains in which the Poincaré inequality still holds was considered recently in [3] while considerations related to global existence of solutions corresponding to particular ('close to a constant') initial data were presented in [11]. Note that the equation (1.1) in $\mathbb{R}^N$ is satisfied by an arbitrary constant function; $u(t,x) \equiv c$. This observation shows that there is no chance for existence of a compact in a reasonable phase space, global attractor for the semigroup generated by (1.1) when $\Omega = \mathbb{R}^N$. A local analysis around fixed stationary solution, as in [11], gives valuable information about the dynamics for such problem.

This paper is devoted to the global solvability and asymptotic behaviour of solutions to the Cauchy problem in $\mathbb{R}^N$ for the *modified viscous Cahn–Hilliard equation*

$$(1.2)\quad \begin{cases} (1-\nu)u_t = (-\Delta + \varepsilon I)((\Delta - \varepsilon I)u + f(x,u) - \nu u_t), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases}$$

where $\nu \in [0,1]$, $\varepsilon > 0$, and the assumption on the nonlinear function f will be made more precise later. We modify the original viscous Cahn–Hilliard equation, for the needs of the Cauchy problem in $\mathbb{R}^N$, by adding the first term εI at the right hand side to deal with the invertible operator $(-\Delta + \varepsilon I)$ (in $L^2(\mathbb{R}^N)$). The spectrum of the sole operator $(-\Delta)$ in $L^2(\mathbb{R})^N$ equals $[0,\infty)$ (see [10, p. 33]) and is purely absolutely continuous. That means $(-\Delta)^{-1}$ exists but is unbounded, with the domain dense in $L^2(\mathbb{R})^N$. In a bounded domain Ω the operator $(-\Delta)$ with Dirichlet boundary condition is invertible and this property is substantial in considerations of (1.1) with 'bi-Dirichlet' boundary data.

Remark 1.1. When we consider long time behaviour of solutions to the Cauchy problem in $\mathbb{R}^N$ (especially in term of the global attractors), due to the lack of the Poincaré inequality, it seems necessary to add a term $-\varepsilon^2 u$, $\varepsilon > 0$, to the nonlinearity of the original viscous Cahn–Hilliard equation to deduce some dissipation, see [16], [18], [20]–[22] for other models (although sometimes this requirement is included implicitly in the assumptions on the nonlinearity).

This paper is devoted to solvability, existence of the global attractors and their *upper semicontinuity* when the problem (1.2) is considered in the Sobolev spaces over $\mathbb{R}^N$. Since the equilibria need not be isolated in general (see [15, p. 680] for the description of the contents of the global attractor in the case $\nu = 1$), the characterization of the global attractor as used in [4] fails to hold and we were unable to prove the *lower semicontinuity* of attractors as in the last reference.

The difficulty we face in this paper is twofold. First, we study the Cauchy problem in $\mathbb{R}^N$, so the (lack of) compact embedding is always a serious problem (we are using the technique of *tail estimate* as in [18], [16] to handle that problem). Second, we deal with the fourth order elliptic operators, which again is much more involved than the case of the second order problems with all its specific tools of comparison and maximum principle type techniques. Dealing with the fourth order problems we are also forced to stay inside the L^2-setting; the general L^p-setting $p \neq 2$ is much more complicated in that case.

This paper is organized as follows. In Section 2 we recall some known auxiliary results including a sufficient condition for the upper semicontinuity of the family of global attractors. In Section 3, for arbitrary $\nu \in [0,1]$, we prove global solvability of (1.2) in $H^1(\mathbb{R}^N)$. Then, after giving some uniform (w.r.t. $\nu \in [0,1]$) a priori estimates in Section 4, we prove existence of the $H^1(\mathbb{R}^N)$ global attractors (Theorem 5.4) in Section 5; and finally, in Section 6, upper semicontinuity of the family of global attractors at $\nu = 0$ is obtained (Theorem 6.1).

2. Preliminaries

In this section, we recall some information used in the main part of the paper.

LEMMA 2.1. *For $A = (-\Delta + \varepsilon I)^{-1}$ with $\varepsilon > 0$ the following estimate holds: there exists a positive constant C such that for any $v \in H^{-1}(\mathbb{R}^N)$,*

$$\|v\|_{H^{-1}(\mathbb{R}^N)} \leq C(\|Av\|_{L^2(\mathbb{R}^N)} + \|\nabla Av\|_{L^2(\mathbb{R}^N)}). \tag{2.1}$$

PROOF. To verify the estimate (2.1), observe that for any $u \in H^1(\mathbb{R}^N)$ and $v \in H^{-1}(\mathbb{R}^N)$

$$\begin{aligned}
\langle u, v\rangle_{H^1(\mathbb{R}^N),H^{-1}(\mathbb{R}^N)} &= \langle u, (-\Delta + \varepsilon I)Av\rangle_{H^1(\mathbb{R}^N),H^{-1}(\mathbb{R}^N)} \\
&= \varepsilon \int_{\mathbb{R}^N} uAv\,dx + \int_{\mathbb{R}^N} \nabla u \cdot \nabla Av\,dx \\
&\leq \varepsilon \|u\|_{L^2(\mathbb{R}^N)}\|Av\|_{L^2(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathbb{R}^N)}\|\nabla Av\|_{L^2(\mathbb{R}^N)} \\
&\leq C(\|Av\|_{L^2(\mathbb{R}^N)} + \|\nabla Av\|_{L^2(\mathbb{R}^N)})\|u\|_{H^1(\mathbb{R}^N)}. \qquad \square
\end{aligned}$$

We recall next a criterion for the upper semicontinuity of attractors taken from [9], [14]:

PROPOSITION 2.2 ([9], [14]). *Let $\{S_\lambda(t)\}_{t\geq 0}$ $(\lambda \in \Lambda)$ be a family of semigroups defined on Banach space X, and for each $\lambda \in \Lambda$, $\{S_\lambda(t)\}_{t\geq 0}$ has a global attractor A_λ. Assume further that λ_0 is a nonisolated point of Λ and there exist $s > 0$, $t_0 > 0$ and a compact set $K \subset X$ such that*

$$\bigcup_{\lambda \in \mathcal{N}_\Lambda(\lambda_0,s)} \mathrm{A}_\lambda \subset K, \ \text{ and if } \lambda_n \to \lambda_0 \text{ and } x_n \to x_0, \ \text{ then } S_{\lambda_n}(t_0)x_n \to S_{\lambda_0}(t_0)x_0.$$

Then the global attractors A_λ are upper semicontinuous on Λ at $\lambda = \lambda_0$; that is,

$$\lim_{\Lambda \ni \lambda \to \lambda_0} \mathrm{dist}_X(\mathrm{A}_\lambda, \mathrm{A}_{\lambda_0}) = 0.$$

Notation. To simplify the notation inside the calculations we agree hereafter that all the unspecified norms are taken over $L^2(\mathbb{R}^N)$, that is, $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^N)}$. Also, from now on we cancel the dependence of the function spaces on $\mathbb{R}^N$; thus L^2 means $L^2(\mathbb{R}^N)$, $W^{s,p}$ means $W^{s,p}(\mathbb{R}^N)$ and so on. Moreover, all the unspecified integrals are taken over $\mathbb{R}^N$. The letter C is used to denote various positive constants and sometimes we also mention explicitly (in the bracket) the quantities on which the constant C depends.

3. Well-posedness in H^1

In this section, we will investigate the solvability of (1.2) in H^1:

$$(1-\nu)u_t = (-\Delta+\varepsilon I)((\Delta-\varepsilon I)u + f(x,u) - \nu u_t), \quad t>0,\ x\in\mathbb{R}^N, \tag{3.1}$$

where $\varepsilon > 0$ and $\nu \in [0,1]$.

We will consider problem (3.1) within the framework of the approach of [10] as an equation with sectorial operator. Rewrite (1.2) in an abstract form

$$u_t = (-\Delta_\varepsilon)B_\nu u + B_\nu(f(\cdot,u)),$$

where $B_\nu = (-\Delta+\varepsilon I)[(1-\nu)I + \nu(-\Delta+\varepsilon I)]^{-1} = -\Delta_\varepsilon[(1-\nu)I - \nu\Delta_\varepsilon]^{-1}$ and $(-\Delta_\varepsilon)$ is the realization in L^2 of the operator $(-\Delta+\varepsilon I)$ considered on the domain $D(-\Delta_\varepsilon) = H^2$. Note also, that the operator B_ν will be, for $\nu\in(0,1]$, written in an equivalent form:

$$B_\nu = \frac{1}{\nu}I - \frac{1-\nu}{\nu}((1-\nu)I - \nu\Delta_\varepsilon)^{-1},$$

which proves that B_ν is a bounded operator from L^2 into itself. Also, the operator $(-\Delta_\varepsilon)B_\nu$ is a bounded perturbation of the sectorial operator $(1/\nu)(-\Delta_\varepsilon)$:

$$(-\Delta_\varepsilon)B_\nu = \frac{1}{\nu}(-\Delta_\varepsilon) - \frac{1-\nu}{\nu^2}I + \frac{(1-\nu)^2}{\nu^2}((1-\nu)I - \nu\Delta_\varepsilon)^{-1},$$

hence is sectorial itself in L^2 with the domain H^2 (see [10, p. 27]).

With the above observations it is easy to get local solvability of the problem (1.2), $\nu\in(0,1]$, in the space $X := L^2$. Indeed, following the approach of D. Henry it is sufficient to check that the nonlinearity

$$\mathcal{F}(\phi) = B_\nu(f(\,\cdot\,,\phi))$$

acting from $X^\alpha = H^{2\alpha}$ into X is Lipschitz continuous on bounded subsets of X^α. Assume that the function $f\colon\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is (locally) Lipschitz continuous with respect to u with the Lipschitz constant uniform in the sets $\mathbb{R}^N\times[-M,M]$. We will study below the $X^{1/2}$ solution of (1.2), $\nu\in(0,1]$ (which will vary in H^1). To obtain such a local solution we need to assume

(3.2) $f(x,\,\cdot\,)$ is continuous w.r.t. s for a.e. $x\in\mathbb{R}^N$ and $f(\,\cdot\,,s)$ is measurable w.r.t. x for each $s\in\mathbb{R}$, also impose, additionally to the local Lipschitz continuity, the following growth restriction on the nonlinear function f; there exists $q \in [1, N/(N-2)]$ such that

(3.3) there exists $0 \le \beta(\cdot) \in L^\infty$ such that for all $s_1, s_2 \in \mathbb{R}$ and all $x \in \mathbb{R}^N$

$$|f(x,s_1) - f(x,s_2)| \le \beta(x)|s_1 - s_2|(1 + |s_1|^{q-1} + |s_2|^{q-1}).$$

Local Lipschitz continuity of the nonlinear term $\mathcal{F}$ then follows from (3.3), Hölder inequality and Sobolev embeddings. Let $B \subset X^{1/2} = H^1$ be bounded and let $\phi, \psi \in B$, then:

$$\begin{aligned}\|\mathcal{F}(\phi) - \mathcal{F}(\psi)\|_X &= \|B_\nu(f(\,\cdot\,,\phi)) - B_\nu(f(\,\cdot\,,\psi))\|_X \\ &\le \|B_\nu\|_{\mathcal{L}(X,X)}\|f(\,\cdot\,,\phi) - f(\,\cdot\,,\psi)\|_X \\ &\le \|B_\nu\|_{\mathcal{L}(X,X)}\|\beta\|_{L^\infty}[\|\phi-\psi\|_{L^2} \\ &\quad + \|\phi-\psi\|_{L^{2N/(N-2)}}(\|\phi\|_{L^{2N/(N-2)}} + \|\psi\|_{L^{2N/(N-2)}})^{2/(N-2)}] \\ &\le \|B_\nu\|_{\mathcal{L}(X,X)} L_B \|\phi-\psi\|_{X^{1/2}}.\end{aligned} \tag{3.4}$$

Therefore local solvability of (1.2) in $X^{1/2}$ follows immediately ([10]). We also have

$$u \in C([0,\tau), X^{1/2}) \cap C((0,\tau), D((-\Delta_\varepsilon))),$$
$$u_t \in C((0,\tau), X^\kappa) \quad \text{for every } \kappa \in [0,1),$$

where $\tau > 0$ is the maximal time of existence of the local $X^{1/2}$-solution $u(t)$. In addition the variation of constants formula is valid for such local solution

$$u(t) = e^{-(-\Delta_\varepsilon)B_\nu t}u_0 + \int_0^t e^{-(-\Delta_\varepsilon)B_\nu(t-s)}\mathcal{F}(u(s))\,ds, \quad \text{for } t \in [0,\tau).$$

In general, for $\nu \in (0,1]$, problem (1.2) behaves as a second order parabolic equation.

For $\nu = 0$ the modified Cahn–Hilliard equation is the fourth order parabolic equation:

$$\begin{cases} u_t = (-\Delta + \varepsilon I)((\Delta - \varepsilon I)u + f(x,u)), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x). \end{cases} \tag{3.5}$$

This will be again treated in the framework of the Henry's approach. We will use the lower index 0 in notation of the fractional order scale corresponding to realization of sectorial operator $(-\Delta + \varepsilon I)^2$ in the space $X_0 = (H^2)^*$, with the domain $X_0^1 = H^2$ and with $X_0^{1/2} = L^2$. We are interested in the $X_0^{3/4}$-solution, that is the case when the phase space equals to H^1. The problem (3.5) is then written abstractly as

$$\begin{cases} u_t = -(-\Delta_\varepsilon)^2 u + (-\Delta_\varepsilon) f(\,\cdot\,,u) & \text{in } X_0, \\ u(0) = u_0 \in X_0^{3/4}. \end{cases} \tag{3.6}$$

To justify local solvability of that problem, we need to check that the nonlinear operator $\overline{\mathcal{F}}(\phi) = (-\Delta_\varepsilon)(f(\,\cdot\,,\phi))$ is locally Lipschitz continuous as a map from $X_0^{3/4}$ to X_0. Take a bounded set $B \subset X_0^{3/4}$ and let $\phi, \psi \in B$. Since $(-\Delta_\varepsilon)\colon X_0^{1/2} \to X_0$ is a linear isomorphism, we have

$$\begin{aligned}\|\overline{\mathcal{F}}(\phi) - \overline{\mathcal{F}}(\psi)\|_{X_0} &= \|(-\Delta_\varepsilon)(f(\,\cdot\,,\phi) - f(\,\cdot\,,\psi))\|_{X_0} \\ &\le c\|f(\,\cdot\,,\phi) - f(\,\cdot\,,\psi)\|_{X_0^{1/2}} \le C_B\|\phi - \psi\|_{X_0^{3/4}},\end{aligned}$$

precisely as in the estimate (3.4) and with the same assumptions on the function f. Hence local solvability of (3.6) follows immediately.

To justify the global solvability of (1.2), we need to impose additional assumptions on the nonlinear term. We will borrow such assumptions from [15], where the set of stationary solutions to the second order parabolic Cauchy problem in Sobolev spaces ($\nu = 1$) was recently analyzed. In many cases this set is rich (compare [15, Theorem 1.1]) and the whole global attractor is located between the minimal and the maximal equilibrium solutions.

Following [15] we will assume that:

(3.7) there exists $\mu \in (0, \varepsilon/2)$ such that for all $s \in \mathbb{R}$, $x \in \mathbb{R}^N$

$$F(x,s) \le \mu s^2 + \alpha_1(x)|s| + \gamma_1(x),$$

and that:

(3.8) there exists $l \in (0,\varepsilon)$, $k \in (0, (\varepsilon - l)/\mu)$ such that for all $s \in \mathbb{R}$, $x \in \mathbb{R}^N$

$$sf(x,s) - kF(x,s) \le ls^2 + \alpha_2(x)|s| + \gamma_2(x),$$

where $\alpha_i(\,\cdot\,), \gamma_i(\,\cdot\,) \ge 0$, $\alpha_i(\,\cdot\,) \in L^2$, $\gamma_i(\,\cdot\,) \in L^1$ and $F(x,s) = \int_0^s f(x,z)\,dz$.

Remark 3.1. (a) The constants μ, l (and so k) depend on ε, which is natural since in $\mathbb{R}^N$ the Poincaré inequality is not true and our dissipation comes from the term $\varepsilon^2 u$; see Remark 1.1;

(b) Note, that such condition is satisfied in particular for the standard nonlinearities $f(s) = \alpha s - s|s|^{p-1}$, $p > 1$ and $\alpha \in (0, \varepsilon)$.

Remark 3.2. Our assumptions allow a rich set of stationary solutions of (1.2), consequently rich global attractor. For example, let

$$L(v) := -(\Delta - \varepsilon I)v, \qquad \overline{v}(x) = \frac{1}{(1+|x|^2)^N}$$

and take

$$f(x,s) = \sin\left(\frac{s}{2\overline{v}(x)}\pi\right)L(\overline{v}(x)) - \cos\left(\frac{s}{2\overline{v}(x)}\pi\right)L(2\overline{v}(x)).$$

Then, one can verify that such nonlinearity f satisfies (3.2)–(3.3) and (3.7)–(3.8) (with arbitrary small positive μ, l, k), and also (4.5) below, with some proper α_i

and γ_i. Obviously, the functions $\overline{v}(x)$ and $2\overline{v}(x)$ are two different solutions of the stationary equation $(\Delta - \varepsilon I)u + f(x,u) = 0$ with the above function $f(x,s)$.

Having already local solvability, combining it with the a priori estimate (4.3) given in Section 4 below, we obtain the global well-posedness of (1.2) in H^1 for every $\nu \in [0,1]$ when f satisfies (3.2)–(3.3) and (3.7)–(3.8), and then we can define the corresponding semigroup in H^1 through the solution of (1.2).

Hereafter, we will use the notation: $\{S_\nu(t)\}_{t\geq 0}$ ($\nu \in [0,1]$) denotes the semigroup generated by the solution of (1.2) in H^1.

4. A priori estimates

4.1. H^1-estimates. Let $A = (-\Delta + \varepsilon I)^{-1}$. Then (1.2) is equivalent to

$$(1-\nu)Au_t + \nu u_t = (\Delta - \varepsilon I)u + f(x,u). \tag{4.1}$$

To get the required a priori estimate in H^1, multiplying (4.1) by u one obtains

$$\begin{aligned}
\frac{1-\nu}{2}\frac{d}{dt}\int |\nabla Au|^2\,dx &+ \frac{\varepsilon(1-\nu)}{2}\frac{d}{dt}\int |Au|^2\,dx + \frac{\nu}{2}\frac{d}{dt}\int |u|^2\,dx \\
&+ \varepsilon\int |u|^2\,dx + \int |\nabla u|^2\,dx - k\int F(x,u)\,dx \\
&\leq l\int |u|^2\,dx + \int \alpha_2(x)|u|\,dx + \int \gamma_2(x)\,dx \\
&\leq (l+\delta)\int |u|^2\,dx + C_\delta\int \alpha_2^2(x)\,dx + \int \gamma_2(x)\,dx,
\end{aligned}$$

where we have used (3.8) and $\delta > 0$ is small enough.

Also, multiplying (4.1) by u_t and integrating, one gets:

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\int |u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx - \frac{d}{dt}\int F(x,u)\,dx \\
&\quad + (1-\nu)\int |\nabla Au_t|^2\,dx + (1-\nu)\varepsilon\int |Au_t|^2\,dx + \nu\int |u_t|^2\,dx = 0.
\end{aligned} \tag{4.2}$$

Adding these two estimates, one obtains (similar to that in the proof of Lemma 5.1 below) the H^1 a priori estimate of the solution u of the form:

$$\|u(t,\cdot)\|_{H^1}^2 \leq E_u'(0)e^{-M't} + \text{const}, \tag{4.3}$$

where

$$\begin{aligned}
E_u'(0) = (1-\nu)\int |\nabla Au_0|^2\,dx &+ \varepsilon(1-\nu)\int |Au_0|^2\,dx \\
&+ (\nu+\varepsilon)\int |u_0|^2\,dx + \int |\nabla u_0|^2\,dx - 2\int F(x,u_0)\,dx,
\end{aligned}$$

where M' depends on μ, ε, l, the *const* depends on μ, ε, l, $\|\alpha_i\|$, $\|\gamma_i\|_{L^1}$, but both M', const are chosen uniformly in $\nu \in [0,1]$.

Remark 4.1. Note that using only the estimate (4.2) we obtain the usual Lyapunov function for the modified viscous Cahn–Hilliard equation. Consequently, global solvability in H^1 phase space follows, since due to (3.7) we have

$$\begin{aligned}\frac{\varepsilon}{2}\int |u|^2\,dx + \frac{1}{2}\int |\nabla u|^2\,dx - \int F(x,u)\,dx \\ \geq \left(\frac{\varepsilon}{2} - \mu - \delta\right)\int |u|^2\,dx + \frac{1}{2}\int |\nabla u|^2\,dx - C_\delta \int \alpha_1^2(x)\,dx - \|\gamma_1\|_{L^1},\end{aligned}$$

which provides a (not necessary uniform in time) H^1-estimate of the solution u.

We summarize the global H^1 estimate (4.3) in the following:

Theorem 4.2. *Let f satisfy* (3.2)–(3.3), (3.7)–(3.8), *then there exists a positive constant $r_1 > 0$ such that: for any bounded set $B \subset H^1$ there is a $T_{1B} = T_1(B)$ (which only depends on $\|B\|_{H^1}$) that*

$$\|S_\nu(t)B\|_{H^1} \leq r_1 \quad \textit{for all } t \geq T_{1B} \textit{ and any } \nu \in [0,1].$$

Remark 4.3. From (4.2), (3.7), (3.3) and Theorem 4.2, we have that for any H^1-bounded subset B, there is a T_{1B} (which only depends on $\|B\|_{H^1}$) such that the following estimate holds: for any $t \geq T_{1B}$,

$$\begin{aligned}(4.4)\quad \int_t^{t+1} \bigg((1-\nu)\int |\nabla A u_t(s)|^2\,dx \\ + (1-\nu)\varepsilon \int |A u_t(s)|^2\,dx + \nu \int |u_t(s)|^2\,dx\bigg)\,ds \\ \leq Q(r_1, \mu, l, k, \|\alpha_i\|, \|\gamma_i\|_{L^1}, \|\beta\|_{L^\infty}),\end{aligned}$$

where $Q(\,\cdot\,)$ is a continuous increasing function in each component.

4.2. H^2-estimates. Multiplying the equation (1.2) by $(\Delta - \varepsilon I)u$ it is possible to obtain a H^2 estimate of the solution u to (1.2) with our previous assumptions on the nonlinear term f only. But such an estimate will not be uniform, as $\nu \to 0^+$.

To obtain the uniform in $\nu \in [0,1]$ estimate of the solution u in H^2 we need to strengthen the assumptions concerning the nonlinear term f to the following one; $f(x,\,\cdot\,) \in C^1(\mathbb{R})$ and

(4.5) there exists $C > 0$ such that for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$

$$|f_s'(x,s)| \leq C(1 + \alpha_3(x) + |s|^{2/(N-2)}),$$

where $0 \leq \alpha_3(\,\cdot\,) \in L^{(N/2)^+}$.

Here and below, the symbol a^- (respectively, a^+) denotes a real number strictly less (respectively, strictly greater) than a (but eventually arbitrarily close to a). As before, B denotes a bounded subset of H^1.

Set $v(t) = u_t(t)$ and differentiate (4.1) with respect to time t, to obtain

$$(1-\nu)Av_t + \nu v_t = (\Delta - \varepsilon I)v + f'_u(x,u)v. \tag{4.6}$$

Multiplying (4.6) by v and integrating over $\mathbb{R}^N$, we find that

$$\begin{aligned}\frac{d}{dt}\int((1-\nu)(|\nabla Av|^2 + \varepsilon|Av|^2) + \nu|v|^2)\,dx + 2\varepsilon\int|v|^2\,dx + 2\int|\nabla v|^2\,dx \\ \le \int|f'_u(x,u)|v^2\,dx \le C\int(1+\alpha_3(x)+|u|^{2/(N-2)})v^2\,dx,\end{aligned} \tag{4.7}$$

where (4.5) was used.

Note that, for each $r \in [2, 2N/(N-2))$, we have an interpolation inequality (see [1, Chapter V])

(4.8) there exists $C > 0$ for all $\phi \in H^1$

$$\|\phi\|_{L^r} \le c\|\phi\|_{H^s} \le C\|\phi\|^{\theta}_{H^{-1}}\|\phi\|^{1-\theta}_{H^1},$$

where $s = N/2 - N/r$ and $\theta = (s+1)/2 \in (0,1)$ depends on r.

So, for any $0 < \delta << 1$,

$$\int v^2\,dx = \|v\|^2 \le C\|v\|_{H^{-1}}\|v\|_{H^1} \le C_\delta\|v\|^2_{H^{-1}} + \delta\|v\|^2_{H^1}, \tag{4.9}$$

and since $\alpha_3 \in L^{(N/2)^+}$, by the Hölder inequality and (4.8) we have

$$\begin{aligned}\int\alpha_3(x)v^2\,dx &\le \|\alpha_3\|_{L^{(N/2)+}}\|v\|^2_{L^{(2N/(N-2))-}} \\ &\le C_\delta\|\alpha_3\|^{1/\theta}_{L^{(N/2)+}}\|v\|^2_{H^{-1}} + \delta\|v\|^2_{H^1},\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\int|u|^{2/(N-2)}v^2\,dx &\le C\|u\|^{2/(N-2)}_{H^1}\|v\|^2_{L^{2N/(N-1)}} \\ &\le C_\delta\|u\|^{(2/(N-2))\cdot 4/3}_{H^1}\|v\|^2_{H^{-1}} + \delta\|v\|^2_{H^1}.\end{aligned} \tag{4.11}$$

Taking δ small enough (depending on ε) and inserting (4.9)–(4.11) into (4.7), we obtain

$$\frac{d}{dt}\int((1-\nu)(|\nabla Av|^2+\varepsilon|Av|^2)+\nu|v|^2)\,dx \le Q(\varepsilon, \|u(t)\|_{H^1}, \|\alpha_3\|_{L^{(N/2)+}})\|v\|^2_{H^{-1}}.$$

Then, from (2.1) and (4.4), applying the Uniform Gronwall Lemma, for $t \ge T_{1B}$ and $u_0 \in B$, we have

$$\begin{aligned}(1-\nu)(\|\nabla Av(t)\|^2 + \varepsilon\|Av(t)\|^2) + \nu\|v(t)\|^2 \\ \le C(r_1, \varepsilon, \mu, l, \|\alpha_3\|_{L^{(N/2)+}}, \|\alpha_i\|, \|\gamma_i\|_{L^1}).\end{aligned} \tag{4.12}$$

Now, returning to (4.1),

$$(\Delta - \varepsilon I)u = (1-\nu)Au_t + \nu u_t - f(x,u),$$

for the right hand side terms, we note that when $t \geq T_{1B}$, (4.12) implies that

$$\|(1-\nu)Au_t(t) + \nu u_t(t)\| \leq C(r_1, \varepsilon, \mu, l, \|\alpha_3\|_{L^{(N/2)+}}, \|\alpha_i\|, \|\gamma_i\|_{L^1}),$$

and (3.3) and Theorem 4.2 imply

$$\|f(\,\cdot\,, u(t))\| \leq C(r_1, \|\beta\|_{L^\infty}).$$

Hence

$$\|u(t)\|_{H^2} \leq C(r_1, \varepsilon, \mu, l, \|\alpha_3\|_{L^{(N/2)+}}, \|\alpha_i\|, \|\gamma_i\|_{L^1}, \|\beta\|_{L^\infty})$$

for all $t \geq T_{1B}$, $u_0 \in B$. That is, we obtain a uniform with respect to $\nu \in [0,1]$ estimate of the solution u in H^2:

Theorem 4.4. *Let f satisfy* (3.2)–(3.3), (3.7)–(3.8) *and* (4.5). *Then there exists a positive constant $r_2 > 0$ such that for any bounded set $B \subset H^1$, there is a time $T_{2B} = T_2(B)$ (depending only on $\|B\|_{H^1}$) such that*

$$\|S_\nu(t)B\|_{H^2} \leq r_2 \quad \textit{for all } t \geq T_{2B} \textit{ and any } \nu \in [0,1].$$

5. H^1-attractors

In this section, based on the a priori estimates obtained in Section 4, we will show that for each $\nu \in [0,1]$, the semigroup $\{S_\nu(t)\}_{t\geq 0}$ has a global H^1-attractor (see [9]).

As shown in [18], [16], in order to obtain the necessary (H^1, H^1)-asymptotic compactness, where thanks to Theorem 4.4, we only need to prove the following *tail estimate*:

Lemma 5.1. *Under the assumption of Theorem* 4.4, *for any $\eta > 0$ and any bounded set $B \subset H^1$, there exist $h = h(\eta, \|B\|_{H^1})$ and $T = t(\eta, \|B\|_{H^1})$ such that*

$$\int_{\mathcal{O}_h} (|S_\nu(t)u_0|^2 + |\nabla S_\nu(t)u_0|^2)\,dx \leq \eta \quad \textit{for all } t \geq T,\ u_0 \in B \textit{ and } \nu \in [0,1],$$

where $\mathcal{O}_h = \{x \in \mathbb{R}^N : |x| \geq h\}$.

Proof. Choose a smooth function θ such that $0 \leq \theta(s) \leq 1$ for any $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \quad \text{for } 0 \leq s \leq 1, \quad \text{and} \quad \theta(s) = 1 \quad \text{for } s \geq 2.$$

Then there exists a constant C such that $|\theta'(s)| + |\theta''(s)| \leq C$ for any $s \in \mathbb{R}^+$.

Multiplying (4.1) by $\lambda(\theta^2(|x|^2/h^2) \cdot u)$ and $(\theta^2(|x|^2/h^2) \cdot u_t)$, respectively, integrating in $\mathbb{R}^N$, adding and using (3.9)–(3.10) we obtain

$$\frac{d}{dt}E_u(t) + 2G_u(t) \leq 2L_u(t), \tag{5.1}$$

where $\lambda \in (0,1)$ is a small constant which will be fixed later,

$$E_u(t) = (\nu\lambda+\varepsilon)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u|^2\,dx + \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla u|^2\,dx - 2\int \theta^2(\frac{|x|^2}{h^2})F(x,u)\,dx, \tag{5.2}$$

$$\begin{aligned} G_u(t) = {} & \lambda(\varepsilon-l-\delta)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u|^2\,dx + \lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla u|^2\,dx \\ & - \lambda k\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)F(x,u)\,dx + (1-\nu)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla Au_t|^2\,dx \\ & + (1-\nu)\varepsilon\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx + \nu\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u_t|^2\,dx, \end{aligned} \tag{5.3}$$

($\delta \in (0, \varepsilon - l - k\mu)$) and, with a small positive constant λ_1 to be determined later,

$$\begin{aligned} L_u(t) = {} & (1-\nu)\lambda_1\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx \\ & + (1-\nu)C_{\lambda_1}\lambda^2\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u|^2\,dx \\ & + (1-\nu)\int Au_t\Big(\nabla Au_t\cdot\nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\Big)\,dx \\ & - \lambda\int u\nabla u\cdot\nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\,dx + \lambda\|\gamma_2\|_{L^1(\mathcal{O}_h)} \\ & - \int u_t\nabla u\cdot\nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\,dx + \lambda C_\delta\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)\alpha_2^2(x)\,dx. \end{aligned} \tag{5.4}$$

At first, denote $\overline{C}_\varepsilon = 1+\varepsilon$, from (5.2) we have

$$E_u(t) \le \overline{C}_\varepsilon\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)(|u|^2+|\nabla u|^2)\,dx - 2\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)F(x,u)\,dx,$$

and from (3.9)–(3.10) we have

$$\varepsilon - l - k\mu > 0 \quad\text{and}\quad \mu s^2 + \alpha_1(x)|s| + \gamma_1(x) - F(x,s) \ge 0 \quad\text{for all } s\in\mathbb{R}.$$

Therefore,

$$\begin{aligned} G_u(t) \ge {} & \lambda(\varepsilon-l-\delta)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u|^2\,dx + \lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla u|^2\,dx \\ & - \lambda k\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)F(x,u)\,dx + (1-\nu)\varepsilon\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx \\ = {} & \lambda(\varepsilon-l-k\mu-\delta)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u|^2\,dx + \lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla u|^2\,dx \\ & + \lambda k\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)(\mu u^2 + \alpha_1(x)|u| + \gamma_1(x) - F(x,u))\,dx \end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&- \lambda k \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)(\alpha_1(x)|u| + \gamma_1(x))\,dx \\
&+ (1-\nu)\varepsilon \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx \\
\geq{}& \lambda\overline{\overline{C}}_{\varepsilon,l,k,\delta}\bigg(\overline{C}_\varepsilon \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)(|u|^2 + |\nabla u|^2)\,dx \\
&+ 2\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)(\mu u^2 + \alpha_1(x)|u| + \gamma_1(x) - F(x,u))\,dx\bigg) \\
&- \lambda k \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)(\alpha_1(x)|u| + \gamma_1(x))\,dx \\
&+ \frac{1}{2}\lambda(\varepsilon - l - \delta)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u|^2\,dx \\
&+ (1-\nu)\varepsilon \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx \\
\geq{}& \lambda\overline{\overline{C}}_{\varepsilon,l,k,\delta}E_u(t) - \lambda k(\|u\|\|\alpha_1\|_{L^2(\mathcal{O}_h)} + \|\gamma_1\|_{L^1(\mathcal{O}_h)}) \\
&+ \frac{1}{2}\lambda(\varepsilon - l - \delta)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|u|^2\,dx \\
&+ (1-\nu)\varepsilon \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx.
\end{aligned}
$$

In the following we will estimate the required terms in (5.4) one by one (recall that the symbol $\|\cdot\|$ denotes the L^2 norm) :

$$
(5.6)\qquad -\int u\nabla u\cdot\nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\,dx \leq \frac{C'}{h}\|u\|\|\nabla u\|;
$$

$$
(5.7)\qquad -\int Au_t\bigg(\nabla Au_t\cdot\nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\bigg)\,dx \leq \frac{C'}{h}\|Au_t\|\|\nabla Au_t\|;
$$

and

$$
\begin{aligned}
(5.8)\qquad \bigg|\int u_t\nabla u\cdot\nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\,dx\bigg| &= \bigg|\int u_t\theta'\Big(\frac{|x|^2}{h^2}\Big)\theta\Big(\frac{|x|^2}{h^2}\Big)\frac{4x}{h^2}\cdot\nabla u\,dx\bigg| \\
&\leq \frac{4}{h}\|u_t\|_{H^{-1}}\bigg\|\theta'\Big(\frac{|x|^2}{h^2}\Big)\theta\Big(\frac{|x|^2}{h^2}\Big)\frac{x}{h}\cdot\nabla u\bigg\|_{H^1},
\end{aligned}
$$

with

$$
\begin{aligned}
\bigg\|\theta'\Big(\frac{|x|^2}{h^2}\Big)\theta\Big(\frac{|x|^2}{h^2}\Big)\frac{x}{h}\cdot\nabla u\bigg\|_{H^1}^2 &= \int\bigg|\theta'\Big(\frac{|x|^2}{h^2}\Big)\theta\Big(\frac{|x|^2}{h^2}\Big)\frac{x}{h}\cdot\nabla u\bigg|^2\,dx \\
&+ \int\bigg|\nabla\bigg(\theta'\Big(\frac{|x|^2}{h^2}\Big)\theta\Big(\frac{|x|^2}{h^2}\Big)\frac{x}{h}\cdot\nabla u)\bigg|^2\,dx,
\end{aligned}
$$

where

$$\int \left|\theta'\left(\frac{|x|^2}{h^2}\right)\theta\left(\frac{|x|^2}{h^2}\right)\frac{x}{h}\cdot\nabla u\right|^2 dx = \int_{h\le|x|\le\sqrt{2}h}\left|\theta'\left(\frac{|x|^2}{h^2}\right)\theta\left(\frac{|x|^2}{h^2}\right)\frac{x}{h}\cdot\nabla u\right|^2 dx \le C_\theta\int_{h\le|x|\le\sqrt{2}h}|\nabla u|^2\,dx,$$

and, similarly,

$$\int \left|\nabla\left(\theta'\left(\frac{|x|^2}{h^2}\right)\theta\left(\frac{|x|^2}{h^2}\right)\frac{x}{h}\cdot\nabla u\right)\right|^2 dx \le C_\theta\int_{h\le|x|\le\sqrt{2}h}(|\nabla u|^2+|\Delta u|^2)\,dx,$$

where the constant C_θ depends only on the cutoff function θ. Hence,

$$\left|\int u_t\nabla u\cdot\nabla\theta^2\left(\frac{|x|^2}{h^2}\right)dx\right| \le \frac{C_\theta}{h}\|u_t\|_{H^{-1}}(\|\nabla u\|+\|\Delta u\|). \tag{5.9}$$

Summarizing (5.6)–(5.9) and inserting into (5.4), we deduce that

$$\begin{aligned}L_u(t) \le{} & \frac{C_\theta}{h}(\|u\|\|\nabla u\|+\||Au_t\|\|\nabla Au_t\|+\|u_t\|_{H^{-1}}(\|\nabla u\|+\|\Delta u\|)) \\ & + C_\delta\|\alpha_2\|^2_{L^2(\mathcal{O}_h)}+\|\gamma_2\|_{L^1(\mathcal{O}_h)}+(1-\nu)\lambda_1\int\theta^2\left(\frac{|x|^2}{h^2}\right)|Au_t|^2\,dx \\ & + (1-\nu)C_{\lambda_1}\lambda^2\int\theta^2\left(\frac{|x|^2}{h^2}\right)|u|^2\,dx,\end{aligned}$$

which, combined with Lemma 2.1, Theorems 4.2 and 4.4, implies that when $t\ge T_{1B}+T_{2B}$,

$$\begin{aligned}L_u(t) \le{} & \frac{C_\theta}{h}(\|Au_t(t)\|^2+\|\nabla Au_t(t)\|^2+r_1^2+r_2^2) \\ & + C_\delta\|\alpha_2\|^2_{L^2(\mathcal{O}_h)}+\|\gamma_2\|_{L^1(\mathcal{O}_h)}+(1-\nu)\lambda_1\int\theta^2\left(\frac{|x|^2}{h^2}\right)|Au_t|^2\,dx \\ & + (1-\nu)C_{\lambda_1}\lambda^2\int\theta^2\left(\frac{|x|^2}{h^2}\right)|u|^2\,dx.\end{aligned} \tag{5.10}$$

Now, noting the final estimate of (5.5), we first fix λ_1 small enough such that $\lambda_1<\varepsilon$, then fix λ small enough such that $(1-\nu)C_{\lambda_1}\lambda<(\varepsilon-l-\delta)/2$. From (5.1), (5.2), (5.5) and (5.10), we have that: as $t\ge T_{1B}+T_{2B}$,

$$\begin{aligned}\frac{d}{dt}E_u(t)+2\lambda\overline{\overline{C}}_{\varepsilon,l,k,\delta}E_u(t) \le{} & \frac{C_\theta}{h}(\|Au_t(t)\|^2+\|\nabla Au_t(t)\|^2+r_1^2+r_2^2) \\ & + C_\delta\|\alpha_2\|^2_{L^2(\mathcal{O}_h)}+\|\gamma_2\|_{L^1(\mathcal{O}_h)}+\lambda k(r_1\|\alpha_1\|_{L^2(\mathcal{O}_h)}+\|\gamma_1\|_{L^1(\mathcal{O}_h)}).\end{aligned} \tag{5.11}$$

On the other hand, (4.7) implies

$$\frac{1}{2}\int_t^{t+1}(\|Au_t(s)\|^2+\|\nabla Au_t(s)\|^2)\,ds \le Q(r_1,\mu,l,k,\|\alpha_i\|,\|\gamma_i\|_{L^1},\|\beta\|_{L^\infty}) \tag{5.12}$$

for all $t\ge T_{1B}+T_{2B}$ and any $\nu\in[0,1]$.

Hence, combining with (5.11) and (5.12), applying the Uniform Gronwall Lemma to (5.11), we have that for any $t \geq t_1 = T_{1B} + T_{2B}$,

$$
\begin{aligned}
(5.13)\quad E_u(t) &\leq e^{-\overline{\overline{C}}(t-t_1)} E_u(t_1) \\
&+ \frac{e^{\overline{\overline{C}}}}{1-e^{-\overline{\overline{C}}}} \bigg[\frac{C_\theta}{h} Q(r_i, \mu, l, k, \|\alpha_i\|, \|\gamma_i\|_{L^1}, \|\beta\|_{L^\infty}) + C_\delta \|\alpha_2\|^2_{L^2(\mathcal{O}_h)} \\
&\qquad + \lambda k \|\alpha_1\|_{L^2(\mathcal{O}_h)} r_1 + \lambda k \|\gamma_1\|_{L^1(\mathcal{O}_h)} + \|\gamma_2\|_{L^1(\mathcal{O}_h)} \bigg],
\end{aligned}
$$

for all $u_0 \in B$. Then, combining with (5.2) and applying (3.7) again we find (note also that $\|\alpha_i\|_{L^2(\mathcal{O}_h)}, \|\gamma_i\|_{L^1(\mathcal{O}_h)} \to 0$ as $h \to \infty$)

$$\int_{\mathcal{O}_h} (|S_\nu(t)u_0|^2 + |\nabla S_\nu(t)u_0|^2)\, dx \leq \eta$$

as t, h are taken large enough. □

Lemma 5.2 ((H^1, H^1)-asymptotic compactness). *Under the assumptions of Theorem* 4.4*, for each $\nu \in [0,1]$, the semigroup $\{S_\nu(t)\}_{t\geq 0}$ is (H^1, H^1)-asymptotically compact.*

Proof. This is a direct consequence of Lemma 5.1, Theorem 4.4 and the compact embedding $H^2(\mathbb{R}^N \setminus \mathcal{O}_h) \hookrightarrow H^1(\mathbb{R}^N \setminus \mathcal{O}_h)$. □

To apply the standard abstract results guarantying existence of the H^1 attractor, we also need (H^1, H^1)-continuity of $\{S_\nu(t)\}_{t\geq 0}$.

Lemma 5.3 ((H^1, H^1)-continuity). *Under the assumptions* (3.7)*,* (3.8) *and* (4.5)*, for each $\nu \in [0,1]$, the semigroup $\{S_\nu(t)\}_{t\geq 0}\colon H^1 \to H^1$ is continuous.*

The above lemma is a direct consequence of our local existence theorems presented in Section 3.

We are now ready to state the main result of this section, which is a direct consequence of Lemmas 5.2 and 5.3 (see [2], [9]):

Theorem 5.4 (H^1-attractor). *Let the conditions* (3.2)–(3.3)*,* (3.7)–(3.8) *and* (4.5) *be satisfies. Then, for each $\nu \in [0,1]$, the semigroup $\{S_\nu(t)\}_{t\geq 0}\colon H^1 \to H^1$ has a H^1 global attractor $\mathcal{A}_\nu$, that is, $\mathcal{A}_\nu$ is compact in H^1, invariant under $\{S_\nu(t)\}_{t\geq 0}$ and attracts every H^1-bounded set with respect to the H^1 norm.*

6. Upper semicontinuity at $\nu = 0$

The main purpose of this section is the following upper semicontinuity result:

Theorem 6.1. *Under the assumptions of Theorem* 5.4*, the global attractors $\{\mathrm{A}_\nu\}_{\nu\in[0,1]}$ obtained in Theorem* 5.4 *are upper semicontinuous at $\nu = 0$:*

$$\operatorname{dist}_{H^1}(\mathcal{A}_\nu, \mathcal{A}_0) \to 0 \quad \textit{as } \nu \to 0^+,$$

where $\mathrm{dist}_{H^1}(\,\cdot\,,\,\cdot\,)$ *denotes the usual Hausdorff semidistance in* H^1.

Let $\nu \in (0,1/2)$ and $u(t) = S_\nu(t)u_0$ with $u_0 \in \mathcal{A}_\nu \subset H^1$. Also let $v(t) = S_0(t)v_0$ with $v_0 \in H^1$, then $u(t)$ and $v(t)$ satisfy the following equations, respectively,

$$\begin{cases} (1-\nu)Au_t + \nu u_t = (\Delta - \varepsilon I)u + f(x,u), \\ u(0) = u_0, \end{cases}$$

when $\nu \in (0,1]$, and the limit equation (1.2) (parameter $\nu = 0$),

$$\begin{cases} Av_t = (\Delta - \varepsilon I)v + f(x,v), \\ v(0) = v_0, \end{cases}$$

where $A = (-\Delta + \varepsilon I)^{-1}$.

Denote $w(t) = u(t) - v(t)$. Then w solves the following problem:

$$\begin{cases} (1-\nu)Aw_t + \nu w_t = (\Delta - \varepsilon I)w + f(x,u) - f(x,v) - \nu(v_t - Av_t), \\ w(0) = u_0 - v_0 \in H^1. \end{cases} \tag{6.1}$$

Multiplying (6.1) by w and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned} \frac{1-\nu}{2}\frac{d}{dt}\int |\nabla Aw|^2\,dx &+ \frac{\varepsilon(1-\nu)}{2}\frac{d}{dt}\int |Aw|^2\,dx \\ &+ \frac{\nu}{2}\frac{d}{dt}\int |w|^2\,dx + \varepsilon\int |w|^2\,dx + \int |\nabla w|^2\,dx \\ &\le \int (f(x,u) - f(x,v))w\,dx - \nu\int (v_t - Av_t)w\,dx. \end{aligned}$$

From the assumption (3.3) we have (here we take $N \ge 3$)

$$\begin{aligned} \left|\int (f(x,u) - f(x,v))w\,dx\right| &\le C\int (1 + |u|^{2/(N-2)} + |v|^{2/(N-2)})|w|^2\,dx \\ &\le C\|w\|^2 + C(1 + \|u\|_{H^1}^{4/(N-2)} + \|v\|_{H^1}^{4/(N-2)})\|w\|_{H^1}^2, \end{aligned}$$

and

$$\begin{aligned} \nu\left|\int (v_t - Av_t)w\,dx\right| &\le \nu(\|v_t\|_{H^{-1}}\|w\|_{H^1} + \|Av_t\|\|w\|) \\ &\le \nu C(\|v_t\|_{H^{-1}} + \|Av_t\|)\|w\|_{H^1} \\ &\le \nu C(\|\nabla Av_t\| + \|Av_t\|)\|w\|_{H^1}, \end{aligned}$$

where we have used (2.1). Therefore, from (4.3) we have

$$\begin{aligned} \frac{1-\nu}{2}\frac{d}{dt}\int |\nabla Aw|^2\,dx &+ \frac{\varepsilon(1-\nu)}{2}\frac{d}{dt}\int |Aw|^2\,dx + \frac{\nu}{2}\frac{d}{dt}\int |w|^2\,dx \\ &\le Q(\|u_0\|_{H^1}, \|v_0\|_{H^1})\|w\|_{H^1}^2 + \nu^2 C(\|\nabla Av_t\|^2 + \|Av_t\|^2), \end{aligned}$$

which implies that, for $\nu \in [0,1/2]$,

$$(6.2)\quad \frac{1}{4}\int |\nabla Aw(t)|^2\,dx + \frac{\varepsilon}{4}\int |Aw(t)|^2\,dx \\ \leq e^{Q(\|u_0\|_{H^1},\|v_0\|_{H^1})t}\bigg(\int |\nabla Aw(0)|^2\,dx + \varepsilon\int |Aw(0)|^2\,dx + \int |w(0)|^2\,dx\bigg) \\ + \nu^2 Ce^{Q(\|u_0\|_{H^1},\|v_0\|_{H^1})t}\int_0^t (\|\nabla Av_t(s)\|^2 + \|Av_t(s)\|^2)\,ds.$$

Hence, combining with (4.4) and the uniform (w.r.t. $\nu \in [0,1]$) boundedness of $\mathcal{A}_\nu$ in H^1 (e.g. see Theorem 4.2), we deduce the following result:

Lemma 6.2. *For any $\nu_m \in (0,1/2]$ and $u_{0m} \in \mathcal{A}_{\nu_m}$, if $\nu_m \to 0$ and $u_{0m} \to v_0$ in H^1 then*

$$S_{\nu_m}(t_0)u_{0m} \to S_0(t_0)v_0 \quad \textit{in } H^1$$

for any fixed $t_0 > 0$.

Proof. For arbitrary sequence $u_{0m} \in \mathcal{A}_{\nu_m}$, $u_{0m} \to v_0$ in H^1 as $\nu_m \to 0$, let $w_m(t) = S_{\nu_m}(t)u_{0m} - S_0(t)v_0$. Then (6.2) implies that

$$\frac{1}{4}\int |\nabla Aw_m(t)|^2\,dx + \frac{\varepsilon}{4}\int |Aw_m(t)|^2\,dx \\ \leq e^{Q(\|u_{0m}\|_{H^1},\|v_0\|_{H^1})t}\bigg(\int |\nabla A(u_{0m}-v_0)|^2\,dx \\ + \varepsilon\int |Aw(u_{0m}-v_0)|^2\,dx + \int |w(u_{0m}-v_0)|^2\,dx\bigg) \\ + \nu_m^2 Ce^{Q(\|u_{0m}\|_{H^1},\|v_0\|_{H^1})t}\int_0^t (\|\nabla A(S_0(s)v_0)(s)\|^2 + \|A(S_0(s)v_0)(s)\|^2)\,ds.$$

So, using (4.2) and Theorem 4.2, since $\nu_m \to 0$ and $u_{0m} \to v_0$ in H^1, for arbitrary fixed $t_0 > 0$ (e.g. $t_0 = 1$), we have

$$\int |\nabla Aw_m(t_0)|^2\,dx + \varepsilon\int |Aw_m(t_0)|^2\,dx \to 0 \quad \text{as } \nu_m \to 0,$$

which, combined with (2.1), implies

$$\|w_m(t_0)\|_{H^{-1}} \to 0 \quad \text{as } \nu_m \to 0.$$

Thanks to the uniform (w.r.t. ν) H^2 estimate of Theorem 4.4, we complete the proof using interpolation inequality. □

To apply the abstract result of Lemma 2.2, we need the following property:

Lemma 6.3. *$\bigcup_{\nu\in[0,1/2]} \mathcal{A}_\nu$ is precompact in H^1.*

Proof. At first, from Theorem 4.4 we know that

$$(6.3)\quad \bigcup_{\nu\in[0,1/2]} \mathcal{A}_\nu \text{ is bounded in } H^2,$$

and $\{u \in H^2 : \|u\|_{H^2} \le r_2\}$ is a uniformly w.r.t. $\nu \in [0,1/2]$ absorbing set for $\{S_\nu(t)\}_{\nu\in[0,1/2]}$.

Next, take $\overline{B} = \{u \in H^2 : \|u\|_{H^2} \le r_2\}$ and apply Lemma 5.1, then from the construction of global attractor (as the ω-limit set of an absorbing set $\overline{B}$), we know that for any $\eta > 0$, there exists $h = h(\eta)$ such that

$$\int_{\mathcal{O}_h} (|u|^2 + |\nabla u|^2)\,dx \le \eta \quad \text{for all } u \in \bigcup_{\nu\in[0,1/2]} \mathcal{A}_\nu, \tag{6.4}$$

where $\mathcal{O}_h = \{x \in \mathbb{R}^N : |x| \ge h\}$.

Hence, precompactness of $\bigcup_{\nu\in[0,1/2]} \mathcal{A}_\nu$ in H^1 follows from (6.3)–(6.4) immediately. □

Proof of Theorem 6.1. Based on the Lemmas 6.2 and 6.3, this is a direct application of Lemma 2.2. □

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