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NONLINEAR SCALAR FIELD EQUATIONS IN $\mathbb{R}^N$: MOUNTAIN PASS AND SYMMETRIC MOUNTAIN PASS APPROACHES

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Abstract. We study the existence of radially symmetric solutions of the following nonlinear scalar field equations in $\mathbb{R}^N$:

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N,$$
$$u \in H^1(\mathbb{R}^N).$$

We give an extension of the existence results due to H. Berestycki, T. Gallouët and O. Kavian [2].

We take a mountain pass approach in $H^1(\mathbb{R}^N)$ and introduce a new method generating a Palais–Smale sequence with an additional property related to Pohozaev identity.

1. Introduction

In this paper we study the existence of radially symmetric solutions of the following nonlinear scalar field equations:

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$
$$u \in H^1(\mathbb{R}^N). \tag{1.2}$$

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Here $N \geq 2$ and $g:\mathbb{R} \to \mathbb{R}$ is a continuous function. This type of problem appears in many models in mathematical physics etc. and almost necessary and sufficient conditions for the existence of non-trivial solutions are obtained by H. Berestycki and P.-L. Lions [3], [4] for $N \geq 3$ and H. Berestycki, T. Gallouët and O. Kavian [2] for $N = 2$. See also W. A. Strauss [16] and S. Coleman, V. Glaser and A. Martin [10] for earlier works.

In [2]–[4] they assume:

(g0) $g(\xi) \in C(\mathbb{R}, \mathbb{R})$ and $g(\xi)$ is odd.

(g1) For $N \geq 3$,

$$\limsup_{\xi\to\infty} \frac{g(\xi)}{\xi^{(N+2)/(N-2)}} \leq 0.$$

For $N = 2$,

$$\limsup_{\xi\to\infty} \frac{g(\xi)}{e^{\alpha\xi^2}} \leq 0 \quad \text{for any } \alpha > 0.$$

(g2) For $N \geq 3$

$$-\infty < \liminf_{\xi\to 0} \frac{g(\xi)}{\xi} \leq \limsup_{\xi\to 0} \frac{g(\xi)}{\xi} < 0. \tag{1.3}$$

For $N = 2$

$$-\infty < \lim_{\xi\to 0} \frac{g(\xi)}{\xi} < 0. \tag{1.4}$$

(g3) There exists a $\zeta_0 > 0$ such that $G(\zeta_0) > 0$, where $G(\xi) = \int_0^\xi g(\tau)\,d\tau$.

Under the above conditions, they show the existence of *a positive solution* and *infinitely many* (*possibly sign changing*) *radially symmetric solutions*.

Remark 1.1. For the existence of a positive solution, it is sufficient to assume (g0)–(g3) just for $\xi > 0$. Namely we assume

(g0') $g(\xi) \in C([0,\infty), \mathbb{R})$, $g(0) = 0$

and (g1), (g3) and (g2) just for a limit as $\xi \to +0$.

Remark 1.2. (a) We refer to [5] (see also Section 11, Chapter II of [18]) for the study of *zero mass case*, when $N \geq 3$. In particular, they assume

$$\limsup_{\xi\to 0} \frac{G(\xi)}{|\xi|^{2N/(N-2)}} \leq 0$$

instead of (g2) and they show the existence of infinitely many solutions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

(b) For the study of the existence of at least one solution, especially the existence of a least energy solution, we also refer to H. Brezis and E. H. Lieb [6], in which they study the system of equations

$$-\Delta u_i = g^i(u) \quad \text{in } \mathbb{R}^d,\ i = 1, \ldots, n,$$

$d \geq 2$ with $u: \mathbb{R}^d \to \mathbb{R}^n$ and $g^i(u) = \partial G/\partial u_i$. Under suitable conditions on G (which differ for $d = 2$ and $d \geq 3$) they prove that the system admits a non-trivial solution of finite action and that this solution also minimizes the action among solutions of finite action. We also refer to E. Bruning [7] for a generalization when $d = 2$.

(g0)–(g3) are natural conditions for the existence of solutions. However we can see a difference between cases $N \geq 3$ and $N = 2$ in the condition (g2). We remark that when $N = 2$, the existence of a limit $\lim_{\xi \to 0} g(\xi)/\xi \in (-\infty, 0)$ is used essentially to show the Palais–Smale compactness condition for the corresponding functional under suitable constraint ([2]).

The aim of this paper is to extend the result of [2] slightly and we prove the existence of positive solution and infinitely many radially symmetric solutions under the conditions (g0), (g1), (g3) and (1.3) (not (1.4)).

We also remark that in [2]–[4] (cf. [6], [7]), they constructed solutions of (1.1)–(1.2) through constraint problems in the space of radially symmetric functions:

- find critical points of

$$\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\} \quad (N \geq 3), \tag{1.5}$$

or

- find critical points of

$$\left\{ \int_{\mathbb{R}^2} |\nabla u|^2 \, dx : \int_{\mathbb{R}^2} G(u) \, dx = 0, \ \int_{\mathbb{R}^2} u^2 \, dx = 1 \right\} \quad (N = 2). \tag{1.6}$$

In fact, if $v(x)$ is a critical point of (1.5) or (1.6), then for a suitable $\lambda > 0$, $u(x) = v(x/\lambda)$ is a solution of (1.1)–(1.2). On the other hand, solutions of (1.1)–(1.2) are also characterized as critical points of the functional $I(u) \in C^1(H^1_r(\mathbb{R}^N), \mathbb{R})$ defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx.$$

Here we denote by $H^1_r(\mathbb{R}^N)$ the space of radially symmetric H^1-functions defined on $\mathbb{R}^N$. It is natural to ask whether it is possible to find critical points through the unconstraint functional $I(u)$.

Our second aim is to give another proof of the results of [2]–[4] using mountain pass and symmetric mountain pass arguments to $I(u)$.

Now we can state our main result.

THEOREM 1.3. *Assume* $N \geq 2$ *and* (g0), (g1), (g3) *and*

(g2') $-\infty < \liminf_{\xi \to 0} \dfrac{g(\xi)}{\xi} \leq \limsup_{\xi \to 0} \dfrac{g(\xi)}{\xi} < 0.$

Then (1.1)–(1.2) *has a positive least energy solution and infinitely many* (*possibly sign changing*) *radially symmetric solutions, which are characterized by the mountain pass and symmetric mountain pass minimax arguments in* $H^1_r(\mathbb{R}^N)$ (*see* (3.1)–(3.2), (5.13) *and* (6.1)–(6.3) *below*).

Remark 1.4. (a) When $N \ge 3$, the existence of solutions of (1.1)–(1.2) is obtained in [3], [4] and we provide another proof and give a minimax characterization of infinitely many solutions using the functional $I(u)$.

(b) When $N = 2$, our existence result extends the result of [2] slightly. Indeed, we show the existence under condition (g2') not (1.4).

In L. Jeanjean and K. Tanaka [13], we give a mountain pass characterization to a least energy solution of (1.1)–(1.2) under the conditions (g0)–(g3). More precisely, let b be the mountain pass minimax value for $I(u)$ and furthermore let m be the least energy level. To show $b = m$, we argued in [13] as follows: To show $b \le m$, for any solution $u(x)$ we constructed a path $\gamma(t) \in C([0,1], H^1_r(\mathbb{R}^N))$ such that $u \in \gamma([0,1])$, $\gamma(0) = 0$, $I(\gamma(1)) < 0$ and $\max_{t\in[0,1]} I(\gamma(t)) = I(u)$. To show $b \ge m$, the existence of a minimizer of the minimization problems (1.5) or (1.6) is essential and we relied on the argument in [2], [3].

We will take mountain pass and symmetric mountain pass approaches to prove Theorem 1.3. In Section 3, we will observe that $I(u)$ is an even functional with a mountain pass geometry and it is possible to define a mountain pass minimax value b_{mp} and symmetric mountain pass values b_n ($n \in \mathbb{N}$) for $I(u)$. By the Ekeland's principle, we can find a Palais–Smale sequence $(u_j)_{j=1}^\infty \subset H^1_r(\mathbb{R}^N)$ at levels b_{mp} and b_n, that is, $(u_j)_{j=1}^\infty$ satisfies

$$I(u_j) \to b_{mp} \quad (\text{or } b_n), \tag{1.7}$$

$$I'(u_j) \to 0 \qquad \text{strongly in } (H^1_r(\mathbb{R}^N))^*. \tag{1.8}$$

However one of the difficulty is the lack of the Palais–Smale compactness condition and it seems difficult to show the existence of strongly convergent subsequence merely under the conditions (1.7)–(1.8). A key of our argument is to find a Palais–Smale sequence with an extra property related to Pohozaev identity. We recall that if $u(x)$ is a critical point of $I(u)$, then $u(x)$ satisfies

$$P(u) = 0, \quad \text{where} \quad P(u) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - N\int_{\mathbb{R}^N} G(u)\,dx.$$

The above equality is called Pohozaev identity. It is natural to ask the existence of a Palais–Smale sequence $(u_j)_{j=1}^\infty$ satisfying (1.7)–(1.8) and $P(u_j) \to 0$. For this purpose in Section 4 we introduce an auxiliary functional:

$$\widetilde{I}(\theta, u) = \frac{e^{(N-2)\theta}}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - e^{N\theta}\int_{\mathbb{R}^N} G(u)\,dx : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R}.$$

We will find a Palais–Smale sequence (θ_j, u_j) in the augmented space $\mathbb{R}\times H^1_r(\mathbb{R}^N)$ satisfying

$$\theta_j \to 0, \tag{1.9}$$

$$\widetilde{I}(\theta_j, u_j) \to b_{mp} \quad (\text{or } b_n), \tag{1.10}$$

$$\widetilde{I}'(\theta_j, u_j) \to 0 \qquad \text{strongly in } (H^1_r(\mathbb{R}^N))^*, \tag{1.11}$$

$$\frac{N-2}{2} e^{(N-2)\theta_j} \int_{\mathbb{R}^N} |\nabla u_j|^2\,dx - N e^{N\theta_j} \int_{\mathbb{R}^N} G(u_j)\,dx \to 0. \tag{1.12}$$

Remark 1.5. We remark that this type of auxiliary functionals was first used in L. Jeanjean [11] for a nonlinear eigenvalue problem. It should be compared with monotonicity method due to M. Struwe [17] and L. Jeanjean [12]. We expect that this type of auxiliary functionals can be applied to other problems.

We remark that our auxiliary functional $\widetilde{I}(\theta, u)$ satisfies

$$\begin{aligned} \widetilde{I}(0, u) &= I(u), \\ \widetilde{I}(\theta, u(x)) &= I(u(e^{-\theta}x)) \quad \text{for all } \theta \in \mathbb{R} \text{ and } u \in H^1_r(\mathbb{R}^N). \end{aligned}$$

Properties (1.9)–(1.12) enable us to obtain boundedness and the existence of strongly convergent subsequence of (u_j).

2. Preliminaries

We will deal with the cases $N = 2$ and $N \geq 3$ in a unified way. In what follows we assume $N \geq 2$ and $g(\xi)$ satisfies (g0), (g1), (g2') and (g3).

2.1. Modification of $g(\xi)$. To give a proof of Theorem 1.3, we modify the nonlinearity $g(\xi)$. First we remark that we can assume

(g1') when $N \geq 3$, $\displaystyle\lim_{\xi\to\infty} \frac{g(\xi)}{|\xi|^{(N+2)/(N-2)}} = 0$,

when $N = 2$, $\displaystyle\lim_{\xi\to\infty} \frac{g(\xi)}{e^{\alpha\xi^2}} = 0$ for any $\alpha > 0$.

In fact, if $g(\xi)$ satisfies $g(\xi) > 0$ for $\xi \geq \zeta_0$, (g1') clearly follows from (g1). If there exists $\zeta_1 > \zeta_0$ such that $g(\zeta_1) = 0$, we set

$$\widetilde{g}(\xi) = \begin{cases} g(\xi) & \text{for } 0 \leq \xi \leq \zeta_1, \\ 0 & \text{for } \xi > \zeta_1, \\ -g(-\xi) & \text{for } \xi < 0. \end{cases}$$

Then $\widetilde{g}(\xi)$ satisfies (g0), (g1'), (g2'), (g3) and solutions of $-\Delta u = \widetilde{g}(u)$ in $\mathbb{R}^N$ satisfy $-\zeta_1 \leq u(x) \leq \zeta_1$ for all $x \in \mathbb{R}^N$, that is, $u(x)$ also solves (1.1). Thus, we may replace $g(\xi)$ with $\widetilde{g}(\xi)$ and assume (g1').

In what follows, we assume that $g(\xi)$ satisfies (g0), (g1'), (g2'), and (g3).

Next we set

$$m_0 = -\frac{1}{2} \limsup_{\xi \to 0} \frac{g(\xi)}{\xi} \in (0, \infty)$$

and rewrite (1.1) as

$$-\Delta u + m_0 u = m_0 u + g(u) \quad \text{in } \mathbb{R}^N.$$

We introduce $h(\xi) \in C(\mathbb{R}, \mathbb{R})$ by

$$h(\xi) = \begin{cases} \max\{m_0\xi + g(\xi), 0\} & \text{for } \xi \geq 0, \\ -h(-\xi) & \text{for } \xi < 0. \end{cases}$$

Furthermore, we choose $p_0 \in (1, (N+2)/(N-2))$ if $N \geq 3$, $p_0 \in (1, \infty)$ if $N = 2$ and set

$$\overline{h}(\xi) = \begin{cases} \xi^{p_0} \sup\limits_{0<\tau\leq\xi} \dfrac{h(\tau)}{\tau^{p_0}} & \text{for } \xi > 0, \\ 0 & \text{for } \xi = 0, \\ -\overline{h}(-\xi) & \text{for } \xi < 0. \end{cases}$$

We also set

$$H(\xi) = \int_0^\xi h(\tau)\, d\tau, \quad \overline{H}(\xi) = \int_0^\xi \overline{h}(\tau)\, d\tau.$$

From the definition of $h(\xi)$, $\overline{h}(\xi)$ and m_0, we have

Lemma 2.1.

(a) *$m_0\xi + g(\xi) \leq h(\xi) \leq \overline{h}(\xi)$ for all $\xi \geq 0$.*
(b) *$h(\xi) \geq 0$, $\overline{h}(\xi) \geq 0$ for all $\xi \geq 0$.*
(c) *There exists $\delta_0 > 0$ such that $h(\xi) = \overline{h}(\xi) = 0$ for $\xi \in [0, \delta_0]$.*
(d) *There exists $\xi_0 > 0$ such that $0 < h(\xi_0) \leq \overline{h}(\xi_0)$.*
(e) *$\xi \mapsto \overline{h}(\xi)/\xi^{p_0}$; $(0, \infty) \to \mathbb{R}$ is non-decreasing.*
(f) *$h(\xi)$, $\overline{h}(\xi)$ satisfies* (g1').

Proof. (a), (b) follow from the definitions of $h(\xi)$ and $\overline{h}(\xi)$.

(c) By the definition of m_0, we can easily see that $\xi g(\xi) \leq -m_0\xi^2$ in a neighbourhood of $\xi = 0$. Thus (c) holds for small $\delta_0 > 0$.

(d) By (g3), there exists $\xi_0 \in (0, \zeta_0)$ such that $g(\xi_0) > 0$. Thus $\overline{h}(\xi_0) \geq h(\xi_0) \geq m_0\xi_0 + g(\xi_0) > 0$ and (d) holds.

(e) Since $\overline{h}(\xi)/\xi^{p_0} = \sup_{\tau \in (0,\xi]} h(\tau)/\tau^{p_0}$, (e) holds.

(f) It is easy to see that $h(\xi)$ satisfies (g1') and we will show (f) for $\overline{h}(\xi)$. We consider the case $N \geq 3$ first. We remark that

$$\begin{aligned} \frac{\overline{h}(\xi)}{\xi^{(N+2)/(N-2)}} &= \xi^{-((N+2)/(N-2)-p_0)} \sup_{0<\tau\leq\xi} \frac{h(\tau)}{\tau^{p_0}} \\ &= \sup_{0<\tau\leq\xi} \frac{h(\tau)}{\tau^{(N+2)/(N-2)}} \frac{\tau^{(N+2)/(N-2)-p_0}}{\xi^{(N+2)/(N-2)-p_0}}. \end{aligned}$$

Since $h(\xi)$ satisfies (g1'), for any $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that

$$\left|\frac{h(\tau)}{\tau^{(N+2)/(N-2)}}\right| < \varepsilon \quad \text{for all } \tau \geq \tau_\varepsilon.$$

Thus, denoting $C_\varepsilon = \sup_{0<\tau\leq\tau_\varepsilon} |h(\tau)/\tau^{(N+2)/(N-2)}|$, we have

$$\begin{aligned}
&\frac{\overline{h}(\xi)}{\xi^{(N+2)/(N-2)}} \\
&\leq \max\left\{\sup_{0<\tau\leq\tau_\varepsilon}\left|\frac{h(\tau)}{\tau^{(N+2)/(N-2)}}\right|\frac{\tau_\varepsilon^{(N+2)/(N-2)-p_0}}{\xi^{(N+2)/(N-2)-p_0}}, \sup_{\tau_\varepsilon\leq\tau\leq\xi}\left|\frac{h(\tau)}{\tau^{(N+2)/(N-2)}}\right|\right\} \\
&\qquad\qquad\leq \max\left\{\frac{C_\varepsilon\tau_\varepsilon^{(N+2)/(N-2)-p_0}}{\xi^{(N+2)/(N-2)-p_0}}, \varepsilon\right\}.
\end{aligned}$$

Therefore we have

$$\limsup_{\xi\to\infty}\frac{\overline{h}(\xi)}{\xi^{(N+2)/(N-2)}} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{\xi\to\infty}\overline{h}(\xi)/\xi^{(N+2)/(N-2)} = 0$.

Next we deal with the case $N = 2$. It suffices to show

$$\lim_{\xi\to\infty}\frac{\overline{h}(\xi)}{\xi^{p_0}e^{\alpha\xi^2}} = 0 \quad \text{for any } \alpha > 0. \tag{2.1}$$

Since

$$\frac{\overline{h}(\xi)}{\xi^{p_0}e^{\alpha\xi^2}} = \frac{1}{e^{\alpha\xi^2}}\sup_{0<\tau\leq\xi}\frac{h(\tau)}{\tau^{p_0}} = \sup_{0<\tau\leq\xi}\frac{h(\tau)}{\tau^{p_0}e^{\alpha\tau^2}}\frac{e^{\alpha\tau^2}}{e^{\alpha\xi^2}}$$

and $h(\xi)$ satisfies $\lim_{\xi\to\infty} h(\xi)/\xi^{p_0}e^{\alpha\xi^2} = 0$, we can show (2.1) in a similar way.□

COROLLARY 2.2.

(a) $m_0|\xi|^2/2 + G(\xi) \leq H(\xi) \leq \overline{H}(\xi)$ *for all* $\xi \in \mathbb{R}$.
(b) $H(\xi)$, $\overline{H}(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}$.
(c) *There exists* $\delta_0 > 0$ *such that* $H(\xi) = \overline{H}(\xi) = 0$ *for* $|\xi| \leq \delta_0$.
(d) $\overline{H}(\zeta_0) - m_0\zeta_0^2/2 > 0$.
(e) $0 \leq (p_0 + 1)\overline{H}(\xi) \leq \xi\overline{h}(\xi)$ *for all* $\xi \in \mathbb{R}$.
(f) $H(\xi)$, $\overline{H}(\xi)$ *satisfies*

$$\lim_{|\xi|\to\infty}\frac{H(\xi)}{|\xi|^{2N/(N-2)}} = \lim_{|\xi|\to\infty}\frac{\overline{H}(\xi)}{|\xi|^{2N/(N-2)}} = 0 \quad \textit{when } N \geq 3,$$
$$\lim_{|\xi|\to\infty}\frac{H(\xi)}{e^{\alpha\xi^2}} = \lim_{|\xi|\to\infty}\frac{\overline{H}(\xi)}{e^{\alpha\xi^2}} = 0 \qquad \textit{for any } \alpha > 0 \textit{ when } N = 2.$$

PROOF. (a)–(c) easily follow from (a)–(c) of Lemma 2.1.

By (a) and (g3), it follows that

$$\overline{H}(\zeta_0) \geq H(\zeta_0) \geq \frac{1}{2}m_0\zeta_0^2 + G(\zeta_0) > 0.$$

Thus (d) holds.

Since $\xi \mapsto \overline{h}(\xi)/\xi^{p_0}$; $(0,\infty) \to \mathbb{R}$ is non-decreasing, we have for $\xi > 0$

$$\begin{aligned}\xi\overline{h}(\xi) - (p_0+1)\overline{H}(\xi) &= \int_0^\xi \overline{h}(\xi) - (p_0+1)\overline{h}(\tau)\,d\tau \\ &= \int_0^\xi \xi^{p_0}\frac{\overline{h}(\xi)}{\xi^{p_0}} - (p_0+1)\tau^{p_0}\frac{\overline{h}(\tau)}{\tau^{p_0}}\,d\tau \\ &\geq \int_0^\xi \xi^{p_0}\frac{\overline{h}(\xi)}{\xi^{p_0}} - (p_0+1)\tau^{p_0}\frac{\overline{h}(\xi)}{\xi^{p_0}}\,d\tau = 0.\end{aligned}$$

Therefore (e) holds.

(f) also follows from (f) of Lemma 2.1. □

2.2. Fundamental properties of $H^1_r(\mathbb{R}^N)$. In what follows, we use notation: for $u \in H^1_r(\mathbb{R}^N)$ and $1 \leq p < \infty$

$$\|u\|_p = \bigg(\int_{\mathbb{R}^N} |u|^p\,dx\bigg)^{1/p}, \qquad \|u\|_\infty = \operatorname*{esssup}_{x\in\mathbb{R}^N} |u(x)|,$$
$$\|u\|_{H^1} = (\|\nabla u\|_2^2 + m_0\|u\|_2^2)^{1/2}.$$

We also write

$$(u,v)_2 = \int_{\mathbb{R}^N} uv\,dx, \quad (u,v)_{H^1} = \int_{\mathbb{R}^N} \nabla u\nabla v + m_0 uv\,dx.$$

We remark that $H^1_r(\mathbb{R}^N)$ is a closed subspace of $H^1(\mathbb{R}^N)$ and equip $\|\cdot\|_{H^1}$ to $H^1_r(\mathbb{R}^N)$.

The following properties are well-known (cf. [3]).

(i) For $N \geq 2$, there exists a $C_N > 0$ such that

$$|u(x)| \leq C_N|x|^{-(N-1)/2}\|u\|_{H^1} \quad \text{for } u \in H^1_r(\mathbb{R}^N) \text{ and } |x| \geq 1. \tag{2.2}$$

(ii) The embedding $H^1_r(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous for $2 \leq p \leq 2N/(N-2)$ if $N \geq 3$, $2 \leq p < \infty$ if $N = 2$ and it is compact for $2 < p < 2N/(N-2)$ if $N \geq 3$, $2 < p < \infty$ if $N = 2$.

(iii) Set $\Phi(s) = e^s - 1$. When $N = 2$, for any $\beta \in (0, 4\pi)$ there exists $\widetilde{C}_\beta > 0$ such that

$$\int_{\mathbb{R}^2} \Phi\bigg(\frac{\beta u^2}{\|\nabla u\|_2^2}\bigg)\,dx \leq \widetilde{C}_\beta \frac{\|u\|_2^2}{\|\nabla u\|_2^2} \quad \text{for all } u \in H^1(\mathbb{R}^2)\setminus\{0\}. \tag{2.3}$$

(cf. [1]).

(iv) In particular, for any $M > 0$

$$\int_{\mathbb{R}^2} \Phi\bigg(\frac{\beta u^2}{M^2}\bigg)\,dx \leq \widetilde{C}_\beta \frac{\|u\|_2^2}{M^2} \quad \text{for all } u \in H^1(\mathbb{R}^2) \text{ with } \|\nabla u\|_2 \leq M. \tag{2.4}$$

In fact, if $\|\nabla u\|_2 \le M$ holds,

$$M^2\Phi\left(\frac{\beta u^2}{M^2}\right) = M^2\sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{\beta u^2}{M^2}\right)^j = \sum_{j=1}^{\infty}\frac{1}{j!}\frac{\beta^j u^{2j}}{M^{2j-2}}$$
$$\le \sum_{j=1}^{\infty}\frac{1}{j!}\frac{\beta^j u^{2j}}{\|\nabla u\|_2^{2j-2}} = \|\nabla u\|_2^2\Phi\left(\frac{\beta u^2}{\|\nabla u\|_2^2}\right).$$

Thus (2.4) follows from (2.3) (cf. J. Byeon, L. Jeanjean and K. Tanaka [9]).

Let $\delta_0 > 0$ be a number given in Lemma 2.1(c) and Corollary 2.2(c). By (2.2), for any $M > 0$ there exists $R_M > 0$ such that

$$(2.5)\qquad |u(x)| \le \delta_0 \quad \text{for all } |x| \ge R_M \text{ and } u \in H_r^1(\mathbb{R}^N) \text{ with } \|u\|_{H^1} \le M.$$

In particular, it follows from (2.5) that

$$(2.6)\quad h(u(x)),\ \overline{h}(u(x)),\ H(u(x)),\ \overline{H}(u(x)) = 0 \quad \text{for } |x| \ge R_M \text{ and } \|u\|_{H^1} \le M.$$

From (2.6) and the compactness of the embedding $H_r^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, we have

Lemma 2.3. *Let $N \ge 2$ and suppose that $(u_j)_{j=1}^\infty \subset H_r^1(\mathbb{R}^N)$ converges to $u_0 \in H_r^1(\mathbb{R}^N)$ weakly in $H_r^1(\mathbb{R}^N)$. Then*

(a) $\int_{\mathbb{R}^N} H(u_j)\,dx \to \int_{\mathbb{R}^N} H(u_0)\,dx$, $\int_{\mathbb{R}^N} \overline{H}(u_j)\,dx \to \int_{\mathbb{R}^N} \overline{H}(u_0)\,dx$.

(b) $h(u_j) \to h(u_0)$, $\overline{h}(u_j) \to \overline{h}(u_0)$ *strongly in* $(H_r^1(\mathbb{R}^N))^*$.

Proof. We show only $h(u_j) \to h(u_0)$ strongly in $(H_r^1(\mathbb{R}^N))^*$ and deal with the case $N = 2$. Other cases can be treated similarly.

Suppose that $\|u_j\|_{H^1} \le M$ for all $j \in \mathbb{N}$. By (2.4), we have

$$\int_{\mathbb{R}^N} \Phi\left(\frac{u_j^2}{M^2}\right) dx \le \frac{\widetilde{C}_1}{M^2}\|u_j\|_2^2 \le \widetilde{C}_1.$$

Since $h(\xi)$ satisfies (g1'), for any $\varepsilon > 0$ there exists $\ell_\varepsilon (\ge \delta_0 > 0)$ such that

$$|h(\xi)| \le \varepsilon\Phi\left(\frac{\xi^2}{2M^2}\right) \quad \text{for } |\xi| \ge \ell_\varepsilon.$$

We set

$$\widetilde{h}(\xi) = \begin{cases} h(\xi) & \text{for } |\xi| \le \ell_\varepsilon, \\ h(\ell_\varepsilon) & \text{for } \xi > \ell_\varepsilon, \\ -h(\ell_\varepsilon) & \text{for } \xi < -\ell_\varepsilon. \end{cases}$$

Then we have

$$|h(\xi) - \widetilde{h}(\xi)| \le 2\varepsilon\Phi\left(\frac{1}{2}\frac{\xi^2}{M^2}\right) \quad \text{for all } \xi \in \mathbb{R}.$$

Since the embedding $H^1_r(\mathbb{R}^N) \to L^2(|x| \le R_M)$ is compact, we have $u_j \to u_0$ strongly in $L^2(|x| \le R_M)$, which implies

$$\widetilde{h}(u_j) \to \widetilde{h}(u_0) \quad \text{strongly in } L^2(|x| \le R_M).$$

Thus, by (2.6) and the definition of $\widetilde{h}(\xi)$, we have $\widetilde{h}(u_j(x)) = 0$ for $|x| \ge R_M$ and

$$\|\widetilde{h}(u_j) - \widetilde{h}(u_0)\|_2 \to 0 \quad \text{as } j \to \infty.$$

On the other hand,

$$\|h(u_j) - \widetilde{h}(u_j)\|_2^2 \le 4\varepsilon^2 \int_{\mathbb{R}^2} \Phi\bigg(\frac{u_j^2}{2M^2}\bigg)^2 dx \le 4\varepsilon^2 \int_{\mathbb{R}^2} \Phi\bigg(\frac{u_j^2}{M^2}\bigg)\, dx \le 4\varepsilon^2 \widetilde{C}_1.$$

Here we used the fact that $\Phi(s/2)^2 \le \Phi(s)$ for all $s \ge 0$. Similarly we also have $\|h(u_0) - \widetilde{h}(u_0)\|_2^2 \le 4\varepsilon^2 \widetilde{C}_1$. Thus

$$\begin{aligned} \|h(u_j) - h(u_0)\|_2 &\le \|h(u_j) - \widetilde{h}(u_j)\|_2 + \|\widetilde{h}(u_j) - \widetilde{h}(u_0)\|_2 + \|\widetilde{h}(u_0) - h(u_0)\|_2 \\ &\le \|\widetilde{h}(u_j) - \widetilde{h}(u_0)\|_2 + 4\varepsilon\sqrt{\widetilde{C}_1} \to 4\varepsilon\sqrt{\widetilde{C}_1} \quad \text{as } j \to \infty. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\|h(u_j) - h(u_0)\|_2 \to 0$. We remark that $H^1_r(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ implies $L^2(\mathbb{R}^N) \subset (H^1_r(\mathbb{R}^N))^*$ and thus $h(u_j) \to h(u_0)$ strongly in $(H^1_r(\mathbb{R}^N))^*$. □

2.3. A comparison functional $J(u)$**.** We define two functionals $I(u)$, $J(u)\colon H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} G(u)\, dx = \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbb{R}^N} \frac{1}{2} m_0 u^2 + G(u)\, dx, \\ J(u) &= \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbb{R}^N} \overline{H}(u)\, dx. \end{aligned}$$

Critical points of $I(u)$ are solutions of our original problem (1.1)–(1.2) and critical points of $J(u)$ are solutions of the following equation: $-\Delta u + m_0 u = \overline{h}(u)$ in $\mathbb{R}^N$. We have the following

Lemma 2.4.

(a) *$I(u)$, $J(u) \in C^1(H^1_r(\mathbb{R}^N), \mathbb{R})$ and, for all u, $\varphi \in H^1_r(\mathbb{R}^N)$,*

$$\begin{aligned} I'(u)\varphi &= (u, \varphi)_{H^1} - \int_{\mathbb{R}^N} m_0 u\varphi + g(u)\varphi\, dx, \\ J'(u)\varphi &= (u, \varphi)_{H^1} - \int_{\mathbb{R}^N} \overline{h}(u)\varphi\, dx. \end{aligned}$$

(b) *$I(u) \ge J(u)$ for all $u \in H^1_r(\mathbb{R}^N)$.*

(c) *There exist $r_0 > 0$ and $\rho_0 > 0$ such that*

$$\begin{aligned} I(u), J(u) &\ge 0 \quad \text{for } \|u\|_{H^1} \le r_0, \\ I(u), J(u) &\ge \rho_0 \quad \text{for } \|u\|_{H^1} = r_0. \end{aligned}$$

(d) *For any* $n \in \mathbb{N}$, *there exists an odd continuous mapping* $\gamma_{0n}: S^{n-1} = \{\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n;\ |\sigma| = 1\} \to H^1_r(\mathbb{R}^N)$ *such that*

$$I(\gamma_{0n}(\sigma)), J(\gamma_{0n}(\sigma)) < 0 \quad \textit{for all } \sigma \in S^{n-1}.$$

Proof. (a) follows from (g1') and (g2').

(b) follows from (a) of Corollary 2.2.

Since $g(0) = 0$, $\overline{h}(0) = 0$, (c) follows from (g1') and Sobolev inequality ($N \geq 3$) or (2.4) ($N = 2$). Since $\overline{h}(\xi)$ is an odd function and satisfies $\overline{H}(\zeta_0) - m_0\zeta_0^2/2 \geq G(\zeta_0) > 0$, we can argue as in Theorem 10 of [4] and find for any $n \in \mathbb{N}$ an odd continuous mapping $\pi_n: S^{n-1} \to H^1_r(\mathbb{R}^N)$ such that

$$0 \notin \pi_n(S^{n-1}), \qquad \int_{\mathbb{R}^N} G(\pi_n(\sigma))\,dx \geq 1 \quad \text{for all } \sigma \in S^{n-1}.$$

For $\ell \geq 1$, set

$$\gamma_{0n}(\sigma)(x) = \pi_n(\sigma)(x/\ell): S^{n-1} \to H^1_r(\mathbb{R}^N).$$

Then

$$I(\gamma_{0n}(\sigma)) = \frac{\ell^{N-2}}{2}\|\nabla \pi_n(\sigma)\|_2^2 - \ell^N \int_{\mathbb{R}^N} G(\pi_n(\sigma))\,dx \leq \frac{\ell^{N-2}}{2}\|\nabla \pi_n(\sigma)\|_2^2 - \ell^N.$$

Thus for sufficiently large $\ell = \ell_n \geq 1$, $\gamma_{0n}(\sigma)$ has the desired property for $I(u)$. Since (b) holds, $\gamma_{0n}(\sigma)$ also has the desired property for $J(u)$. □

By the above lemma, $I(u)$ and $J(u)$ have symmetric mountain pass geometry and we can define symmetric mountain pass values. We will give them in Section 3.

One of the virtue of our comparison functional $J(u)$ is the following:

Lemma 2.5. *$J(u)$ satisfies the Palais–Smale compactness condition.*

Proof. Since $\overline{h}(\xi)$ satisfies the global Ambrosetti–Rabinowitz condition (see Corollary 2.2(e)), we can easily verify the Palais–Smale condition. Indeed, let $(u_j)_{j=1}^\infty \subset H^1_r(\mathbb{R}^N)$ be a sequence satisfying

$$J(u_j) \to b, \tag{2.7}$$

$$\|J'(u_j)\|_{(H^1_r(\mathbb{R}^N))^*} \to 0. \tag{2.8}$$

From Corollary 2.2(e), we have

$$\begin{aligned} J(u_j) - \frac{1}{p_0+1}J'(u_j)u_j &= \left(\frac{1}{2} - \frac{1}{p_0+1}\right)\|u_j\|_{H^1}^2 - \int_{\mathbb{R}^N} \overline{H}(u_j) - \frac{1}{p_0+1}\overline{h}(u_j)u_j\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p_0+1}\right)\|u_j\|_{H^1}^2. \end{aligned}$$

Thus we can get boundedness of $(u_j)_{j=1}^{\infty}$ in $H_r^1(\mathbb{R}^N)$ from (2.7)–(2.8) and extract a subsequence such that $u_{j_k} \rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^N)$. By Lemma 2.3(b), we have $\overline{h}(u_{j_k}) \to \overline{h}(u_0)$ strongly in $(H_r^1(\mathbb{R}^N))^*$, thus by (2.8), u_{j_k} converges strongly in $H_r^1(\mathbb{R}^N)$. □

3. Minimax arguments

By Lemma 2.4, $I(u)$ and $J(u)$ have a symmetric mountain pass geometry and we can define mountain pass and symmetric mountain pass values. Here we follow [15, Chapter 9] essentially and set for $n \in \mathbb{N}$

$$b_n = \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} I(\gamma(\sigma)), \quad c_n = \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} J(\gamma(\sigma)). \tag{3.1}$$

Here $D_n = \{\sigma = (\sigma_1, \ldots , \sigma_n) \in \mathbb{R}^n : |\sigma| \le 1\}$ and a family of mappings Γ_n is defined by

$$\begin{aligned}\Gamma_n = \{\gamma \in C(D_n, H_r^1(\mathbb{R}^N)) : \gamma(-\sigma) = -\gamma(\sigma) \text{ for all } \sigma \in D_n,\\ \gamma(\sigma) = \gamma_{0n}(\sigma) \text{ for all } \sigma \in \partial D_n\},\end{aligned} \tag{3.2}$$

where $\gamma_{0n}(\sigma): \partial D_n = S^{n-1} \to H_r^1(\mathbb{R}^N)$ is given in Lemma 2.4. We remark that

$$\gamma(\sigma) = \begin{cases} |\sigma|\gamma_{0n}\left(\dfrac{\sigma}{|\sigma|}\right) & \text{for } \sigma \in D_n \setminus \{0\},\\ 0 & \text{for } \sigma = 0,\end{cases}$$

belongs to Γ_n and $\Gamma_n \ne \emptyset$ for all n.

Remark 3.1. We can define mountain pass minimax values b_{mp}, c_{mp} for $I(u)$, $J(u)$ by

$$b_{mp} = \inf_{\gamma\in\Gamma_{mp}} \max_{t\in[0,1]} I(\gamma(t)), \quad c_{mp} = \inf_{\gamma\in\Gamma_{mp}} \max_{t\in[0,1]} J(\gamma(t)), \tag{3.3}$$

where $\Gamma_{mp} = \{\gamma(t) \in C([0, 1], H_r^1(\mathbb{R}^N)) : \gamma(0) = 0,\ \gamma(1) = e_0\}$ and $e_0 \in H_r^1(\mathbb{R}^N)$ is chosen so that $I(e_0) < 0$. We will show in Section 6 that b_{mp}, c_{mp} do not depend on the choice of e_0 (see Lemma 6.1). Thus, recalling $S^0 = \{\pm1\}$ and choosing $e_0 = \gamma_{01}(1)$, we can see $b_{mp} = b_1$, $c_{mp} = c_1$. We will also show that b_{mp} is corresponding to a positive least energy solution of (1.1)–(1.2) in Section 6.

We can easily see that $\gamma(D_n) \cap \{u \in H_r^1(\mathbb{R}^N) : \|u\|_{H^1} = r_0\} \ne \emptyset$ for all $\gamma \in \Gamma_n$. Thus, it follows from Lemma 2.4(b) and (c) that

$$b_n \ge c_n \ge \rho_0 > 0. \tag{3.4}$$

Moreover, we have:

Lemma 3.2.

(a) *c_n ($n \in \mathbb{N}$) are critical values of $J(u)$.*
(b) *$c_n \to \infty$ as $n \to \infty$.*

Proof. (a) By Lemma 2.5, $J(u)$ satisfies the Palais–Smale condition. Thus (a) holds (see for example [15]).

(b) We apply an argument in [15, Chapter 9]. We set

$$\underline{\Gamma}_n = \{h(\overline{D_m \setminus Y}) : h \in \Gamma_m,\ m \geq n,\ Y \in \mathcal{E}_m \text{ and } \operatorname{genus}(Y) \leq m-n\}.$$

Here $\mathcal{E}_m$ is the family of closed sets $A \subset \mathbb{R}^m \setminus \{0\}$ such that $-A = A$ and genus(A) is the Krasnoselski's genus of A. We define another sequence of minimax values by

$$d_n = \inf_{A \in \underline{\Gamma}_n} \max_{u \in A} J(u).$$

Then we have $c_n \geq d_n$ for all $n \in \mathbb{N}$, $d_1 \leq d_2 \leq \ldots \leq d_n \leq d_{n+1} \leq \ldots$. Moreover, since $J(u)$ satisfies the Palais–Smale condition, modifying the argument in [15, Chapter 9] slightly, we have $d_n \to \infty$ as $n \to \infty$. Thus $c_n \to \infty$ as $n \to \infty$. □

By (3.4) and Lemma 3.2, the minimax values b_n satisfy

$$b_n > 0 \quad (n \in \mathbb{N}), \qquad b_n \to \infty \quad \text{as } n \to \infty.$$

In the following sections we will see b_n are critical values of $I(u)$.

4. Functional $\widetilde{I}(\theta, u)$

It seems difficult to show the Palais-Smale compactness condition for $I(u)$ directly and it is a main difficulty in showing b_n are critical values of $I(u)$.

As stated in Introduction, we introduce an auxiliary functional $\widetilde{I}(\theta, u) \in C^1(\mathbb{R} \times H_r^1(\mathbb{R}^N), \mathbb{R})$ by

$$\widetilde{I}(\theta, u) = \frac{1}{2} e^{(N-2)\theta} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - e^{N\theta} \int_{\mathbb{R}^N} G(u) \, dx.$$

$\widetilde{I}(\theta, u)$ is introduced based on the scaling properties of $\|\nabla u\|_2^2$, $\int_{\mathbb{R}^N} G(u)\,dx$ and has the following properties:

$$\widetilde{I}(0, u) = I(u), \tag{4.1}$$

$$\widetilde{I}(\theta, u(x)) = I(u(e^{-\theta} x)) \quad \text{for all } \theta \in \mathbb{R} \text{ and } u \in H_r^1(\mathbb{R}^N). \tag{4.2}$$

We equip a standard product norm $\|(\theta, u)\|_{\mathbb{R} \times H^1} = (|\theta|^2 + \|u\|_{H^1}^2)^{1/2}$ to $\mathbb{R} \times H_r^1(\mathbb{R}^N)$.

We define minimax values $\widetilde{b}_n$ for $\widetilde{I}(\theta, u)$ by

$$\widetilde{b}_n = \inf_{\widetilde{\gamma} \in \widetilde{\Gamma}_n} \max_{\sigma \in D_n} \widetilde{I}(\widetilde{\gamma}(\sigma)),$$

$$\begin{aligned} \widetilde{\Gamma}_n = \{\widetilde{\gamma}(\sigma) \in C(D_n, \mathbb{R} \times H_r^1(\mathbb{R}^N)) : \widetilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \text{ satisfies} \\ (\theta(-\sigma), \eta(-\sigma)) = (\theta(\sigma), -\eta(\sigma)) \quad \text{for all } \sigma \in D_n, \\ (\theta(\sigma), \eta(\sigma)) = (0, \gamma_{0n}(\sigma)) \quad \text{for all } \sigma \in \partial D_n\}. \end{aligned}$$

Then we have

Lemma 4.1. *$\widetilde{b}_n = b_n$ for all $n \in \mathbb{N}$.*

Proof. For any $\gamma \in \Gamma_n$ we can see that $(0, \gamma(\sigma)) \in \widetilde{\Gamma}_n$ and we may regard $\Gamma_n \subset \widetilde{\Gamma}_n$. Thus by the definitions of b_n, $\widetilde{b}_n$ and (4.1), we have $\widetilde{b}_n \le b_n$. Next, for any given $\widetilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \in \widetilde{\Gamma}_n$, we set $\gamma(\sigma) = \eta(\sigma)(e^{-\theta(\sigma)}x)$. We can verify that $\gamma(\sigma) \in \Gamma_n$ and, by (4.2), $I(\gamma(\sigma)) = \widetilde{I}(\widetilde{\gamma}(\sigma))$ for all $\sigma \in D_n$. Thus we also have $\widetilde{b}_n \ge b_n$. □

As a virtue of $\widetilde{I}(\theta, u)$ we can obtain a Palais–Smale sequence (θ_j, u_j) in the augmented space $\mathbb{R} \times H^1_r(\mathbb{R}^N)$ with an additional property (d) in Proposition 4.2 below. Namely we have:

Proposition 4.2. *For any $n \in \mathbb{N}$ there exists a sequence $(\theta_j, u_j)_{j=1}^{\infty} \subset \mathbb{R} \times H^1_r(\mathbb{R}^N)$ such that:*

(a) $\theta_j \to 0$.
(b) $\widetilde{I}(\theta_j, u_j) \to b_n$.
(c) $\widetilde{I}'(\theta_j, u_j) \to 0$ *strongly in* $(H^1_r(\mathbb{R}^N))^*$.
(d) $\dfrac{\partial}{\partial\theta}\widetilde{I}(\theta_j, u_j) \to 0$.

To prove Proposition 4.2, we need the following lemma, which is a version of Ekeland's principle. In the following lemma we use notation:

$$\begin{aligned} D\widetilde{I}(\theta, u) &= \left(\frac{\partial \widetilde{I}}{\partial\theta}(\theta, u), \widetilde{I}'(\theta, u)\right), \\ \operatorname{dist}_{\mathbb{R}\times H^1_r(\mathbb{R}^N)}((\theta, u), A) &= \inf_{(\tau, v)\in A} (|\theta - \tau|^2 + \|u - v\|^2_{H^1})^{1/2} \end{aligned}$$

for $A \subset \mathbb{R} \times H^1_r(\mathbb{R}^N)$.

Lemma 4.3. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Suppose $\widetilde{\gamma} \in \widetilde{\Gamma}_n$ satisfies*

$$\max_{\sigma \in D_n} \widetilde{I}(\widetilde{\gamma}(\sigma)) \le \widetilde{b}_n + \varepsilon.$$

Then there exists $(\theta, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N)$ such that:

(a) $\operatorname{dist}_{\mathbb{R}\times H^1_r(\mathbb{R}^N)}((\theta, u), \widetilde{\gamma}(D_n)) \le 2\sqrt{\varepsilon}$.
(b) $\widetilde{I}(\theta, u) \in [b_n - \varepsilon, b_n + \varepsilon]$.
(c) $\|D\widetilde{I}(\theta, u)\|_{\mathbb{R}\times(H^1_r(\mathbb{R}^N))^*} \le 2\sqrt{\varepsilon}$.

Proof. Since $\widetilde{I}(\theta, u)$ satisfies

$$\widetilde{I}(\theta, -u) = \widetilde{I}(\theta, u) \quad \text{for all } (\theta, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N),$$

we can see that the family $\widetilde{\Gamma}_n$ is stable under the pseudo-deformation flow generated by $\widetilde{I}(\theta, u)$. Moreover, since $\widetilde{b}_n = b_n > 0$, $I(0) = 0$ and $\max_{\sigma\in\partial D_n} \widetilde{I}(0, \gamma_{0n}(\sigma)) < 0$, we can show Lemma 4.3 in a standard way. □

Proof of Proposition 4.2. For any $j \in \mathbb{N}$ we can find a $\gamma_j \in \Gamma_n$ such that

$$\max_{\sigma \in D_n} I(\gamma_j(\sigma)) \le b_n + \frac{1}{j}.$$

Since $\widetilde{b}_n = b_n$, $\widetilde{\gamma}_j(\sigma) = (0, \gamma_j(\sigma)) \in \widetilde{\Gamma}_n$ satisfies $\max_{\sigma \in D_n} \widetilde{I}(\widetilde{\gamma}_j(\sigma)) \le \widetilde{b}_n + 1/j$. Applying Lemma 4.3, we can find a (θ_j, u_j) such that

$$\text{dist}_{\mathbb{R} \times H^1_r(\mathbb{R}^N)}((\theta_j, u_j), \widetilde{\gamma}_j(D_n)) \le \frac{2}{\sqrt{j}}, \tag{4.3}$$

$$\widetilde{I}(\theta_j, u_j) \in \left[b_n - \frac{1}{j}, b_n + \frac{1}{j} \right], \tag{4.4}$$

$$\| D\widetilde{I}(\theta_j, u_j) \|_{\mathbb{R} \times H^1} \le \frac{2}{\sqrt{j}}. \tag{4.5}$$

Since $\widetilde{\gamma}_j(D_n) \subset \{0\} \times H^1_r(\mathbb{R}^N)$, (4.3) implies $|\theta_j| \le 2/\sqrt{j}$, in particular, (a). Clearly (4.4) implies (b) and (4.5) implies (c) and (d). Thus the proof of Proposition 4.2 is completed. □

In the following section, we consider boundedness and compactness properties of the sequence $(\theta_j, u_j)_{j=1}^{\infty}$ satisfying (a)–(d) of Proposition 4.2.

5. Boundedness and compactness of (θ_j, u_j)

Let $(\theta_j, u_j) \subset \mathbb{R} \times H^1_r(\mathbb{R}^N)$ be a sequence given in Proposition 4.2. In particular, u_j satisfies (a)–(d) of Proposition 4.2. First we observe that (b) and (d) imply the following

$$\frac{1}{2} e^{(N-2)\theta_j} \|\nabla u_j\|_2^2 - e^{N\theta_j} \int_{\mathbb{R}^N} G(u_j)\, dx \to b_n,$$

$$\frac{N-2}{2} e^{(N-2)\theta_j} \|\nabla u_j\|_2^2 - N e^{N\theta_j} \int_{\mathbb{R}^N} G(u_j)\, dx \to 0 \quad \text{as } j \to \infty.$$

Thus we have

$$\|\nabla u_j\|_2^2 \to N b_n, \tag{5.1}$$

$$\int_{\mathbb{R}^N} G(u_j)\, dx \to \frac{N-2}{2} b_n. \tag{5.2}$$

First we show boundedness of (u_j) in $H^1_r(\mathbb{R}^N)$.

Proposition 5.1. *Let (θ_j, u_j) be a sequence satisfying* (a)–(d) *of Proposition* 4.2. *Then (u_j) is bounded in $H^1_r(\mathbb{R}^N)$.*

Proof (cf. Proof of Proposition 5.5 of [14]). We set

$$\varepsilon_j = \|\widetilde{I}'(\theta_j, u_j)\|_{(H^1_r(\mathbb{R}^N))^*}.$$

By Proposition 4.2(c) we have $\varepsilon_j \to 0$ and, for any $\psi \in H^1_r(\mathbb{R}^N)$,

$$|\widetilde{I}'(\theta_j, u_j)\psi| \le \varepsilon_j \|\psi\|_{H^1},$$

that is,

$$\left|e^{(N-2)\theta_j}\int_{\mathbb{R}^N}\nabla u_j\nabla\psi\,dx - e^{N\theta_j}\int_{\mathbb{R}^N} g(u_j)\psi\,dx\right| \le \varepsilon_j\sqrt{\|\nabla\psi\|_2^2+m_0\|\psi\|_2^2}. \tag{5.3}$$

We argue indirectly and assume $\|u_j\|_2\to\infty$. We remark that $\|\nabla u_j\|_2$ is bounded by (5.1). We set $t_j=\|u_j\|_2^{-2/N}\to 0$ and $v_j(y)=u_j(y/t_j)$. Then we have

$$\|v_j\|_2=1 \quad\text{and}\quad \|\nabla v_j\|_2^2=t_j^{N-2}\|\nabla u_j\|_2^2. \tag{5.4}$$

In particular, (v_j) is bounded in $H_r^1(\mathbb{R}^N)$ and we can extract a subsequence $v_j\to v_0$ weakly in $H_r^1(\mathbb{R}^N)$. First we claim:

Step 1. $v_0=0$.

Let $\varphi\in H_r^1(\mathbb{R}^N)$ be a function with compact support. Setting $\psi(x)=\varphi(t_jx)$ in (5.3), we have

$$\begin{aligned}\left|e^{(N-2)\theta_j}t_j^{-(N-2)}\int_{\mathbb{R}^N}\nabla v_j\nabla\varphi\,dy - e^{N\theta_j}t_j^{-N}\int_{\mathbb{R}^N} g(v_j)\varphi\,dx\right|& \\ \le \varepsilon_j\sqrt{t_j^{-(N-2)}\|\nabla\varphi\|_2^2+m_0t_j^{-N}\|\varphi\|_2^2}&.\end{aligned}$$

Multiplying t_j^N,

$$\begin{aligned}\left|e^{(N-2)\theta_j}t_j^2\int_{\mathbb{R}^N}\nabla v_j\nabla\varphi\,dy - e^{N\theta_j}\int_{\mathbb{R}^N} g(v_j)\varphi\,dx\right|& \\ \le \varepsilon_j t_j^{N/2}\sqrt{t_j^2\|\nabla\varphi\|_2^2+m_0\|\varphi\|_2^2}\to 0&.\end{aligned}$$

Thus $v_0\in H_r^1(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} g(v_0)\varphi\,dy=0 \quad\text{for all } \varphi\in H_r^1(\mathbb{R}^N) \text{ with compact support}, \tag{5.5}$$

which implies $g(v_0)\equiv 0$. Since $\xi=0$ is an isolated solution of $g(\xi)=0$ by (g2'), it follows from (5.5) that $v_0(y)\equiv 0$.

Step 2. Conclusion.

Next we set $\psi(x)=u_j(x)$ in (5.3). We have

$$\begin{aligned}\left|e^{(N-2)\theta_j}t_j^{-(N-2)}\|\nabla v_j\|_2^2 - e^{N\theta_j}t_j^{-N}\int_{\mathbb{R}^N} g(v_j)v_j\,dx\right|& \\ \le \varepsilon_j\sqrt{t_j^{-(N-2)}\|\nabla v_j\|_2^2+m_0t_j^{-N}\|v_j\|_2^2}&.\end{aligned}$$

Again, multiplying t_j^N, we have

$$\delta_j\equiv e^{(N-2)\theta_j}t_j^2\|\nabla v_j\|_2^2 - e^{N\theta_j}\int_{\mathbb{R}^N} g(v_j)v_j\,dx\to 0.$$

Thus,

$$e^{(N-2)\theta_j}t_j^2\|\nabla v_j\|_2^2+m_0e^{N\theta_j}\|v_j\|_2^2 = e^{N\theta_j}\int_{\mathbb{R}^N} m_0v_j^2+g(v_j)v_j\,dx+\delta_j \le e^{N\theta_j}\int_{\mathbb{R}^N} h(v_j)v_j\,dx+\delta_j. \tag{5.6}$$

Here we used Lemma 2.1(a). Since $v_j \to 0$ weakly in $H_r^1(\mathbb{R}^N)$, Lemma 2.3(b) implies $\int_{\mathbb{R}^N} h(v_j)v_j\,dx \to 0$. Thus (5.6) implies $\|v_j\|_2 \to 0$, which is in contradiction to (5.4). Therefore (u_j) is bounded in $H_r^1(\mathbb{R}^N)$. □

REMARK 5.2. When $N \ge 3$, we can prove Proposition 5.1 in a direct way. Indeed, by the definition of $h(\xi)$, we have for some constant $C > 0$

$$|h(\xi)| \le C|\xi|^{(N+2)/(N-2)} \quad \text{for all } \xi \in \mathbb{R}.$$

It follows from $\varepsilon_j = \|\widetilde{I}'(\theta_j,u_j)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0$ that $|\widetilde{I}'(\theta_j,u_j)u_j| \le \varepsilon_j\|u_j\|_{H^1}$. Thus

$$\begin{aligned} e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 &+ m_0e^{N\theta_j}\|u_j\|_2^2 \\ &\le e^{N\theta_j}\int_{\mathbb{R}^N} m_0u_j^2+g(u_j)u_j\,dx+\varepsilon_j\|u_j\|_{H^1} \\ &\le e^{N\theta_j}\int_{\mathbb{R}^N} h(u_j)u_j\,dx+\varepsilon_j\|u_j\|_{H^1} \\ &\le Ce^{N\theta_j}\|u_j\|_{2N/(N-2)}^{2N/(N-2)}+\varepsilon_j\|u_j\|_{H^1}. \end{aligned} \tag{5.7}$$

Since $\|\nabla u_j\|_2$ is bounded, we can observe that $\|u_j\|_{2N/(N-2)}$ is also bounded. Thus (5.7) implies boundedness of $\|u_j\|_2$, that is, (u_j) is bounded in $H_r^1(\mathbb{R}^N)$.

Lastly in this section, we prove that (u_j) has a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$.

PROPOSITION 5.3. *Let (θ_j,u_j) be a sequence satisfying* (a)–(d) *of Proposition* 4.2. *Then (θ_j,u_j) has a strongly convergent subsequence in $\mathbb{R}\times H_r^1(\mathbb{R}^N)$.*

PROOF. It suffices to prove (u_j) has a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$. By Proposition 5.1, (u_j) is bounded in $H_r^1(\mathbb{R}^N)$ and we may assume $u_j \to u_0$ weakly in $H_r^1(\mathbb{R}^N)$ as $j \to \infty$.

It follows from Proposition 4.2(c) that $\widetilde{I}'(\theta_j,u_j)\varphi \to 0$ as $j \to \infty$ for any $\varphi \in H_r^1(\mathbb{R}^N)$, that is,

$$\int_{\mathbb{R}^N} e^{(N-2)\theta_j}\nabla u_j\nabla\varphi - e^{N\theta_j}g(u_j)\varphi\,dx \to 0 \quad \text{as } j \to \infty. \tag{5.8}$$

Thus u_0 satisfies $\int_{\mathbb{R}^N} \nabla u_0\nabla\varphi - g(u_0)\varphi\,dx = 0$ for all $\varphi \in H_r^1(\mathbb{R}^N)$ and $u_0(x)$ is a solution of (1.1)–(1.2). In particular we have $\|\nabla u_0\|_2^2 - \int_{\mathbb{R}^N} g(u_0)u_0\,dx = 0$, that is,

$$\|u_0\|_{H^1}^2 - \int_{\mathbb{R}^N} m_0 u_0^2 + g(u_0)u_0\,dx = 0. \tag{5.9}$$

Setting $\varphi = u_j$ in (5.8), we have $e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 - e^{N\theta_j}\int_{\mathbb{R}^N} g(u_j)u_j\,dx \to 0$. Thus

$$\begin{aligned} e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + m_0 e^{N\theta_j}\|u_j\|_2^2 &= e^{N\theta_j}\int_{\mathbb{R}^N} m_0 u_j^2 + g(u_j)u_j\,dx + o(1) \\ &= e^{N\theta_j}\int_{\mathbb{R}^N} h(u_j)u_j\,dx - e^{N\theta_j}\int_{\mathbb{R}^N} h(u_j)u_j - m_0u_j^2 - g(u_j)u_j\,dx + o(1) \\ &= e^{N\theta_j}(\mathrm{I}) - e^{N\theta_j}(\mathrm{II}) + o(1) \quad \text{as } j\to\infty. \end{aligned} \tag{5.10}$$

By Lemma 2.3(b), we have

$$(\mathrm{I}) \to \int_{\mathbb{R}^N} h(u_0)u_0\,dx. \tag{5.11}$$

On the other hand, by Lemmma 2.1(a) we have $h(u_j(x))u_j(x) - m_0u_j(x)^2 - g(u_j(x))u_j(x) \geq 0$ for all $j \in \mathbb{N}$ and $x \in \mathbb{R}$. Thus by Fatou's lemma,

$$\liminf_{j\to\infty}(\mathrm{II}) \geq \int_{\mathbb{R}^N} h(u_0)u_0 - m_0u_0^2 - g(u_0)u_0\,dx. \tag{5.12}$$

It follows from (5.10)–(5.12) that

$$\begin{aligned} \limsup_{j\to\infty}\|u_j\|_{H^1}^2 &= \limsup_{j\to\infty}\Big(e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + m_0e^{N\theta_j}\|u_j\|_2^2\Big) \\ &\leq \int_{\mathbb{R}^N} m_0u_0^2 + g(u_0)u_0\,dx. \end{aligned}$$

Thus by (5.9) we have

$$\limsup_{j\to\infty}\|u_j\|_{H^1} \leq \|u_0\|_{H^1},$$

which implies $u_j \to u_0$ strongly in $H_r^1(\mathbb{R}^N)$. □

Now we can prove

Theorem 5.4. *Assume* $N \geq 2$ *and* (g0), (g1'), (g2'), (g3). *Then* b_n $(n \in \mathbb{N})$ *defined in* (3.1)–(3.2) *is a critical value of* $I(u)$. *That is, for any* $n \in \mathbb{N}$ *there exists a critical point* $u_{0n}(x) \in H_r^1(\mathbb{R}^N)$, *which is a solution of* (1.1)–(1.2), *such that*

$$I(u_{0n}) = b_n, \quad I'(u_{0n}) = 0. \tag{5.13}$$

Proof. Let (θ_j, u_j) be a sequence obtained in Proposition 4.2. By Proposition 5.3, we may assume $u_j \to u_{0n}$ strongly in $H_r^1(\mathbb{R}^N)$. Then u_{0n} satisfies

$$\widetilde{I}(0,u_{0n}) = b_n \quad \text{and} \quad \widetilde{I}'(0,u_{0n}) = 0,$$

that is nothing but (5.13). Thus b_n is a critical value of $I(u)$ which completes the proof. □

6. Least energy solutions

In this section we show that a mountain pass value b_{mp} is corresponding to a positive solution of (1.1)–(1.2), which has the least energy among all non-trivial solutions.

We start with the following lemma.

Lemma 6.1. *Suppose $N \geq 2$ and assume* (g0), (g1'), (g2') *and* (g3). *Let $O = \{u \in H^1_r(\mathbb{R}^N) : I(u) < 0\}$. Then O is arc-wise connected.*

We will give a proof of Lemma 6.1 in the Appendix. By Lemma 6.1, we can easily see that the mountain pass minimax value b_{mp} given in (3.3) does not depend on the end point e_0 and we may write

$$b_{mp} = \inf_{\gamma \in \Gamma_{mp}} \max_{t \in [0,1]} I(\gamma(t)), \tag{6.1}$$

$$\Gamma_{mp} = \{\gamma \in C([0,1], H^1_r(\mathbb{R}^N)) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}. \tag{6.2}$$

This fact is also used in Remark 3.1.

Remark 6.2. Lemma 6.1 is also obtained in Byeon [8] (but with a different proof). We learned [8] from Professor J. Byeon and the referee after submission of this paper.

Our main result in this section is the following

Theorem 6.3. *Suppose $N \geq 2$ and assume* (g0), (g1'), (g2'), (g3). *Then for b_{mp} defined in* (6.1)–(6.2) *we have:*

(a) *There exists a positive solution $u_0(x) > 0$ of* (1.1)–(1.2) *such that*

$$I(u_0) = b_{mp}. \tag{6.3}$$

(b) *For any non-trivial solution $v(x)$ of* (1.1)–(1.2), *we have*

$$b_{mp} \leq I(v), \tag{6.4}$$

that is, $u_0(x)$ is the least energy solution of (1.1)–(1.2) *and b_{mp} is the least energy level.*

Proof. (a) We argue as in previous sections and for any $\gamma_j \in \Gamma_{mp}$ satisfying

$$\max_{t \in [0,1]} I(\gamma_j(t)) \leq b_{mp} + \frac{1}{j} \tag{6.5}$$

we can find a $(\theta_j, u_j) \in \mathbb{R} \times H^1_r(\mathbb{R}^N)$ such that

$$\operatorname{dist}_{\mathbb{R}\times H^1_r(\mathbb{R}^N)}((\theta_j, u_j), \{0\} \times \gamma_j([0,1])) \le \frac{2}{\sqrt{j}}, \tag{6.6}$$

$$u_j(x) \to u_0(x) \quad \text{strongly in } H^1_r(\mathbb{R}^N). \tag{6.7}$$

Here u_0 is a critical point of $I(u)$ satisfying $I(u_0) = b_{mp}$. Since $I(u) = I(|u|)$ for all $u \in H^1_r(\mathbb{R}^N)$, we may assume $\gamma_j \in \Gamma_{mp}$ in (6.5) satisfies

$$\gamma_j(t)(x) \ge 0 \quad \text{for all } t \in [0,1] \text{ and } x \in \mathbb{R}^N.$$

Then it follows from (6.6) that

$$\|u_{j-}\|_{H^1} \le \operatorname{dist}_{\mathbb{R}\times H^1_r(\mathbb{R}^N)}((\theta_j, u_j), \{0\} \times \gamma_j([0,1])) \to 0,$$

where $u_-(x) = \max\{0, -u(x)\}$. Thus we have $u_{0-}(x) = 0$ and by the maximal principle $u_0(x) > 0$ in $\mathbb{R}^N$ and (a) is proved.

(b) To see (6.4), we can use argument in [13] and for any given non-trivial solution $v \in H^1_r(\mathbb{R}^N)$ we can construct a path $\gamma \in \Gamma_{mp}$ such that

$$v(x) \in \gamma([0,1]), \qquad \max_{t\in[0,1]} I(\gamma(t)) = I(v).$$

Thus, we have (b) and the proof of Theorem 6.3 is completed. □

End of proof of Theorem 1.3. Theorem 1.3 clearly follows from Theorems 5.4 and 6.3. □

7. Appendix

The aim of this appendix is to give a proof of Lemma 6.1. We will show that for any u_0, $u_1 \in O$ there exists a continuous path $\gamma(t)$ in O joining u_0 and u_1.

In this appendix, we write $r = |x|$ and we identify $u(r)$ and a radially symmetric function $u(x) = u(|x|)$. We set for $R \ge 1$, $t \ge 0$

$$\eta(R, t : r) = \begin{cases} 0 & \text{for } r \in [0, R], \\ \zeta_0(r - R) & \text{for } r \in [R, R+1], \\ \zeta_0 & \text{for } r \in [R+1, R+1+t], \\ \zeta_0(R+2+t-r) & \text{for } r \in [R+1+t, R+2+t], \\ 0 & \text{for } r \in [R+2+t, \infty). \end{cases}$$

Here $\zeta_0 > 0$ is given in (g3). In particular, we have $G(\zeta_0) > 0$.

We will see that $\eta(R, T; r) \in O$ for large R, T and there exist continuous curves joining u_i $(i = 0, 1)$ and $\eta(R, T; r)$ in O. Clearly this proves our Lemma 6.1.

We start with the following lemma.

LEMMA 7.1. *There exist* $R_0 \geq 1$ *and* C_0, $C_1 > 0$ *independent of* R *and* t *such that:*

(a) $I(\eta(R,t;r)) \leq -C_0 G(\zeta_0) t^N$ *for all* (R,t) *with* $t \geq R \geq R_0$.
(b) $\sup_{t\in[0,\infty)} I(\eta(R,t;r)) \leq C_1 R^{N-1}$ *for all* $R \geq R_0$.
(c) $\max_{s\in[0,1]} I(s\eta(R,0;r)) \leq C_1 R^{N-1}$ *for all* $R \geq R_0$.

PROOF. For $R \geq 1$, $t \geq 0$, a direct computation gives us

$$\begin{aligned}
&I(\eta(R,t;r)) \\
&= \omega_{N-1}\biggl(\int_R^{R+1} + \int_{R+1}^{R+1+t} + \int_{R+1+t}^{R+2+t}\biggr)\biggl(\frac{1}{2}|\eta_r(R,t;r)|^2 - G(\eta(R,t;r))\biggr) r^{N-1}\,dr \\
&\leq \frac{\omega_{N-1}}{N} B((R+1)^N - R^N + (R+2+t)^N - (R+1+t)^N) \\
&\quad - \frac{\omega_{N-1}}{N} G(\zeta_0)((R+1+t)^N - (R+1)^N),
\end{aligned}$$

where ω_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$ and B is defined by

$$B = \frac{1}{2}\zeta_0^2 + \max_{\xi\in[0,\zeta_0]} |G(\xi)|. \tag{7.1}$$

We remark for $R \geq 1$ and $t \geq 0$

$$\begin{aligned}
(R+1)^N - R^N &= {}_N C_1 R^{N-1} + {}_N C_2 R^{N-2} + \ldots + {}_N C_N \\
&\leq ({}_N C_1 + \ldots + {}_N C_N) R^{N-1} = (2^N - 1) R^{N-1}, \\
(R+2+t)^N - (R+1+t)^N &\leq (2^N - 1)(R+1+t)^{N-1} \\
&\leq 2^{N-1}(2^N - 1)(R+t)^{N-1}, \\
(R+1+t)^N - (R+1)^N &\geq t^N.
\end{aligned}$$

Thus there exists a constant $C_2 > 0$ independent of $R \geq 1$, $t \geq 0$ such that

$$I(\eta(R,t;r)) \leq C_2(R^{N-1} + (R+t)^{N-1}) - \frac{\omega_{N-1}}{N} G(\zeta_0) t^N. \tag{7.2}$$

(a)–(c) follow from (7.2). Indeed, if $t \geq R$, it follows from (7.2) that

$$I(\eta(R,t;r)) \leq C_2(t^{N-1} + (2t)^{N-1}) - \frac{\omega_{N-1}}{N} G(\zeta_0) t^N.$$

Thus for sufficiently large $R_0 \geq 1$, (a) holds.

By (a), for $R \geq R_0$ we have $\sup_{t\in[0,\infty)} I(\eta(R,t;r)) = \max_{t\in[0,R]} I(\eta(R,t;r))$. From (7.2) we have

$$I(\eta(R,t;r)) \leq C_2(R^{N-1} + (2R)^{N-1}) \quad \text{for } t \in [0,R].$$

Thus we have (b).

For (c), recalling (7.1), we have

$$\begin{aligned} I(s\eta(R,0;r)) &\le \omega_{N-1}\int_R^{R+2}\bigg(\frac{1}{2}|s\eta_r(R;0;r)|^2 - G(s\eta(R;0;r))\bigg)r^{N-1}\,dr \\ &\le \frac{\omega_{N-1}}{N}B\bigg((R+2)^N - R^N\bigg) \quad \text{for } s\in[0,1]. \end{aligned}$$

Thus, choosing $C_1 > 0$ larger if necessary, we get (c). □

Now suppose u_0, $u_1 \in O$ and we try to join u_0 and u_1 through $\eta(R_1,T_1;r)$ ($T_1 \ge R_1 \gg 1$) in O. We remark that we may assume that u_0, u_1 have compact supports and

$$\operatorname{supp} u_0(r),\ \operatorname{supp} u_1(r) \subset [0,L_0] \quad \text{for some constant } L_0 > 0.$$

We consider the following curves:

$$\begin{aligned} &\gamma_1\colon [L_0,R_1] \to H_r^1(\mathbb{R}^N), && R \mapsto u_0(L_0 r/R), \\ &\gamma_2\colon [0,1] \to H_r^1(\mathbb{R}^N), && s \mapsto u_0(L_0 r/R_1) + s\eta(R_1,0;r), \\ &\gamma_3\colon [0,T_1] \to H_r^1(\mathbb{R}^N), && t \mapsto u_0(L_0 r/R_1) + \eta(R_1,t;r), \\ &\gamma_4\colon [0,1] \to H_r^1(\mathbb{R}^N), && s \mapsto (1-s)u_0(L_0 r/R_1) + \eta(R_1,T_1;r). \end{aligned}$$

Joining these curves, we get the desired path joining $u_0(r)$ and $\eta(R_1,T_1;r)$. We need to show with suitable choices of R_1, T_1, our path is included in O.

Lemma 7.2.

(a) $I(u_0(L_0 r/R)) < 0$ *for all* $R \in [L_0,\infty)$.

(b) *There exists* $R_1 \ge R_0$ *such that*

$$I(u_0(L_0 r/R_1) + s\eta(R_1,0;r)) < 0 \quad \text{for all } s \in [0,1], \tag{7.3}$$

$$I(u_0(L_0 r/R_1) + \eta(R_1,t;r)) < 0 \quad \text{for all } t \in [0,\infty). \tag{7.4}$$

(c) *There exists* $T_1 \ge R_1$ *such that*

$$I((1-s)u_0(L_0 r/R_1) + \eta(R_1,T_1;r)) < 0 \quad \text{for all } s \in [0,1]. \tag{7.5}$$

Proof. (a) Since $u_0 \in O$, we have $\int_{\mathbb{R}^N} G(u_0)\,dx > 0$ and we can see $R \mapsto I(u_0(r/R))$, $[1,\infty) \to H_r^1(\mathbb{R}^N)$ is strictly decreasing. Thus (a) holds.

(b) We mainly deal with (7.4). Suppose $R_1 \ge R_0$, where $R_0 \ge 1$ is given in Lemma 7.1. We remark

$$\operatorname{supp} u_0(L_0 r/R_1) \subset [0,R_1], \qquad \operatorname{supp} \eta(R_1,t;r) \subset [R_1, R_1+2+t].$$

Thus, for all $t \geq 0$, $R_1 \geq R_0$,

$$\begin{aligned} I(u_0(L_0 r/R_1) + \eta(R_1, t; r)) &= I(u_0(L_0 r/R_1)) + I(\eta(R_1, t; r)) \\ &\leq \frac{1}{2}\left(\frac{R_1}{L_0}\right)^{N-2} \|\nabla u_0\|_2^2 - \left(\frac{R_1}{L_0}\right)^N \int_{\mathbb{R}^N} G(u_0)\,dx + C_1 R_1^{N-1}. \end{aligned}$$

Here we used Lemma 7.1(b). Thus for sufficiently large $R_1 \geq R_0$ we have (7.4). Using Lemma 7.1(c), we also get (7.3).

(c) As in the proof of (b), for $T_1 \geq R_1$ we have from Lemma 7.1(a)

$$\begin{aligned} I((1-s)u_0(L_0 r/R_1) + \eta(R_1, T_1; r)) &= I((1-s)u_0(L_0 r/R_1)) + I(\eta(R_1, T_1; r)) \\ &\leq I((1-s)u_0(L_0 r/R_1)) - C_0 T_1^N. \end{aligned}$$

Taking $T_1 \geq R_1$ large, we have (7.5). □

END OF THE PROOF OF LEMMA 6.1. We choose $R_1 \geq R_0$ and $T_1 \geq R_1$ as in Lemma 7.2. We can see $\gamma_1([L_0, R_1])$, $\gamma_2([0,1])$, $\gamma_3([0,T_1])$, $\gamma_4([0,1]) \subset O$ and thus $u_0(r)$ and $\eta(R_1, T_1; r)$ are connected by a continuous path in O. We can also join $u_1(r)$ and $\eta(R_1, T_1; r)$ in O in a similar way. Thus Lemma 6.1 is proved. □

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