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TWIN POSITIVE SOLUTIONS FOR SINGULAR NONLINEAR ELLIPTIC EQUATIONS

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Abstract. For a bounded domain $Z \subseteq \mathbb{R}^N$ with a C^2-boundary, we prove the existence of an ordered pair of smooth positive strong solutions for the nonlinear Dirichlet problem

$$-\Delta_p\, x(z) = \beta(z)x(z)^{-\eta} + f(z, x(z)) \quad \text{a.e. on } Z \text{ with } x \in W_0^{1,p}(Z),$$

which exhibits the combined effects of a singular term ($\eta \geq 0$) and a $(p-1)$-linear term $f(z, x)$ near $+\infty$, by using a combination of variational methods, with upper-lower solutions and with suitable truncation techniques.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We study the existence of positive solutions of the following nonlinear singular Dirichlet problem:

$$\begin{cases} -\Delta_p\, x(z) = \beta(z)x(z)^{-\eta} + f(z, x(z)) & \text{a.e. on } Z, \\ x|_{\partial Z} = 0, \end{cases} \tag{1.1}$$

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where Δ_p denotes (as usual) the p-Laplace differential operator, defined by $\Delta_p x = \operatorname{div}(\|Dx\|^{p-2}Dx)$, $1 < p < \infty$, for all $x \in W_0^{1,p}(Z)$. Here, $\beta: Z \to \mathbb{R}$ is a measurable function, $\beta \geq 0$, $\eta \geq 0$ and $f: Z \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. it is measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$). Problem (1.1) was studied primarily within the context of semilinear equations (i.e. $p = 2$). Among the first works in this direction are the papers of M. Crandall, P. Rabinowitz, L. Tártar [6] and C. Stuart [15]. Since then, there have been several other papers on the subject. We mention the relevant works of M. M. Coclite and G. Palmieri [5], J. I. Diaz, J.-M. Morel and L. Oswald [7], A. V. Lair and A. W. Shaker [9], A. W. Shaker [13], J. Shi and M. Yao [14], Y. Sun, S. Wu and Y. Long [16], Z. Zhang [18]. In particular, A. V. Lair and A. W. Shaker [9] assumed that $f \equiv 0$ and $\beta \in L^2(Z)$ and established the existence of a unique positive weak solution. Their result was extended by J. Shi and M. Yao [14] to the case of a "sublinear" reaction, namely when

$$f(z,x) = \lambda x^{r-1} \quad \text{with } \lambda > 0 \tag{1.2}$$

and $1 < r \leq 2$. The case of a "superlinear-subcritical" reaction, i.e. when (1.2) holds for $2 < r < 2^*$, where 2^* is the critical Sobolev exponent, was investigated by M. M. Coclite and G. Palmieri [5] under the assumption that $\beta \equiv 1$. In both works (i.e. [5], [14]), it is shown that there exists a critical value $\lambda^* > 0$ of the parameter, such that for every $\lambda \in (0, \lambda^*)$, the problem admits a nontrivial positive solution. Subsequently, Y. Sun, S. Wu and Y. Long [16] using the Ekeland variational principle, obtained two nontrivial positive weak solutions for more general functions β. Z. Zhang in [18] extended their result to more general nonnegative superlinear perturbations, using critical point theory on closed convex sets.

Recently, there have been some works on singular elliptic problems driven by the p-Laplacian. We mention the works of R. P. Agarwal, H. Lü and D. O'Regan [2], R. P. Agarwal and D. O'Regan [3], where $N = 1$ (ordinary differential equations), and K. Perera, E. A. B. Silva [11], K. Perera, Z. Zhang [12], where $N \geq 2$ (partial differential equations) and the reaction term has the form $\beta(z)x^{-\eta} + \lambda f(z,x)$ with $\lambda > 0$. For such a parametric nonlinearity, the authors prove existence and multiplicity results (two positive weak solutions), valid for all $\lambda \in (0, \lambda^*)$. Moreover, the perturbation term $f(z,\cdot)$ exhibits a strict $(p-1)$ superlinear growth near $+\infty$ and, more precisely, it satisfies on $\mathbb{R}_+ = [0,\infty)$, the well-known Ambrosetti–Rabinowitz condition.

In this paper, the reaction term is nonparametric and the perturbation $f(z,\,\cdot\,)$ is $(p-1)$-linear near $+\infty$. In detail, consider the following hypothesis on the singular term:

H(β): There exists $\vartheta \in C_+$ such that $\beta(\,\cdot\,)\vartheta(\,\cdot\,)^{-\eta} \in L^q(Z)$ for some $q > N$.

Here, C_+ denotes the cone of positive functions in the ordered Banach space $C_0^1(\overline{Z})$ and $\operatorname{int} C_+$ its nonempty interior (see Section 2).

Remark 1.1. If $\beta \in L^\infty(Z)$ and $\eta < 1/\max\{N/p, 1\}$, then we can take any $\vartheta \in \operatorname{int} C_+$ and $q < 1/\eta$. However, as it was observed by K. Perera and Z. Zhang [12], who where the first to use this hypothesis, H(β) does not require that $\eta < 1$, a restriction common in the literature. For example, if $Z \equiv B_1$ (the open unit ball in $\mathbb{R}^N$), $\beta(z) = (1 - \|z\|^2)^{\beta_0}$ with $\beta_0 \geq 0$, and $\eta < \beta_0 + 1/N$, then we can choose $\vartheta(z) = 1 - \|z\|^2$ and $q < 1/(\eta - \beta_0)$ if $\eta > \beta_0$ (but no additional restriction on q if $\eta \leq \beta_0$).

The hypotheses on the perturbation term $f(z, x)$ are the following:

H(f): $f\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(a) for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable;

(b) for almost all $z \in Z$, $x \mapsto f(z, x)$ is continuous and $f(z, 0) = 0$;

(c) for almost all $z \in Z$ and all $x \in \mathbb{R}$,
$$|f(z, x)| \leq a(z) + c|x|^{p-1}$$
with $a \in L^\infty(Z)_+$ and $c > 0$;

(d) there exist functions $\eta, \widehat{\eta} \in L^\infty(Z)_+$ such that
$$\lambda_1 \leq \eta(z) \quad \text{a.e. on } Z,$$
with strict inequality on a set of positive measure, and
$$\eta(z) \leq \liminf_{x\to\infty} \frac{f(z, x)}{x^{p-1}} \leq \limsup_{x\to\infty} \frac{f(z, x)}{x^{p-1}} \leq \widehat{\eta}(z)$$
uniformly almost everywhere on Z;

(e) there exist $\alpha_0 > 0$ and $\delta_0 \in (0, \min\{\alpha_0, 1\})$ such that
$$\beta(z)\alpha_0^{-\eta} + f(z, \alpha_0) \leq 0 \quad \text{a.e. on } Z,$$
and $f(z, x) \geq 0$ for almost all $z \in Z$ and all $x \in (0, \delta_0)$.

Here, λ_1 denotes the first eigenvalue of the negative Dirichlet p-Laplacian (see Section 2). Note that, for convenience, we use both notations: "for a.a. $z \in Z$" meaning "for almost all $z \in Z$", and "a.e. on Z" meaning "almost everywhere on Z".

Remark 1.2. Since we are looking for positive solutions and hypotheses H(f)(d)–(e) only concern the positive semiaxis $\mathbb{R}_+ = [0, \infty)$, we may (and will) assume that $f(z, x) = 0$ for almost all $z \in Z$ and all $x \leq 0$. Hypothesis H(f)(d) dictates a $(p - 1)$-linear growth of $x \mapsto f(z, x)$ near ∞. Hypotheses H(f)(a)–(c) are standard conditions which ensure that problem (1.1) is well defined and of variational nature.

A strong solution of problem (1.1) is a function $x \in W_0^{1,p}(Z)$ which satisfies (1.1) almost everywhere in Z. By using a combination of variational arguments based on critical point theory, with the method of upper-lower solutions and with suitable truncation techniques, we show that problem (1.1) has two nontrivial positive strong solutions. In particular, we prove the following theorem.

THEOREM 1.3. *If* H(β) *and* H(f) *hold, then problem* (1.1) *has* (*at least*) *two positive strong solutions such that* $x_0, \widehat{x} \in \operatorname{int} C_+$, $x_0 \leq \widehat{x}$ *and* $x_0 \neq \widehat{x}$.

As an example, consider the following nonlinearity (where for the sake of simplicity, we drop the z-dependence):

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^{\mu-1} - cx^{\tau-1} & \text{if } 0 \leq x \leq 1, \\ \eta_0 x^{p-1} - \xi x & \text{if } 1 < x, \end{cases}$$

where $\lambda_1 < \eta_0 < \infty$, $\xi = \eta_0 + c - 1$, $c > 1$ and $\mu < \tau$. If $\beta \in L^\infty(Z)$ satisfies $\|\beta\|_\infty < c - 1$, then the associated singular Dirichlet elliptic problem (1.1) has (at least) two positive strong solutions, as stated in Theorem 1.3. Note that, in this example, $\alpha_0 = 1$.

The rest of the paper is organized as follows. In the Section 2, we briefly recall the mathematical background relevant in subsequent Sections. In Section 3, we prove the intermediate results needed in the proof of Theorem 1.3, which will be concluded in Section 4.

2. Mathematical background

We recall some basic facts from critical point theory. So, let X be a Banach space and X^* its topological dual. By $\langle \cdot\,, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Let $\varphi \in C^1(X)$. We say that φ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (the PS_c-condition for short) if the following holds:

- every sequence $\{x_n\}_{n \geq 1} \subset X$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad \varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty$$

  has a strongly convergent subsequence.

If this is true at every level $c \in \mathbb{R}$, then we say that φ satisfies the Palais–Smalle condition (PS-condition for short).

Using this compactness-type condition, we have the following minimax characterization of certain critical values of a C^1-functional. The result is the well-known “mountain pass theorem”.

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$, $x_0, x_1 \in X$, $r > 0$, $\|x_0 - x_1\| > r$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \leq \inf\{\varphi(x) : \|x - x_0\| = r\} = c_0,$$
$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0 \text{ and } \gamma(1) = x_1\},$$

$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ and φ satisfies the C_c-condition, then $c \geq c_0$ and c is a critical value of φ. Moreover, if $c = c_0$, then there exists a critical point $x \in X$ of φ with critical value c and $\|x\| = r$.

Besides the Banach space $W_0^{1,p}(Z)$ and its dual $W^{-1,p'}(Z)$, other Banach spaces will be relevant in our approach. Namely, the space $C_0^1(\overline{Z}) = \{x \in C^1(\overline{Z}) : x|_{\partial Z} = 0\}$ is an ordered Banach space, so admits a positive cone

$$C_+ = \{x \in C_0^1(\overline{Z}) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \left\{x \in C_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial \nu}(z) < 0 \text{ for all } z \in \partial Z\right\},$$

where $\nu(\,\cdot\,)$ denotes the outward unit normal on ∂Z.

We also recall some basic facts about the first eigenvalue of the negative Dirichlet p-Laplacian satisfying the nonlinear weighted eigenvalue problem:

$$-\Delta_p u(z) = \widehat{\lambda}\, m(z)|u(z)|^{p-2}u(z) \quad \text{a.e. on } Z,\ u|_{\partial Z} = 0, \tag{2.1}$$

with $m \in L^\infty(Z)$, $m \geq 0$, $m \neq 0$. Every $\widehat{\lambda} \in \mathbb{R}$ for which (2.1) has a nontrivial solution u, is said to be a eigenvalue of $(-\Delta_p, W_0^{1,p}(Z), m)$, and u is a corresponding eigenfunction. The smallest $\widehat{\lambda}$ for which this is true, is the first eigenvalue and is denoted by $\widehat{\lambda}_1(m)$ (to emphasize the dependence on the weight m). The following results are known about $\widehat{\lambda}_1(m)$:

- (P_1) $\widehat{\lambda}_1(m) > 0$;
- (P_2) $\widehat{\lambda}_1(m)$ is isolated, i.e. for some $\varepsilon > 0$, there are no eigenvalues in $[\widehat{\lambda}_1(m), \widehat{\lambda}_1(m) + \varepsilon]$;
- (P_3) $\widehat{\lambda}_1(m)$ is simple, i.e. the corresponding eigenspace is one-dimensional;
- (P_4) $\widehat{\lambda}_1(m) = \min\{\|Du\|_p^p / \int_Z m|u|^p\,dz : u \in W_0^{1,p}(Z), u \neq 0\}$.

The minimum in (P_4) is attained on the corresponding one dimensional eigenspace. If u_1 is an eigenvalue corresponding to $\widehat{\lambda}_1(m)$, then from (P_4) it is clear that u_1 does not change sign. So, we may assume that $u_1 \geq 0$. Nonlinear regularity theory (see e.g. L. Gasinski and N. S. Papageorgiou [8, pp. 737–738]) implies that $u_1 \in C_+$, so the nonlinear strong maximum principle of J. Vazquez [17] implies $u_1 \in \operatorname{int} C_+$. Note that every eigenfunction, corresponding to an eigenvalue $\widehat{\lambda}(m) \neq \widehat{\lambda}_1(m)$, is necessarily nodal (i.e. sign changing). If $m \equiv 1$, then we write $\widehat{\lambda}_1(m) \equiv \widehat{\lambda}_1$ and by $\widetilde{u}_1 \in \operatorname{int} C_+$ we denote the L^p-normalized corresponding eigenfunction (i.e. $\|\widetilde{u}_1\|_p = 1$). As a function of the weight m, $\widehat{\lambda}_1(m) > 0$ exhibits the following monotonicity property, which can be easily deduced from (P_4):

(P_5) If $m(z) \le m'(z)$ almost everywhere on Z with strict inequality on a set of positive measure, then $\widehat{\lambda}_1(m') < \widehat{\lambda}_1(m)$.

Throughout the paper, for all $u \in W_0^{1,p}(Z)$, $\|u\| = \|Du\|_p$. Also, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and $r^{\pm} = \max\{\pm r, 0\}$ for $r \in \mathbb{R}$.

3. Auxiliary results

In this section, we establish several intermediate results which will be crucial in the proof of Theorem 1.3 for problem (1.1).

A function $\overline{u} \in W^{1,p}(Z)$ with $\overline{u}|_{\partial Z} \ge 0$ is said to be an *upper solution* for problem (1.1), if

$$\int_Z \|D\overline{u}\|^{p-2}(D\overline{u}, Dv)_{\mathbb{R}^N}\, dz \ge \int_Z \beta\, \overline{u}^{\,-\eta} v\, dz + \int_Z f(z,\overline{u}) v\, dz \tag{3.1}$$

for all $v \in W_0^{1,p}(Z)$ with $v \ge 0$. In a similar way, a function $\underline{u} \in W^{1,p}(Z)$ with $\underline{u}|_{\partial Z} \le 0$ is said to be a *lower solution* for problem (1.1), if we apply (3.1) to $\underline{u}$ and reverse the inequality.

In what follows, $A\colon W_0^{1,p}(Z) \to W^{-1,p'}(Z)$, with $1/p + 1/p' = 1$, is the nonlinear map defined by

$$\langle A(u), v\rangle = \int_Z \|Du\|^{p-2}(Du, Dv)_{\mathbb{R}^N}\, dz \quad \text{for all } u, v \in W_0^{1,p}(Z). \tag{3.2}$$

This map is maximal monotone, strictly monotone, and of type $(S)_+$, i.e. if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(Z)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p}(Z)$ (see F. Browder [4]).

Proposition 3.1. *If hypotheses* $\mathrm{H}(\beta)$ *and* $\mathrm{H(f)}$ *hold, then problem* (1.1) *has a lower solution* $\underline{u} \in \operatorname{int} C_+$.

Proof. We consider the following auxiliary Dirichlet problem

$$-\Delta_p\, u(z) = \beta(z) \quad \text{a.e. on } Z, \quad u|_{\partial Z} = 0. \tag{3.3}$$

Invoking the Proposition 2.1 of K. Perera and Z. Zhang [12], we know that problem (3.3) has a unique solution $\widehat{\underline{u}} \in \operatorname{int} C_+$. We choose $t \in (0,1)$ small such that

$$t\widehat{\underline{u}}(z) \in (0, \delta_0) \quad \text{for all } z \in Z.$$

Let $\underline{u} = t\widehat{\underline{u}} \in \operatorname{int} C_+$. Then, for a.a. $z \in Z$, we have

$$\begin{aligned} -\Delta_p\, \underline{u}(z) &= t^{p-1}(-\Delta_p \widehat{\underline{u}}(z)) = t^{p-1}\beta(z) \le \beta(z) \qquad (\text{since } t \in (0,1)) \\ &\le \beta(z)\underline{u}(z)^{-\eta} \quad (\text{since } \delta_0 < 1,\ 0 < \underline{u}(z) \le \delta_0 \text{ for all } z \in Z). \end{aligned} \tag{3.4}$$

This shows that $\underline{u} \in \operatorname{int} C_+$ is a lower solution for problem (1.1). $\square$

By virtue of hypothesis H(f)(e), it is evident that $\overline{u} \equiv \alpha_0$ is an upper solution for problem (1.1) and we have $\underline{u} \leq \overline{u}$. Using the ordered pair $\{\underline{u}, \alpha_0\}$ and suitable truncations, we produce one positive smooth solution for problem (1.1). In particular, for obtaining the first solution, we truncate the reaction term of (1.1) at the ordered pair $\{\underline{u}, \alpha_0\}$. So, we introduce the following function

$$g(z,x) = \begin{cases} \beta(z)\underline{u}(z)^{-\eta} + f(z,\underline{u}(z)) & \text{if } x < \underline{u}(z), \\ \beta(z)x^{-\eta} + f(z,x) & \text{if } \underline{u}(z) \leq x \leq \alpha_0 \\ & \text{for all } (z,x) \in Z \times \mathbb{R}, \\ \beta(z)\alpha_0^{-\eta} + f(z,\alpha_0) & \text{if } \alpha_0 < x. \end{cases} \tag{3.5}$$

Evidently g is a Carathéodory function. We set $G(z,x) = \int_0^x g(z,s)\,ds$ and consider the functional $\varphi_0 \colon W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p - \int_Z G(z,u(z))\,dz \quad \text{for all } u \in W_0^{1,p}(Z).$$

Note that, although g has a singular term, φ_0 is a C^1-functional, since the term $\beta(z)\xi^{-\eta}$ is evaluated with $\xi \in [\underline{u}, \alpha_0]$ and $\underline{u} > 0$ (recall $\underline{u} \in \operatorname{int} C_+$). Also $\beta u^{-\eta} \in L^q(Z)$, see H($\beta$). The truncated functional φ_0 will be useful in the proof of the following result.

Proposition 3.2. *If* H(β) *and* H(f) *hold, then problem* (1.1) *has a strong solution* $x_0 \in \operatorname{int} C_+$.

Proof. It is clear from (3.5) that φ_0 is coercive and sequentially w-sequentially w-lower semicontinuous. So, by the Weierstrass theorem, we can find $x_0 \in W_0^{1,p}(Z)$ such that

$$\varphi_0(x_0) = \inf_{W_0^{1,p}(Z)} \varphi_0,$$

hence $\varphi_0'(x_0) = 0$ and

$$A(x_0) = N_g(x_0), \tag{3.6}$$

where A is defined by (3.2) and $N_g(u)(\,\cdot\,) = g(\,\cdot\,, u(\,\cdot\,))$ for all $u \in W_0^{1,p}(Z)$. On (3.6), we act with $(\underline{u} - x_0)^+ \in W_0^{1,p}(Z)$. Then

$$\begin{aligned} \langle A(x_0), (\underline{u} - x_0)^+\rangle &= \int_Z g(z,x_0)(\underline{u} - x_0)^+ dz \\ &= \int_Z (\beta\underline{u}^{-\eta} + f(z,\underline{u}))(\underline{u} - x_0)^+\,dz \quad \text{(see (3.5))} \\ &\geq \langle A(\underline{u}), (\underline{u} - x_0)^+\rangle \quad \text{(see (3.4))}. \end{aligned}$$

Denoting the Lebesgue measure by $|\cdot|_N$, we have

$$\int_{\{\underline{u}>x_0\}} (\|Dx_0\|^{p-2}Dx_0 - \|D\underline{u}\|^{p-2}D\underline{u}, D\underline{u} - Dx_0)_{\mathbb{R}^N}\, dz \geq 0,$$

which implies $|\{z \in Z : \underline{u}(z) > x_0(z)\}|_N = 0$ and

$$\underline{u} \leq x_0. \tag{3.7}$$

Next, on (3.6), we act with $(x_0 - \alpha_0)^+ \in W_0^{1,p}(Z)$. Then

$$\begin{aligned} \langle A(x_0), (x_0 - \alpha_0)^+\rangle &= \int_Z g(z, \alpha_0)(x_0 - \alpha_0)^+ dz \\ &= \int_Z (\beta\alpha_0^{-\eta} + f(z, \alpha_0))(x_0 - \alpha_0)^+\, dz \quad \text{(see (3.5))} \\ &\leq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(see H(f)(e)).} \end{aligned}$$

Thus

$$\int_{\{x_0>\alpha_0\}} \|Dx_0\|^p\, dz \leq 0,$$

which implies $|\{z \in Z : x_0(z) > \alpha_0\}|_N = 0$ and

$$x_0 \leq \alpha_0. \tag{3.8}$$

From (3.6)–(3.8) and (3.5), it follows that

$$-\Delta_p\, x_0(z) = \beta(z)x_0(z)^{-\eta} + f(z, x_0(z)) \quad \text{a.e. on } Z, \quad x_0|_{\partial Z} = 0. \tag{3.9}$$

Note that

$$\frac{\beta(z)}{\alpha^\eta} \leq \frac{\beta(z)}{x_0(z)^\eta} \leq \frac{\beta(z)}{\underline{u}(z)^\eta} \quad \text{for a.a. } z \in Z, \tag{3.10}$$

see (3.7) and (3.8). Since $\underline{u} \in \operatorname{int} C_+$, we can find $\mu > 0$ such that $\vartheta \leq \mu\underline{u}$, where $\vartheta \in C_+$ is as in the hypothesis H(β). Then

$$\beta(z)(\mu\underline{u}(z))^{-\eta} \leq \beta(z)\vartheta(z)^{-\eta} \quad \text{for a.a. } z \in Z,$$

from where, by using hypothesis H(β), we have $\beta\underline{u}^{-\eta} \in L^q(Z)$, which gives

$$\beta x_0^{-\eta} \in L^q(Z), \tag{3.11}$$

(see (3.10)). Therefore, from (3.11) and hypothesis H(f)(c), we notice that the right hand-side of (3.9) is a function which belongs to $L^q(Z)$. Invoking, once again, Proposition 2.1 of K. Perera and Z. Zhang [12], we obtain $x_0 \in \operatorname{int} C_+$. This is a positive strong solution of problem (1.1) (see (3.9)). $\square$

Now, we use $x_0 \in \operatorname{int} C_+$ to produce a second positive solution for problem (1.1). For this purpose, we introduce the following truncation of the reaction term:

$$h(z,x) = \begin{cases} \beta(z)x_0(z)^{-\eta} + f(z,x_0(z)) & \text{if } x < x_0(z), \\ \beta(z)x^{-\eta} + f(z,x) & \text{if } x \geq x_0(z). \end{cases} \tag{3.12}$$

The function h is also a Carathéodory function and, as usual, we set $H(z,x) = \int_0^x h(z,s)\,ds$ and consider the functional $\psi\colon W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p - \int_Z H(z,u(z))\,dz \quad \text{for all } u \in W_0^{1,p}(Z). \tag{3.13}$$

Note that, although h has a singular term, ψ is a C^1-functional, since the term $\beta(z)\xi^{-\eta}$ is evaluated with $\xi \in [x_0,\infty)$ and $x_0 > 0$ (recall $x_0 \in \operatorname{int} C_+$).

Proposition 3.3. *If hypotheses* H(β) *and* H(f) *hold, then* ψ *satisfies the* PS-*condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ be a sequence such that $\{\psi(x_n)\}_{n\geq 1} \subset \mathbb{R}$ is bounded and

$$\psi'(x_n) \to 0 \quad \text{in } W^{-1,p'}(Z) \text{ as } n \to \infty. \tag{3.14}$$

From (3.14), we have for all $v \in W_0^{1,p}(Z)$

$$\left| \langle A(x_n), v\rangle - \int_Z h(z,x_n)v\,dz \right| \leq \varepsilon_n\|v\| \quad \text{with } \varepsilon_n \downarrow 0. \tag{3.15}$$

In (3.15), we choose $v = -x_n^- \in W_0^{1,p}(Z)$ and obtain $\|Dx_n^-\|_p^p \leq c_1\|x_n^-\|$ for some $c_1 > 0$ and all $n \geq 1$ (see (3.12)). This implies that

$$\{x_n^-\}_{n\geq 1} \subset W_0^{1,p}(Z) \quad \text{is bounded.} \tag{3.16}$$

Suppose $\{x_n^+\}_{n\geq 1} \subset W_0^{1,p}(Z)$ is unbounded. By passing to a suitable subsequence, if necessary, we may assume that $\|x_n^+\| \to \infty$. We set $y_n = x_n^+/\|x_n^+\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \quad y_n \to y \quad \text{in } L^r(Z) \tag{3.17}$$

for all $1 \leq r < p^*$. In (3.15), we choose $v = y_n - y \in W_0^{1,p}(Z)$ and multiply by $\|x_n^+\|^{1-p}$. Using (3.16), we obtain

$$\left| \langle A(y_n), y_n - y\rangle - \int_Z \frac{h(z,x_n^+)}{\|x_n^+\|^{p-1}}(y_n - y)\,dz \right| \leq c_2\|y_n - y\| \tag{3.18}$$

for some $c_2 > 0$ and all $n \geq 1$. Hypothesis H(f)(c) implies that

$$\left\{ \xi_n(z) = \frac{h(z,x_n^+(z))}{\|x_n^+\|^{p-1}} \right\}_{n\geq 1} \subset L^r(Z) \quad \text{is bounded with } r = \min\{p',q\}. \tag{3.19}$$

If $r = p'$ and since $y_n \to y$ in $L^p(Z)$ (see (3.17)), then from (3.18) we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0. \tag{3.20}$$

If $r = q > N \geq 2$ then, again from (3.17), we see that (3.20) holds. When $N = 1$, $y_n \to y$ in $C_0(\overline{Z})$, (3.20) holds again.

Now, from (3.20) and recalling that A is an operator of type $(S)_+$, we obtain

$$y_n \to y \quad \text{in } W_0^{1,p}(Z), \qquad \text{hence } \|y\| = 1. \tag{3.21}$$

Because of (3.19), we may assume

$$\xi_n \xrightarrow{w} \xi \quad \text{in } L^r(Z) \quad \text{with } r = \min\{p', q\}. \tag{3.22}$$

Using (3.22) and hypothesis H(f)(d), we can show (see D. Motreanu, V. Motreanu, N. S. Papageorgiou [10, proof of Proposition 5]) that

$$\xi = \eta_0 y^{p-1} \quad \text{with } \eta \leq \eta_0 \leq \widehat{\eta},\ y \geq 0,\ \|y\| = 1, \tag{3.23}$$

(see (3.21)). Recall that

$$\left| \langle A(y_n), v\rangle - \int_Z \xi_n v\, dz \right| \leq \varepsilon_n \|v\| \quad \text{for all } v \in W_0^{1,p}(Z).$$

So, if we pass to the limit as $n \to \infty$ and use (3.20)–(3.23), we obtain

$$\langle A(y), v\rangle = \int_Z \eta_0 y^{p-1} v\, dz \text{ for all } v \in W_0^{1,p}(Z),$$

so $A(y) = \eta_0 y^{p-1}$, that implies

$$-\Delta_p\, y(z) = \eta_0(z) y(z)^{p-1} \quad \text{for a.a. } z \in Z,\ y|_{\partial Z} = 0. \tag{3.24}$$

We know (see (P_5)) that $\widehat{\lambda}_1(\eta_0) \leq \widehat{\lambda}_1(\eta) < \widehat{\lambda}_1(\lambda_1) = 1$. Hence, from (3.24), it follows that $y \in C_0^1(\overline{Z})$ must be nodal, a contradiction (see (3.23)). Therefore,

$$\{x_n^+\}_{n\geq 1} \subset W_0^{1,p}(Z) \quad \text{is bounded.} \tag{3.25}$$

From (3.16) and (3.25), it follows that $\{x_n\}_{n\geq 1}$ is bounded, So, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \quad x_n \to x \quad \text{in } L^r(Z).$$

So, if in (3.15), we choose $v = x_n - x \in W_0^{1,p}(Z)$ and pass to the limit as $n \to \infty$, then

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0,$$

that implies that

$$x_n \to x \text{ in } W_0^{1,p}(Z) \quad (\text{by the } (S)_+ \text{ property of } A).$$

Therefore, ψ satisfies the PS-condition. $\square$

Recall that $\widetilde{u}_1$ denotes the L^p-normalized principal eigenfunction of the nonlinear eigenvalue problem $(-\Delta_p, W_0^{1,p}(Z))$.

Proposition 3.4. *If hypotheses* H(β) *and* H(f) *hold, then* $\psi(t\widetilde{u}_1) \to -\infty$ *as* $t \to \infty$.

Proof. By virtue of hypotheses $H(f)$(ii)-(iv), given $\varepsilon > 0$, we can find $\xi_\varepsilon \in L^\infty(Z)_+$ such that

$$f(z,x) \geq (\eta(z) - \varepsilon)x^{p-1} - \xi_\varepsilon(z) \quad \text{for a.a. } z \in Z, \quad \text{all } x \geq 0,$$

that implies

$$F(z,x) = \int_0^x f(z,s)\,ds \geq \frac{1}{p}\left(\eta(z) - \varepsilon\right)x^p - \xi_\varepsilon(z)x \tag{3.26}$$

for almost all $z \in Z$ and all $x \geq 0$. Then, for $t > 0$, we have

$$\begin{aligned}
\psi(t\widetilde{u}_1) &= \frac{t^p}{p}\lambda_1 - \int_Z H(z,t\widetilde{u}_1)\,dz \\
&= \frac{t^p}{p}\lambda_1 - \int_{\{t\widetilde{u}_1 \leq x_0\}} H(z,t\widetilde{u}_1)\,dz - \int_{\{t\widetilde{u}_1 > x_0\}} H(z,t\widetilde{u}_1)\,dz \\
&\leq \frac{t^p}{p}\lambda_1 - \int_{\{t\widetilde{u}_1 \leq x_0\}} f(z,x_0)t\widetilde{u}_1\,dz \\
&\quad - \int_{\{t\widetilde{u}_1 > x_0\}} (F(z,t\widetilde{u}_1) - F(z,x_0) + f(z,x_0)x_0)\,dz
\end{aligned} \tag{3.27}$$

(see (3.12) and recall $\beta \geq 0$ and $x_0 \in \operatorname{int} C_+$)

$$\leq \frac{t^p}{p}\int_Z (\lambda_1 - \eta)\widetilde{u}_1^p\,dz + \varepsilon\frac{t^p}{p}\int_{\{t\widetilde{u}_1 > x_0\}} F(z,x_0) - f(z,x_0)x_0 dz + c_1,$$

for some $c_1 > 0$ (see (3.26)). Note that, from hypothesis H(f)(d) and since $\widetilde{u}_1 \in \operatorname{int} C_+$, we have

$$\widehat{\xi} = \int_Z (\lambda_1 - \eta(z))\widetilde{u}_1^p\,dz < 0.$$

Choose $\varepsilon \in (0, -\widehat{\xi})$. Then, from (3.26), we have that

$$\psi(t\widetilde{u}_1) = -\frac{t^p}{p}c_2 + c_3 \quad \text{for some } c_2 > 0 \text{ and } c_3 > 0,$$

which implies the desired result $\psi(t\widetilde{u}_1) \to -\infty$ as $t \to \infty$. □

4. Proof of Theorem 1.3

We are now ready to prove the multiplicity result (Theorem 1.3) concerning problem (1.1).

From Proposition (3.2), we already have one solution $x_0 \in \operatorname{int} C_+$. We consider the functional ψ (defined in (3.13)) but restricted to the order interval

$[x_0,\alpha_0]$. Evidently, $\widetilde{\psi} \equiv \psi|_{[x_0,\alpha_0]}$ is coercive and sequentially w-lower semicontinuous. Therefore, by the Weierstrass theorem, we can find $\widehat{x}_0 \in [x_0,\alpha_0]$ such that

$$\widetilde{\psi}(\widehat{x}_0) = \inf_{[x_0,\alpha_0]} \psi = \widehat{m}. \tag{4.1}$$

For any $y \in [x_0,\alpha_0]$, we set $\gamma(t) = \psi(ty + (1-t)\widehat{x}_0)$, $t \in [0,1]$. From (4.1), we have $0 \geq \gamma'(0)$, which implies

$$0 \leq \langle A(\widehat{x}_0), y - \widehat{x}_0\rangle - \int_Z h(z,\widehat{x}_0)(y - \widehat{x}_0)\,dz. \tag{4.2}$$

Let $v \in W_0^{1,p}(Z)$ and $\varepsilon > 0$. We set $\mathcal{M}_- = \{z \in Z : \widehat{x}_0(z) + \varepsilon v(z) \leq x_0(z)\}$, $\mathcal{M}_0 = \{z \in Z : x_0(z) < \widehat{x}_0(z) + \varepsilon v(z) < \alpha_0\}$, $\mathcal{M}_+ = \{z \in Z : \alpha_0 \leq \widehat{x}_0(z) + \varepsilon v(z)\}$, and

$$y(z) = \begin{cases} x_0(z) & \text{if } z \in \mathcal{M}_-, \\ \widehat{x}_0(z) + \varepsilon v(z) & \text{if } z \in \mathcal{M}_0, \\ \alpha_0 & \text{if } z \in \mathcal{M}_+. \end{cases}$$

Evidently $y \in [\widehat{x},\alpha_0] = \{w \in W_0^{1,p}(Z) : \widehat{x}_0(z) \leq w(z) \leq \alpha_0$ almost everywhere on $Z\}$. We use it as a test function in (4.2). Then

$$\begin{aligned} 0 \leq{}& \varepsilon \int_Z \|D\widehat{x}_0\|^{p-2}(D\widehat{x}_0, Dv)_{\mathbb{R}^N}\,dz - \varepsilon \int_Z h(z,\widehat{x}_0)v\,dz \\ &+ \int_{\mathcal{M}_-} \|Dx_0\|^{p-2}(Dx_0, D(x_0 - \widehat{x} - \varepsilon v))_{\mathbb{R}^N} - h(z,x_0)(x_0 - \widehat{x} - \varepsilon v)\,dz \\ &+ \int_{\mathcal{M}_+} h(z,\alpha_0)(\widehat{x}_0 + \varepsilon v - \alpha_0)\,dz \\ &+ \int_{\mathcal{M}_-} (h(z,x_0) - h(z,\widehat{x}_0))(x_0 - \widehat{x}_0 - \varepsilon v)\,dz \\ &+ \int_{\mathcal{M}_+} (h(z,\alpha_0) - h(z,\widehat{x}_0))(\alpha_0 - \widehat{x}_0 - \varepsilon v)\,dz \\ &- \int_{\mathcal{M}_-} (\|D\widehat{x}_0\|^{p-2}D\widehat{x}_0 - \|Dx_0\|^{p-2}Dx_0, D\widehat{x}_0 - Dx_0)_{\mathbb{R}^N}\,dz \\ &- \varepsilon \int_{\mathcal{M}_-} (\|D\widehat{x}_0\|^{p-2}D\widehat{x}_0 - \|Dx_0\|^{p-2}Dx_0, Dv)_{\mathbb{R}^N}\,dz \\ &- \int_{\mathcal{M}+} \|D\widehat{x}_0\|^p\,dz - \varepsilon \int_{\mathcal{M}_+} \|D\widehat{x}_0\|^{p-2}(D\widehat{x}_0, Dv)_{\mathbb{R}^N}\,dz. \end{aligned} \tag{4.3}$$

We have

$$\int_{\mathcal{M}_-} \|Dx_0\|^{p-2}(Dx_0, D(x_0 - \widehat{x} - \varepsilon v))_{\mathbb{R}^N} - h(z,x_0)(x_0 - \widehat{x} - \varepsilon v)\,dz = 0, \tag{4.4}$$

since $x_0 \in \operatorname{int} C_+$ is a solution of (1.1), and

$$\int_{\mathcal{M}_+} h(z,\alpha_0)(\widehat{x}_0 + \varepsilon v - \alpha_0)\, dz \le 0 \quad \text{for a.a. } z \in Z \tag{4.5}$$

by virtue of hypothesis H(f)(e). We know that the map $\xi \mapsto \|\xi\|^{p-2}\xi$, $\xi \in \mathbb{R}^N$, is monotone. Hence

$$\int_{\mathcal{M}_-} (\|D\widehat{x}_0\|^{p-2} D\widehat{x}_0 - \|Dx_0\|^{p-2} Dx_0, D\widehat{x}_0 - Dx_0)_{\mathbb{R}^N}\, dz \ge 0. \tag{4.6}$$

Since $\widehat{x}_0 \in [x_0, \alpha_0]$ and using hypothesis H(f)(c), we have

$$\int_{\mathcal{M}_-} (h(z,x_0) - h(z,\widehat{x}_0))(x_0 - \widehat{x}_0 - \varepsilon v)\, dz \le -\varepsilon c_4 \int_{\{\widehat{x}_0 + \varepsilon v \le x_0 < \widehat{x}_0\}} v\, dz, \tag{4.7}$$

$$\int_{\mathcal{M}_+} (h(z,\alpha_0) - h(z,\widehat{x}_0))(\alpha_0 - \widehat{x}_0 - \varepsilon v)\, dz \le \varepsilon c_5 \int_{\{\widehat{x}_0 < \alpha_0 \le \widehat{x}_0 + \varepsilon v\}} v\, dz, \tag{4.8}$$

for some $c_4 > 0$ and $c_5 > 0$.

We return to (4.3), use (4.4) in (4.8) and then divide by $\varepsilon > 0$, obtaining

$$\begin{aligned} 0 \le & \int_Z \|D\widehat{x}\|^{p-2}(D\widehat{x}, Dv)_{\mathbb{R}^N}\, dz - \int_Z h(z,\widehat{x}) v\, dz \\ & - c_4 \int_{\{\widehat{x}_0 + \varepsilon v \le x_0 < \widehat{x}_0\}} v\, dz + c_5 \int_{\{\widehat{x}_0 < \alpha_0 \le \widehat{x}_0 + \varepsilon v\}} v\, dz \\ & - \int_{\mathcal{M}_+} \|D\widehat{x}_0\|^{p-2}(D\widehat{x}_0, Dv)_{\mathbb{R}^N}\, dz. \end{aligned} \tag{4.9}$$

Note that, as $\varepsilon \downarrow 0$,

$$|\{\widehat{x}_0 + \varepsilon v \le x_0 < \widehat{x}_0\}|_N \to 0 \quad \text{and} \quad |\{\widehat{x}_0 < \alpha_0 \le \widehat{x}_0 + \varepsilon v\}|_N \to 0.$$

Moreover, from Stampacchia's theorem (see e.g. L. Gasinski and N. S. Papageorgiou [8, pp. 195–196]), we have

$$D\widehat{x}_0(z) = Dx_0(z) \quad \text{a.e. on } \{\widehat{x}_0 = x_0\} \quad \text{and} \quad D\widehat{x}_0(z) = 0 \quad \text{a.e. on } \{\widehat{x}_0 = \alpha_0\}.$$

Hence, if in (4.9), we let $\varepsilon \downarrow 0$, then

$$0 \le \int_Z \|D\widehat{x}_0\|^{p-2}(D\widehat{x}_0, Dv)_{\mathbb{R}^N}\, dz - \int_Z h(z,\widehat{x}_0) v\, dz \tag{4.10}$$

Recall that $v \in W_0^{1,p}(Z)$ was arbitrary. So from (4.5), it follows that

$$-\Delta_p \widehat{x}_0(z) = h(z,\widehat{x}_0(z)) \quad \text{a.e. on } Z, \quad \widehat{x}_0|_{\partial Z} = 0,$$

which means that $\widehat{x}_0$ solves problem (1.1) (see (3.12)), $\widehat{x}_0 \in \operatorname{int} C_+$ by the nonlinear regularity theory, and $x_0 \le \widehat{x}_0$.

If $x_0 \ne \widehat{x}_0$, then $\widehat{x}_0$ is the desired second positive strong solution of (1.1). So, suppose $x_0 \equiv \widehat{x}_0$. It is straightforward to check that all the critical point x of ψ satisfy $x_0 \le x$. Hence, we may assume that $x_0 \equiv \widehat{x}_0$ is an isolated critical point (and local minimizer) of ψ, or otherwise we have a whole sequence of distinct positive strong solutions of (1.1), and so we are done. Arguing as in the proof of Theorem 6 of D. Motreanu, V. Motreanu and N. S. Papageorgiou [10], we can find $\rho \in (0,1)$ small, such that

$$\psi(x_0) < \inf\{\psi(x) : \|x - x_0\| = \rho\} = c_\rho. \tag{4.11}$$

Then (4.11), together with Propositions 3.3 and 3.4, permit the use of the mountain pass theorem (see Theorem 2.1), which gives $\widehat{x} \in W_0^{1,p}(Z)$, $\widehat{x} \neq x_0$ such that $\psi'(\widehat{x}) = 0$, which implies

$$A(\widehat{x}) = N_h(\widehat{x}) \tag{4.12}$$

where $N_h(u)(\,\cdot\,) = h(\,\cdot\,, u(\,\cdot\,))$ for all $u \in W_0^{1,p}(Z)$. As we already mentioned, $\widehat{x} \geq x_0$. So, from (4.12), we have

$$-\Delta_p\,\widehat{x}(z) = \beta(z)\widehat{x}(z)^{-\eta} + f(z, \widehat{x}(z)) \quad \text{a.e. on } Z, \quad \widehat{x}|_{\partial Z} = 0,$$

which implies that $\widehat{x} \in \operatorname{int} C_+$ (by nonlinear regularity theory) is a strong solution of problem (2.1), with $x_0 \leq \widehat{x}$ and $x_0 \neq \widehat{x}$. This concludes the proof of Theorem 1.3.

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