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EXISTENCE AND NONEXISTENCE OF POSITIVE PERIODIC SOLUTIONS TO A DIFFERENTIAL INCLUSION

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Abstract. In this paper, the existence and nonexistence of positive periodic solutions for a second-order differential inclusion are considered. Some existence and nonexistence results are established by the use of Bohnenblust–Karlin's fixed-point theorem for multivalued operators and Sobolev constant.

1. Introduction

Differential inclusions arise in many situations including differential variational inequalities, projected dynamical systems, dynamic Coulomb friction problems and fuzzy set arithmetic.

For example, the basic rule for Coulomb friction is that the friction force has magnitude N in the direction opposite to the direction of slip, where N denotes the normal force and N is a constant (the friction coefficient). However, if the slip is zero, the friction force can be any force in the correct plane with magnitude smaller than or equal to N. Thus, writing the friction force as a function of position and velocity leads to a set-valued function.

Differential inclusions have been widely investigated because of theirs importance in these fields. In the past several decades, there arose many beautiful

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methods and results concerning the solvability of differential inclusions. For example, in [4], M. Benchohora and S.K. Ntouyas discussed the solvability of first order differential inclusions with periodic boundary conditions. Recently in [7], Y. Chang and J.J. Nieto extended the study to the fractional differential inclusions with boundary conditions. By using Bohnenblust–Karlin's fixed point theorem, existence theorem is obtained. While in [9], G. Grammel considered the boundary value problems for semi-continuous delayed differential inclusions on Riemannian manifolds. B.C. Dhage proved some existence theorems for hyperbolic differential inclusions in Banach algebras in [8]; N.S. Papageorgiou and V. Staicu established the upper-lower solutions method for nonlinear second order differential inclusions in [12]. All these results show the *existence* of *solutions* for differential inclusions. For details, see [1]–[4], [7]–[10], [12], [13] and references therein.

In this paper, we aim to give some *existence and nonexistence* results of *positive solutions* for a differential inclusion as follows

$$\text{(P)}\qquad \begin{cases} u''(t)+a(t)u(t)\in F(t,u(t)), \\ u(0)=u(T), u'(0)=u'(T), \end{cases}$$

where $F\colon[0,T]\times\mathbb{R}\to 2^{\mathbb{R}}\backslash\emptyset$ is a multi-valued mapping. Sobolev constant and fixed point theorem for multi-valued operators are crucial in our proof. Differential inclusions(equations) with periodical boundary conditions are of importance in dynamic system study, see [6] and references therein.

In Section 2, some basic definitions and facts from multi-valued analysis and differential equations are introduced. Then the existence of positive solutions are established by the use of Sobolev constant and fixed point theorem in Section 3. The nonexistence results are presented in the last section.

2. Preliminaries

Let $(X,\|\cdot\|)$ be a Banach space. A multi-valued mapping $H\colon X\to 2^X\setminus\emptyset$ is convex (closed) valued if $H(x)$ is convex (closed) for each $x\in X$. H is bounded on bounded sets if $H(B)=\bigcup_{x\in B}H(x)$ is bounded in X for any bounded set B of X.

H is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $H(x_0)$ is nonempty closed subset of X, and if for each open subset B of X containing $H(x_0)$, there exists an open neighbourhood N of x_0 such that $H(N)\subset B$. H is said to be completely continuous if $H(B)$ is relatively compact for every bounded subset B of X.

If the multi-valued mapping H is completely continuous with nonempty compact values, then H is u.s.c. if and only if H has a closed graph, i.e. $x_n\to\overline{x}$, $y_n\to\overline{y}$, $y_n\in H(x_n)$ imply $\overline{y}\in H(\overline{x})$.

In the following, $\mathrm{BCC}(X)$ denotes the set of all nonempty bounded, closed and convex subset of X. H has a fixed point if there is $x \in X$ such that $x \in H(x)$. For more details on multi-valued mapping see the books of Aubin [2] and Hu and Papageorgious [10].

The following lemmas are crucial in this paper.

Lemma 2.1 ([5]). *Let X be a Banach space, D a nonempty subset of X, which is bounded, closed and convex. Suppose $H\colon D \to \mathrm{BCC}(X)$ such that $H(D) \subset D$ and $\overline{H(D)}$ is compact. Then H has a fixed point.*

Lemma 2.2 ([11]). *Let $H\colon [0,T] \times \mathbb{R} \to \mathrm{BCC}(\mathbb{R})$, $(t,x) \to H(t,x)$ is measurable with respect to t for each $x \in \mathbb{R}$, u.s.c. with respect to x for almost every $t \in [0,T]$, and for each fixed $x \in R$ the set*

$$S_{H,x} := \{h(t) \in L^1[0,T] : h(t) \in H(t,x) \text{ for a.e. } t \in [0,T]\}$$

is nonempty. Let Γ be a linear continuous operator from $L^1[0,T]$ to $C[0,T]$, then the operator

$$\Gamma \circ S_H\colon C[0,T] \to \mathrm{BCC}(C[0,T]), \qquad y \to (\Gamma \circ S_H)(y) = \Gamma(S_{H,y})$$

is a closed graph operator in $C[0,T] \times C[0,T]$.

Definition 2.3. A function $u \in C[0,T]$ is called a solution of problem (P) if there exists a function $v \in L^1[0,T]$ such that $v(t) \in F(t,u(t))$ for almost every $t \in [0,1]$ and

$$u(t) = \int_0^1 G(t,s)v(s)\,ds,$$

where $G(t,s)$ denotes the Green function associated with problem (Q). In addition, if $u(t) > 0$, $t \in [0,T]$, $u(t)$ is called a positive solution.

Throughout this paper, we assume the following condition holds:

(C) The Green function $G(t,s)$, associated with the following problem

$$\text{(Q)} \qquad \begin{cases} u''(t) + a(t)u(t) = h(t), \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases}$$

has a definite sign on $[0,T]^2$.

In other words, the (strict) maximum principle or the (strict) anti-maximum principle holds for (Q). In this case, the solution of problem (Q) is given by:

$$u(t) = (Lh)(t) = \int_0^T G(t,s)h(s)\,ds.$$

Some classes of potentials $a(t)$ for (C) to hold have been found recently in [13]. When $a \in \Lambda^+ \bigcup \Lambda^-$, where

$$\begin{aligned} \Lambda^+ &= \{\alpha \in L^1(0,T) : \alpha \succ 0,\ \|\alpha\|_p \le K(2p*),\ 1 \le p \le +\infty\}, \\ \Lambda^- &= \{\alpha \in L^1(0,T) : \alpha \prec 0\}, \end{aligned}$$

$G(t,s)$ is positive for $a \in \Lambda^+$, $G(t,s) < 0$ for $a \in \Lambda^-$.

Given $a \in L^p[0,T]$, $a \succ 0$ means $a(t) \ge 0$ for almost every $t \in [0,T]$ and it is positive in a set of positive measure. Similarly, $a \prec 0$ implies $-a \succ 0$. $\|a\|_p$ denotes the usual L^p norm over $[0,T]$ for any given component $p \in [1,+\infty]$, the conjugate exponent of p is $p* = p/(1-p)$ if $1 < p < +\infty$ and $p* = 1$ if $p = \infty$. $K(p)$ denotes the best Sobolev constant in the following inequality:

$$C\|u\|_p^2 \le \|u'\|_2^2$$

for all $u \in H_1^0(0,T)$. The explicit formula of $K(p)$ is:

$$K(p) = \begin{cases} \dfrac{2\pi}{pT^{1-2/p}} \left(\dfrac{2}{2+p}\right)^{1-2/p} \left(\dfrac{\Gamma(1/p)}{\Gamma(1/2+1/p)}\right)^2, & 1 \le p < +\infty, \\ \dfrac{4}{T} & p = +\infty. \end{cases}$$

3. Existence of positive periodic solutions

In this section, the existence results of positive solutions to problem (P) are established.

We give the following assumptions:

(A1) $a \in \Lambda^+$.

(A2) $F\colon [0,T] \times \mathbb{R} \to \mathrm{BCC}(\mathbb{R}^+)$; $(t,x) \to F(t,x)$ is measurable with respect to t for each $x \in R$, u.s.c. with respect to x for almost every $t \in [0,T]$.

(A3) For each $l > 0$, there exists a function $m_l \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,y)\| = \sup\{|v| : v(t) \in F(t,y)\} \le m_l(t)$$

for each $(t,y) \in [0,T] \times \mathbb{R}$ with $|y| \le l$ and

$$\inf_{l>0} \frac{1}{l} \int_0^T m_l(t)\,dt = \lambda < \infty.$$

(A4) $\lambda B < 1$, where $B = \max_{0 \le t,\ s \le T} G(t,s)$ and $G(t,s)$ is the Green function associated with problem (Q).

(A5) For almost every $t \in [0,T]$,

$$\langle F(t,0)\rangle = \inf\{|v| : v(t) \in F(t,0)\} > 0.$$

The main results in this paper are presented as follows:

Theorem 3.1. *Assume conditions* (A1)–(A5) *hold, then problem* (P) *has at least one positive solution on* $[0,T]$.

Proof. Let $\|u\|$ denote the maximum norm for $u \in C[0,T]$, then $(C[0,T], \|\cdot\|)$ is Banach space. Let $C^+[0,T] = \{u \in C[0,T] \mid u(t) \geq 0,\ t \in [0,T]\}$. Since $a \in \Lambda^+$, the Green function $G(t,s)$ associated with problem (Q) is positive on $[0,T]^2$.

Now, let us transform the differential inclusion problem (P) into a fixed point problem. By (A2) and (A3), for each fixed $x \in R$ the set

$$S_{F,x} := \{f(t) \in L^1[0,T] : f(t) \in F(t,x) \text{ for a.e. } t \in [0,T]\}$$

is nonempty. Define a multi-valued operator $N\colon C[0,T] \to 2^{C[0,T]} \setminus \emptyset$ as follows:

$$N(u) = \left\{ h \in C[0,T] \;\middle|\; h(t) = \int_0^T G(t,s)v(s)\,ds,\ v \in S_{F,u} \right\},$$

where $v \in S_{F,u} = \{w \in L^1[0,T] \mid w(t) \in F(t,u(t)) \text{ for a.e. } t \in [0,T]\}$. Then the fixed-point of N is a solution of problem (P).

The existence of fixed point is verified by several steps:

Step 1. For every $u \in C[0,T]$, the multi-valued operator $N(u)$ is convex.

In fact, let $h_1, h_2 \in N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for arbitrary $t \in [0,T]$,

$$h_1(t) = \int_0^T G(t,s)v_1(s)\,ds, \qquad h_2(t) = \int_0^T G(t,s)v_2(s)\,ds.$$

For each $\alpha \in (0,1)$,

$$\alpha h_1(t) + (1-\alpha)h_2(t) = \int_0^T G(t,s)(\alpha v_1(s) + (1-\alpha)v_2(s))\,ds,$$

Since F has convex values, $S_{F,u}$ is convex, $\alpha v_1 + (1-\alpha)v_2 \in S_{F,u}$ and $\alpha h_1 + (1-\alpha)h_2 \in N(u)$.

Step 2. There exists a positive number l such that N maps B_l into B_l, where $B = \{x \in C[0,T] \mid \|x\| \leq l\}$.

Otherwise, suppose for each $l > 0$, there exists a function $u_l \in B_l, h_l \in N(u_l)$, such that $\|h_l\| > l$. Then there exists $v_l \in S_{F,u_l}$ such that $h_1(t) = \int_0^T G(t,s)v_1(s)\,ds$, then

$$\begin{aligned} l < \|h_l\| &= \max_{t\in[0,T]} |h_l(t)| = \max_{t\in[0,T]} \left| \int_0^T G(t,s)v_l(s)\,ds \right| \\ &\leq \max_{t\in[0,T]} \int_0^T G(t,s)|v_l(s)|\,ds \leq B \int_0^T m_l(s)\,ds. \end{aligned}$$

Hence, we have

$$\frac{B}{l} \int_0^T m_l(s)\,ds > 1 \quad \text{for every } l > 0,$$

which implies $\lambda B \geq 1$. This contradicts assumption (A4). Then there exists a positive number l such that N maps B_l into B_l.

Step 3. $N(B_l)$ is compact.

In fact, by step 2, we know that $N(B_l)$ is bounded. In what follows, we only need to show that $N(B_l)$ is equi-continuous.

Let $u \in B_l, h \in N(u)$, then there exists $v \in S_{F,u}$ such that

$$h(t) = \int_0^T G(t,s)v(s)\,ds.$$

For any $t_1, t_2 \in [0,T]$, $t_1 < t_2$,

$$\begin{aligned}|h(t_1) - h(t_2)| &= \bigg|\int_0^T G(t_1,s)v(s)\,ds - \int_0^T G(t_2,s)v(s)\,ds\bigg| \\ &\leq \int_0^T |G(t_1,s) - G(t_2,s)| m_l(s)\,ds.\end{aligned}$$

Since $G(t,s)$ is equi-continuous on $[0,T]^2$ and $m_l \in L[0,T]$, then for every $\varepsilon > 0$, there exists $\delta > 0$, such that $|t_1 - t_2| < \delta$ implies $|h(t_1) - h(t_2)| < \varepsilon$ for any $t_1, t_2 \in [0,T]$. Hence, $N(B_l)$ is equi-continuous.

Step 4. N has closed graph.

Let $u_n \to \overline{u}$, $h_n \in N(u_n)$ and $h_n \to \overline{h}$. In what follows, we need to show that $\overline{h} \in N(\overline{u})$.

Since $h_n \in N(u_n)$, there exists $v_n \in S_{F,u_n}$ such that

$$h_n(t) = \int_0^T G(t,s)v_n(s)\,ds, \quad n = 1,2,\ldots, \quad \text{for } t \in [0,T].$$

We should verify that there exists $\overline{v} \in S_{F,\overline{u}}$ such that $\overline{h}(t) = \int_0^T G(t,s)\overline{v}(s)\,ds$, $n = 1,2,\ldots,$ for $t \in [0,T]$. Define a linear operator $\Gamma\colon L[0,T] \to C[0,T]$ as follows:

$$(\Gamma v)(t) = \int_0^T G(t,s)v(s)\,ds, \quad \text{for all } v \in L[0,T]$$

Then Γ is continuous and, by Lemma 2.2,

$$\Gamma \circ S_F\colon C[0,T] \to \mathrm{BCC}(C[0,T]), \qquad u \to (\Gamma \circ S_F)(u) = \Gamma(S_{F,u})$$

is a closed graph operator in $C[0,T] \times C[0,T]$.

Note that $h_n \in \Gamma(S_{F,u_n})$, $\Gamma \circ S_F$ is a closed graph operator and $h_n \to \overline{h}$, there exists $\overline{v} \in S_{F,\overline{u}}$ such that $\overline{h}(t) = \int_0^T G(t,s)\overline{v}(s)\,ds$, $n = 1,2,\ldots$ for $t \in [0,T]$.

Therefore, N is a compact multi-valued mapping and u.s.c. with convex closed values. As a consequence of Lemma 2.1, we know that N has a fixed point y which is a solution of the problem (P). Since $F\colon [0,T] \times \mathbb{R} \to \mathrm{BCC}(\mathbb{R}^+)$, $y(t) \geq 0$ for every $t \in [0,T]$.

Step 5. y is positive solution.

In fact, let $v \in L^1[0,T]$, $v(t) \in F(t, y(t))$ satisfies

$$y(t) = \int_0^T G(t,s)v(s)\,ds.$$

Since $v(t) \geq 0$, and if $v(t) = 0$ almost everywhere on $[0,T]$, then

$$y(t) = \int_0^T G(t,s)v(s)\,ds = 0, \quad \text{a.e. on } [0,T].$$

Hence, $d(0, f(t,0)) = 0$ almost everywhere on $[0,T]$, which contradicts assumption (A5).

Therefore, there exists a subset D of $[0,T]$ with positive measure such that $v(t) > 0$ for $t \in D$. Thanks to the positivity of $G(t,s)$, it is obvious that $y(t) = \int_0^T G(t,s)v(s)\,ds > 0$ on $[0,T]$. □

In what follows, some concrete cases are discussed to describe the existence and properties of the function m_l.

Corollary 3.2 (Sub-linear Growth). *Suppose* (A1), (A2), (A5) *and the following condition hold*:

(H1) *There exist functions* $\phi(t), \psi(t) \in L^1([0,T], \mathbb{R}^+)$, *0*$< \delta < 1$ *such that*

$$\|F(t,y)\| \leq \phi(t)|y|^\delta + \psi(t), \quad \textit{for } (t,y) \in [0,T] \times \mathbb{R}.$$

Then problem (P) *has at least one positive solution on* $[0,T]$.

Proof. In this case, let $m_l(t) = \phi(t)l^\delta + \psi(t)$. Then

$$\lambda = \inf_{l>0} \frac{1}{l} \int_0^T m_l(t)\,dt = 0 \quad \text{and} \quad \lambda B = 0,$$

where λ is defined in (A3). Hence an application of Theorem 3.1 asserts the conclusion. □

Corollary 3.3 (Quadratic Controlled Growth). *Suppose* (A1), (A2), (A5) *and the following condition hold*:

(H2) *There exist functions* $\phi(t), \psi(t) \in L^1([0,T], \mathbb{R}^+)$ *such that*

$$\|F(t,y)\| \leq \phi(t)|y|^2 + \psi(t), \quad \textit{for } (t,y) \in [0,T] \times \mathbb{R}.$$

Then problem (P) *has at least one positive solution on* $[0,T]$ *provided*

$$4\|\phi\|_{L^1}\|\psi\|_{L^1}B^2 < 1.$$

Proof. In this case, let $m_l(t) = \phi(t)l^2 + \psi(t)$. Then

$$\lambda = \inf_{l>0} \frac{1}{l} \int_0^T m_l(t)\,dt = 2\sqrt{\|\phi\|_{L^1}\|\psi\|_{L^1}}.$$

Theorem 3.1 claims that problem (P) has at least one positive solution provided

$$2B\sqrt{\|\phi\|_{L^1}\|\psi\|_{L^1}} < 1. \qquad \square$$

Corollary 3.4 (Linear Growth). *Suppose* (A1), (A2), (A5) *and the following condition hold*:

(H3) *There exist functions* $\phi(t), \psi(t) \in L^1([0,T],\mathbb{R}^+)$, $0 < \delta < 1$ *such that*

$$\|F(t,y)\| \leq \phi(t)|y| + \psi(t), \quad \textit{for } (t,y) \in [0,T] \times \mathbb{R}.$$

Then (P) *has at least one positive solution on* $[0,T]$ *provided* $\|\phi\|_{L^1} < 1/B$.

Proof. In this case, let $m_l(t) = \phi(t)l + \psi(t)$. Then

$$\lambda = \inf_{l>0} \frac{1}{l}\int_0^T m_l(t)\,dt = \|\phi\|_{L^1}. \qquad \square$$

Analogously, we give the following hypothesizes:

(A1') $a \in \Lambda^-$.
(A2') $F\colon [0,T] \times \mathbb{R} \to \mathrm{BCC}(\mathbb{R}^-)$; $(t,x) \to F(t,x)$ is measurable with respect to t for each $x \in \mathbb{R}$, u.s.c. with respect to x for almost every $t \in [0,T]$.
(A4') $\lambda B < 1$, where $B = -\min_{0\leq t,\ s\leq T} G(t,s)$ and $G(t,s)$ is the Green function associated with the problem (Q).

Theorem 3.5. *Assume conditions* (A1'), (A2'), (A3), (A4'), (A5) *hold, then problem* (P) *has at least one positive solution on* $[0,T]$.

Proof. Due to (A2') and (A3), we know that:

(a) for each fixed $x \in \mathbb{R}$ the set

$$S_{F,x} := \{f(t) \in L^1[0,T] : f(t) \in F(t,x) \text{ for a.e. } t \in [0,T]\}$$

is nonempty,

(b) the multi-valued operator

$$N(u) = \left\{ h \in C[0,T] \;\middle|\; h(t) = \int_0^T G(t,s)v(s)\,ds,\ v \in S_{f,u} \right\}$$

maps $C^+[0,T]$ to $C^+[0,T]$, where

$$v \in S_{F,u} = \{w \in L^1[0,T] \mid w(t) \in F(t,u(t)) \text{ for a.e. } t \in [0,T]\}.$$

Since the rest of the proof follows the same manner as Theorem 3.1, we omit it. $\square$

4. Nonexistence of positive periodic solutions

Given the following assumptions:

(A6) For each $l > 0$, there exists a function $m_l \in L([0,T], \mathbb{R}^+)$ such that

$$\|F(t,y)\| = \sup\{|v| : v(t) \in F(t,y)\} \leq m_l(t)$$

for each $(t,y) \in [0,T] \times \mathbb{R}$ with $|y| \leq l$ and

$$\sup_{l>0} \frac{1}{l} \int_0^T m_l(t)\,dt = \omega < \infty.$$

(A7) For each $l > 0$, there exists a function $c_l \in L([0,T], \mathbb{R}^+)$ such that

$$\langle F(t,y) \rangle = \inf\{|v| : v(t) \in F(t,y)\} \geq c_l(t)$$

for $t \in [0,T]$, $bl/B \leq |y| \leq l$ and

$$\inf_{l>0} \frac{1}{l} \int_0^T c_l(t)\,dt = \Delta < \infty,$$

where $B = \max\limits_{0 \leq t,s \leq T} G(t,s)$, $b = \min\limits_{0 \leq t,s \leq T} G(t,s)$ and $G(t,s)$ is the Green function associated with problem (Q).

The nonexistence results are as follows:

Theorem 4.1. *Assume conditions* (A1), (A2), (A6) *hold, then problem* (P) *has no positive solution on* $[0,T]$ *provided* $\omega < b/B^2$, *where* $B = \max\limits_{0 \leq t,s \leq T} G(t,s)$, $b = \min\limits_{0 \leq t,s \leq T} G(t,s)$ *and* $G(t,s)$ *is the Green function associated with problem* (Q).

Proof. Suppose that y is a positive solution to problem (P). Let $l = \|y\|$, then $l > 0$. There exists $v \in S_{F,y}$, $v(t) \geq 0$ such that

$$y(t) = \int_0^T G(t,s)v(s)\,ds.$$

Since $B = \max\limits_{0 \leq t,s \leq T} G(t,s)$, $b = \min\limits_{0 \leq t,s \leq T} G(t,s)$,

$$\begin{aligned} y(t) &= \int_0^T G(t,s)v(s)\,ds = \frac{1}{B} \int_0^T BG(t,s)v(s)\,ds \\ &\geq \frac{b}{B} \int_0^T Bv(s)\,ds \geq \frac{b}{B} \max_{t \in [0,T]} \int_0^T G(t,s)v(s)\,ds = \frac{b}{B}\|y\|. \end{aligned}$$

Hence we have

$$\frac{b}{B}\,l \leq y(t) = \int_0^T G(t,s)v(s)\,ds \leq \int_0^T G(t,s)|m_l(s)|\,ds \leq B \int_0^T m_l(s)\,ds.$$

Then

$$\frac{1}{l} \int_0^T m_l(t)\,dt \geq \frac{b}{B^2},$$

which contradicts the assumption $\omega < b/B^2$. $\square$

Corollary 4.2. *Suppose* (A1), (A2) *and the following condition hold*:

(H4) *There exists a function* $\phi(t) \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,y)\| \leq \phi(t)|y|, \quad \text{for } (t,y) \in [0,T] \times \mathbb{R}.$$

Then problem (P) *has no positive solution provided* $\|\phi\|_{L^1} < b/B^2$.

Proof. In this case, let $m_l(t) = \phi(t)l$, then

$$\omega = \sup_{l>0} \frac{1}{l} \int_0^T m_l(t)\,dt = \|\phi\|_{L^1}.$$

When $\|\phi\|_{L^1} < b/B^2$, Theorem 4.1 asserts that problem (P) has no positive solution. □

Theorem 4.3. *Assume conditions* (A1), (A2), (A7) *hold, then problem* (P) *has no positive solution on* $[0,T]$ *provided* $\Delta > 1/b$.

Proof. Suppose that y is a positive solution to problem (P). Let $l = \|y\|$. By the proof of Theorem 4.1, we have $bl/B \leq y(t) \leq l$ for $t \in [0,1]$. There exists $v \in S_{F,y}$, $v(t) \geq 0$ such that

$$y(t) = \int_0^T G(t,s)v(s)\,ds.$$

Then

$$l \geq y(t) = \int_0^T G(t,s)v(s)\,ds \geq \int_0^T G(t,s)c_l(s)\,ds \geq b\int_0^T c_l(s)\,ds.$$

Hence we have

$$\frac{1}{l}\int_0^T c_l(t)\,dt \leq \frac{1}{b},$$

which contradicts the assumption $\Delta > 1/b$. □

Corollary 4.4. *Suppose* (A1), (A2) *and the following condition hold*:

(H4) *There exists a function* $\phi(t) \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\langle F(t,y)\rangle \geq e^{ky}, \quad t \in [0,T], \quad \frac{bl}{B} \leq |y| \leq l.$$

Then problem (P) *has no positive solution provided* $k > B/b^2eT$.

Proof. In this case, let $c_l(t) = e^{kbl/B}$, then

$$\langle F(t,y)\rangle \geq c_l(t), \quad t \in [0,T], \quad \frac{bl}{B} \leq |y| \leq l.$$

$$\Delta = \inf_{l>0} \frac{1}{l}\int_0^T c_l(t)\,dt = \frac{kbTe}{B}.$$

When $kbTe/B > 1/b$, i.e. $k > B/b^2eT$, Theorem 4.2 asserts that problem (P) has no positive solution. □

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