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COINCIDENCE OF MAPS FROM TWO-COMPLEXES INTO GRAPHS

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Abstract. The main theorem of this article provides a necessary and sufficient condition for a pair of maps from a two-complex into a one-complex (a graph) can be homotoped to be coincidence free. As a consequence of it, we prove that a pair of maps from a two-complex into the circle can be homotoped to be coincidence free if and only if the two maps are homotopic. We also obtain an alternative proof for the known result that every pair of maps from a graph into the bouquet of a circle and an interval can be homotoped to be coincidence free. As applications of the main theorem, we characterize completely when a pair of maps from the bi-dimensional torus into the bouquet of a circle and an interval can be homotoped to be coincidence free, and we prove that every pair of maps from the Klein bottle into such a bouquet can be homotoped to be coincidence free.

1. Introduction

Let $f_1, f_2\colon X \to Y$ be continuous maps between topological spaces. The general coincidence problem for such a pair of maps (f_1, f_2) is concerned, roughly, with the study of the minimal cardinality of the coincidence set $\mathrm{Coin}(f_1', f_2') = \{x \in X : f_1'(x) = f_2'(x)\}$, among all maps f_1' and f_2' homotopic to f_1 and f_2, respectively. Such minimal cardinality is denoted by $\mu(f_1, f_2)$ and it is called the minimum coincidence number between f_1 and f_2 (or of the pair (f_1, f_2)).

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When $\mathrm{Coin}(f_1, f_2) = \emptyset$, we say that the pair (f_1, f_2) is *coincidence free*. When $\mathrm{Coin}(f_1, f_2)$ is not necessarily empty but $\mu(f_1, f_2) = 0$ we say that the pair (f_1, f_2) *can be homotoped to be coincidence free*, what means that there are maps f_1' homotopic to f_1 and f_2' homotopic to f_2 such that the pair (f_1', f_2') is coincidence free.

When X and Y are well-behaved topological spaces, the Nielsen coincidence theory provides a number, called the Nielsen coincidence number, denoted by $N(f_1, f_2)$, which is a lower bound for $\mu(f_1, f_2)$, but this number is, in general, not easy to compute. Moreover, there are examples in which the nullity of the number $N(f_1, f_2)$ does not implies the nullity of the number $\mu(f_1, f_2)$. For the general topological coincidence theory we suggest [4].

The main theorem of this article (Theorem 2.1) presents a necessary and sufficient condition for a pair (f_1, f_2) of maps from a finite and connected two-dimensional CW complex into a finite and connected one-dimensional CW complex (a graph) can be homotoped to be coincidence free. Such condition involves the existence of certain lifting in a classical diagram on fundamental groups. This approach of the problem is strongly inspired by a similar study of the root problem developed in [3] for the so-called convenient maps.

As a consequence of the main theorem we obtain, in Section 4, a particular case of the main result of [5] that states that if $f_1, f_2 \colon X \to Y$ are maps on (connected) graphs with Y not homeomorphic to the circle, then f_1 and f_2 can be changed by homotopy to be coincidence free. Before that, in Section 3, we regain the well known result that two self-maps of the circle S^1 can be homotoped to be coincidence free if and only if they have the same Brouwer degree; in fact, it is well known that for maps $f_1, f_2 \colon S^1 \to S^1$, the minimum coincidence number $\mu(f_1, f_2)$ is equal to the Nielsen coincidence number $N(f_1, f_2) = |\deg(f_1) - \deg(f_2)|$. More general, we prove that a pair (f_1, f_2) of maps from a two-dimensional CW complex into the circle S^1 can be homotoped to be coincidence free if and only if f_1 and f_2 are homotopic.

For maps into the bouquet $S^1 \vee I$ of a circle and an interval, we prove that this condition may be more flexibility in a sense. Specifically, we prove in Section 5 that if (f_1, f_2) is a pair of maps from the bi-dimensional torus $\mathbb{T} = S^1 \times S^1$ into the bouquet $S^1 \vee I$, then the condition "f_1 is homotopic to f_2" is sufficient to $\mu(f_1, f_2) = 0$, but it is not a necessary condition. In fact, we show that $\mu(f_1, f_2) = 0$ if and only if f_1 and f_2 are *algebraically-pseudo-homotopic* (see Definition 5.1 in Section 5).

For maps from the Klein bottle into S^1 or $S^1 \vee I$, we prove in Section 6 that the situation is similar to that of maps between graphs, that is, there are pairs of maps from the Klein bottle into S^1 that can not be homotoped to be coincidence free, but every pair of maps from the Klein bottle into the bouquet $S^1 \vee I$ can be homotoped to be coincidence free.

Throughout the text, K (respectively L) denotes a finite and connected two-dimensional (respectively one-dimension) CW complex. We simplify this by saying K is a two-complex and L is a one-complex (or a graph). We also simplify f is a continuous map to f is a map. The homomorphism induced by a map f on fundamental groups is denoted by $f_{\#}$.

2. The main theorem

Throughout this section, K is a two-complex and L is a one-complex, both finite and connected. Given two maps $f_1, f_2 \colon K \to L$ we define the so-called *pair-map*

$$F = (f_1, f_2)\colon K \to L \times L \quad \text{by} \quad F(x) = (f_1(x), f_2(x)).$$

The maps f_1 and f_2 induce homomorphisms $f_{1\#}\colon \pi_1(K) \to \pi_1(L)$ and $f_{2\#}\colon \pi_1(K) \to \pi_1(L)$ on fundamental groups and the pair-map induces the homomorphism

$$\begin{aligned} F_{\#}\colon \pi_1(K) &\to \pi_1(L \times L) \approx \pi_1(L) \oplus \pi_1(L), \\ x &\mapsto F_{\#}(x) = f_{1\#}(x) \oplus f_{2\#}(x). \end{aligned}$$

It is obvious that $\mathrm{Coin}(f_1, f_2) = \emptyset$ if and only if the image of F does not meet the diagonal $\Delta = \{(y, y) : y \in L\}$ of the product $L \times L$. Thus, $\mathrm{Coin}(f_1, f_2) = \emptyset$ if and only if there exists a factorization of F through $L \times L \setminus \Delta$, that is, there exists a map $\widetilde{F}\colon K \to L \times L \setminus \Delta$ such that $F = l \circ \widetilde{F}$, where $l\colon L \times L \setminus \Delta \hookrightarrow L \times L$ is the natural inclusion. Now, if this occurs, then we have the following commutative diagram on fundamental groups:

$$\begin{tikzcd} & \pi_1(L \times L \setminus \Delta) \arrow[d, "l_{\#}"] \\ \pi_1(K) \arrow[ur, "\widetilde{F}_{\#}"] \arrow[r, "F_{\#}"'] & \pi_1(L \times L) \end{tikzcd}$$

This makes trivial the "only if" part of the main theorem presented below. The "if" part is not equally trivial; its proof is similar to that of Theorem 2.6 of [3].

Theorem 2.1 (Main Theorem). *Let $f_1, f_2\colon K \to L$ be two maps from a two-complex into a one-complex, both finite and connected, and let consider the pair-map $F = (f_1, f_2)\colon K \to L \times L$. The pair (f_1, f_2) can be homotoped to be coincidence free if and only if there exists a homomorphism $\Phi\colon \pi_1(K) \to \pi_1(L \times L \setminus \Delta)$*

making commutative the diagram:

$$\begin{tikzcd} & \pi_1(L \times L \setminus \Delta) \arrow[d, "l_\#"] \\ \pi_1(K) \arrow[ur, dashed, "\Phi"] \arrow[r, "F_\#"'] & \pi_1(L \times L) \end{tikzcd}$$

We remark that Theorem 2.1 can not be extended for maps between two-complexes, even if they are surfaces. In fact, let us consider a self-map $f\colon S^2 \to S^2$ of the sphere S^2 and let $c\colon S^2 \to S^2$ be the constant map $c(x) = c \in S^2$ for all $x \in S^2$. Suppose that f has degree $\deg(f) \neq 0$. Then, every map homotopic to f is surjective and there exists a map φ homotopic to f such that the cardinality of the set $\varphi^{-1}(c) = \mathrm{Coin}(f, c)$ is equal to one. Therefore, we have $\mu(f, c) = 1$. On the other hand, every homomorphism from $\pi_1(S^2)$ into $\pi_1(S^2 \times S^2)$ lifts through the homomorphism $l_\#\colon \pi_1(S^2 \times S^2 \setminus \Delta) \to \pi_1(S^2 \times S^2)$, since these homomorphisms are all trivial.

In order to prove Theorem 2.1, we first prove some preliminary results.

For each $n \geq 1$, we denote the bouquet $\bigvee_{i=1}^n S^1$ of n circles by Σ_n^1. To avoid confusion, we extend this notation for $n = 0$ by defining Σ_0^1 to be the single point space. Summarizing

$$\Sigma_n^1 = \bigvee_{i=1}^n S^1 \quad \text{for } n \geq 1 \quad \text{and} \quad \Sigma_0^1 = \{e_*^0\}.$$

Let L be a connected and finite one-complex (a graph). Then the fundamental group $\pi_1(L)$ is a free group of rank n for some $n \geq 0$ and it is well known that, in this case, L is homotopy equivalent to the bouquet Σ_n^1. Moreover, there exists a homotopy equivalence $\xi_L\colon L \to \Sigma_n^1$ that is a cellular map. The inverse homotopy equivalence of ξ_L will be denoted by $\xi'_L\colon \Sigma_n^1 \to L$. Summarizing, we write such homotopy equivalences by

$$\xi_L\colon L \twoheadrightarrow \Sigma_n^1 \quad \text{and} \quad \xi'_L\colon \Sigma_n^1 \rightarrowtail L.$$

Let e_*^0 be the unique 0-cell of the bouquet Σ_n^1 and let K be a two-complex. Given a map $f\colon K \to \Sigma_n^1$, it is well known that f has a cellular approximation, that is, there exists a map $f'\colon K \to \Sigma_n^1$ such that $f' \simeq f$ and $f'(K^1) = \{e_*^0\}$, where K^1 is the 1-skeleton of K. In particular, every 0-cell of K is mapped onto e_*^0. Choose a 0-cell of K to be the base-point and denote it by e_K^0. Consider e_*^0 as the base-point of Σ_n^1.

LEMMA 2.2. *Let $f, g\colon K \to \Sigma_n^1$ be based maps. Then f is based homotopic to g if and only if the homomorphisms $f_\#$ and $g_\#$ induced by f and g on fundamental groups are equal.*

Proof. It follows from Theorem 2.1 of [3] (see also Corollary 4.13 on page 95 of [1]), since the second homotopy group $\pi_2(\Sigma_n^1)$ of the bouquet Σ_n^1 is trivial. □

Lemma 2.3. *Let $f, g\colon K \to L$ be two cellular maps and define $f', g'\colon K \to \Sigma_n^1$ to be the compositions $f' = \xi_L \circ f$ and $g' = \xi_L \circ g$, where $\xi_L\colon L \twoheadrightarrow \Sigma_n^1$. Then f' and g' are based maps. Moreover, if $f' \simeq g'$, then $f \simeq g$.*

Proof. That f' and g' are based maps follows from the fact that f, g and ξ_L are cellular maps. In order to prove the second part, suppose that f' is homotopic to g' and that $H'\colon K \times I \to \Sigma_n^1$ is a homotopy starting at f' and ending at g'. Define $H\colon K \times I \to L$ by $H(x,t) = (\xi'_L \circ H')(x,t)$, where $\xi'_L\colon \Sigma_n^1 \rightarrowtail L$ is as before. Then H is a homotopy starting at $\xi'_L \circ \xi_L \circ f$ and ending at $\xi'_L \circ \xi_L \circ g$. Since ξ_L and ξ'_L are inverse homotopy equivalences, $\xi'_L \circ \xi_L$ is homotopic to the identity map of L. Therefore, we have $f \simeq \xi'_L \circ \xi_L \circ f \simeq \xi'_L \circ \xi_L \circ g \simeq g$. □

Proposition 2.4. *Let $f, g\colon K \to L$ be maps. Then $f \simeq g$ if and only if $f_\# = g_\#$.*

Proof. Let take cellular approximation f_c and g_c for f and g, respectively, so that $f \simeq g$ if and only if $f_c \simeq g_c$. Let $f' = \xi_L \circ f_c$ and $g' = \xi_L \circ g_c$ where $\xi_L\colon L \twoheadrightarrow \Sigma_n^1$. By Lemma 2.3, $f_c \simeq g_c$ if and only $f' \simeq g'$. By Lemma 2.2, $f' \simeq g'$ if and only if $f'_\# = g'_\# : \pi_1(K) \to \pi_1(\Sigma_n^1)$. Since $f'_\# = (\xi_L)_\# \circ f_\#$ and $g'_\# = (\xi_L)_\# \circ g_\#$ and $(\xi_L)_\#$ is an isomorphism, it follows that $f'_\# = g'_\#$ if and only if $f_\# = g_\#$. Therefore $f \simeq g$ if and only if $f_\# = g_\#$. □

The next result is a version of Lemma 2.5 of [3].

Lemma 2.5. *Let $\Pi = \pi_1(K)$ and $\Xi = \pi_1(L)$. Every homomorphism $\beta\colon \Pi \to \Xi$ can be obtained as an induced homomorphism on fundamental groups by a cellular map $f\colon K \to L$.*

Henceforward, we consider $L \times L$ with its natural cellular structure. For a well-defined subdivision of $L \times L$, the space $L \times L \setminus \Delta$ retracts by strong deformation to a (well-characterized smaller) subcomplex L^Δ. This retraction (that is a cellular map) will be denoted by R_L^Δ and the inclusion of L^Δ into $L \times L \setminus \Delta$ will be denoted by i_L^Δ. Thus, we have inverse homotopy equivalences

$$R_L^\Delta\colon L \times L \setminus \Delta \twoheadrightarrow L^\Delta \quad \text{and} \quad i_L^\Delta\colon L^\Delta \rightarrowtail L \times L \setminus \Delta.$$

Proof of the Main Theorem. As we have seen, the "only if" part follows from the explanation made before the statement of the theorem.

In order to prove the "if" part, let suppose that $\Phi\colon \pi_1(K) \to \pi_1(L \times L \setminus \Delta)$ is a homomorphism verifying $F_\# = l_\# \circ \Phi$. Let consider the strong deformation retract $R_L^\Delta\colon L \times L \setminus \Delta \twoheadrightarrow L^\Delta$ and the inclusion $i_L^\Delta\colon L^\Delta \rightarrowtail L \times L \setminus \Delta$ as described above. Then $(R_L^\Delta)_\#$ and $(i_L^\Delta)_\#$ provide inverse isomorphisms between the groups $\pi_1(L \times L \setminus \Delta)$ and $\pi_1(L^\Delta)$. Let consider the composition $(R_L^\Delta)_\# \circ \Phi\colon \pi_1(K) \to$

$\pi_1(L^\Delta)$. By Lemma 2.5, there exists a cellular map $\varphi\colon K \to L^\Delta$ such that $\varphi_\# = (R_L^\Delta)_\# \circ \Phi$. Let $F'\colon K \to L \times L$ be the composition $F' = l \circ i_L^\Delta \circ \varphi$. Then

$$F'_\# = l_\# \circ (i_L^\Delta)_\# \circ \varphi_\# = l_\# \circ (i_L^\Delta)_\# \circ (R_L^\Delta)_\# \circ \Phi = l_\# \circ \Phi = F_\#.$$

It follows from Proposition 2.4 that F' is homotopic to F.

Let consider $f'_i\colon K \to L$ given by $f'_i = p_i \circ F'$, where $p_i\colon L \times L \to L$ is the i-th projection, for $i = 1, 2$. Then, the map F' can be considered as the pair-map $F' = (f'_1, f'_2)$. Since F' lifts to $i_L^\Delta \circ \varphi$ through l, we have $\mathrm{Coin}(f'_1, f'_2) = \emptyset$. Moreover, we have $f'_i \simeq f_i$ for $i = 1, 2$, since $f_i = p_i \circ F$ for $i = 1, 2$ and $F' \simeq F$. Therefore, the pair (f_1, f_2) can be homotoped (to the pair (f'_1, f'_2)) to be coincidence free. $\square$

In the next sections, we will study consequences of Theorem 2.1 for particular choices of L. We start with the more simple case $L = S^1$ and after we consider L as the bouquet of a circles and a closed interval. For this study, it is essential to determine the complex L^Δ and the homomorphism $l_\#\colon \pi_1(L^\Delta) \approx \pi_1(L \times L \setminus \Delta) \to \pi_1(L \times L)$. Henceforward, we use the letter l indistinctly to denote the inclusion $L \times L \setminus \Delta \hookrightarrow L \times L$ and $L^\Delta \hookrightarrow L \times L$.

3. Maps into the circle S^1

The product of the circle S^1 with itself is the two-dimensional torus $S^1 \times S^1$. If we consider S^1 as the quotient space obtained from the closed interval $[0, 1]$ by identifying the points 0 and 1, the space $S^1 \times S^1 \setminus \Delta$ can be considered as the space obtained from the square $[0, 1] \times [0, 1]$ with the vertices and a diagonal deleted, by identifying the opposite edges. As it is illustrated in Figure 1, it is easy to see that this space retracts by strong deformation to a circle S^1. Therefore, for $L = S^1$, we have $L^\Delta = S^1$.

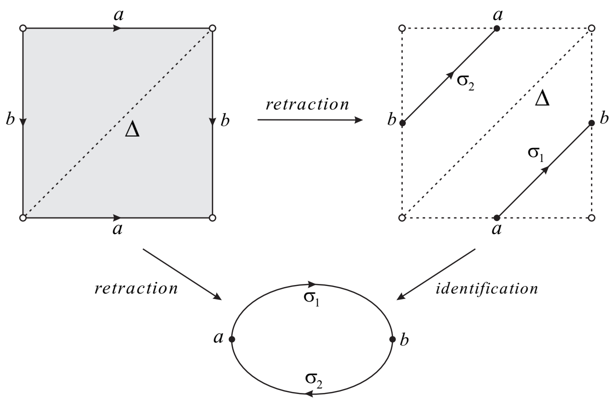

Figure 1. The retraction of $S^1 \times S^1 \setminus \Delta$ onto $(S^1)^\Delta = S^1$

According to the notation of Figure 1, the class $[\sigma] = [\sigma_1\sigma_2]$ is a generator of $\pi_1((S^1)^\Delta)$ and the inclusion $l\colon (S^1)^\Delta \hookrightarrow S^1 \times S^1$ maps σ to a loop that describe a longitudinal and a meridional rounds on the torus $S^1 \times S^1$. Thus, $l(\sigma)$ is homotopic to the loop ab. Therefore, the homomorphism

$$\mathbb{Z} \approx \pi_1((S^1)^\Delta) \approx \pi_1(S^1 \times S^1 \setminus \Delta) \xrightarrow{l_\#} \pi_1(S^1 \times S^1) \approx \mathbb{Z} \oplus \mathbb{Z}$$

induced by the natural inclusion $l\colon S^1 \times S^1 \setminus \Delta \hookrightarrow S^1 \times S^1$ is given by $l_\#[\sigma] = [ab]$, where $[ab]$ is the class of the word ab in the quotient group $F(a,b)/\langle [a,b]\rangle \approx \pi_1(S^1 \times S^1) \approx \mathbb{Z} \oplus \mathbb{Z}$, where $F(a,b)$ is the free group (of rank two) generated by the letters a and b. By considering the identification $[\sigma] \equiv 1 \in \mathbb{Z}$ and $[a] \equiv (1,0) \in \mathbb{Z} \oplus \mathbb{Z}$ and $[b] \equiv (0,1) \in \mathbb{Z} \oplus \mathbb{Z}$, we come to $l_\#\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ given by $l_\#(1) = (1,1)$.

Theorem 3.1. *Let $f_1, f_2\colon K \to S^1$ be maps from a two-complex into the circle. The pair (f_1, f_2) can be homotoped to be coincidence free if and only if f_1 and f_2 are homotopic.*

Proof. Let F be the pair-map $F = (f_1, f_2)\colon K \to S^1 \times S^1$. Then the homomorphism $F_\#\colon \pi_1(K) \to \pi_1(S^1 \times S^1) \approx \pi_1(S^1) \oplus \pi_1(S^1)$ is given by $F_\# = f_{1\#} \oplus f_{2\#}$. We need to know when exists a homomorphism $\Phi\colon \pi_1(K) \to \mathbb{Z}$ making commutative the diagram:

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, "l_\#"] \\ \pi_1(K) \arrow[ur, dashed, "\Phi"] \arrow[r, "f_{1\#}\oplus f_{2\#}"'] & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd}$$

Since $l_\#(1) = (1,1)$, it is clear that such a homomorphism Φ exists if and only if $f_{1\#} = f_{2\#}$. But by Proposition 2.4, $f_{1\#} = f_{2\#}$ if and only if $f_1 \simeq f_2$. Therefore, Theorem 2.1 implies that (f_1, f_2) can be homotoped to be coincidence free if and only if $f_1 \simeq f_2$. □

As a consequence of this theorem, we obtain the following well known result:

Corollary 3.2. *A pair (f_1, f_2) of self-maps of the circle S^1 can be homotoped to be coincidence free if and only if* $\deg(f_1) = \deg(f_2)$.

By following the classical terminology, we say that a map $f\colon X \to Y$ is *self-coincidence free* if it can be homotoped to a map f' such that the pair (f, f') is coincidence free. As a trivial consequence of Theorem 3.1 we have:

Corollary 3.3. *Every map from a two-complex into S^1 is self-coincidence free.*

4. Maps from graphs into the bouquet $S^1 \vee I$

In this section, we illustrate the applicability of the main theorem (Theorem 2.1) presenting an alternative proof for the main result of [5] for the particular case of maps from a (connected) graph into the bouquet $S^1 \vee I$, where I is the closed interval $[0, 1]$. To simplify the notation, we will write often

$$\mathcal{W} = S^1 \vee I.$$

Since $\mathcal{W}$ has the homotopy type of the circle S^1, the product $\mathcal{W} \times \mathcal{W}$ has the homotopy type of the torus $S^1 \times S^1$. In fact, $\mathcal{W} \times \mathcal{W}$ retracts by strong deformation to $S^1 \times S^1$. However, $\mathcal{W} \times \mathcal{W} \setminus \Delta$ does not have the homotopy type of $S^1 \times S^1 \setminus \Delta$. In fact, we have seen in the previous section that $S^1 \times S^1 \setminus \Delta$ retracts to the circle S^1. On the other hand, by Figure 2 we see that $\mathcal{W} \times \mathcal{W} \setminus \Delta$ retracts to a space homotopy equivalent to the bouquet of three circles. In this figure, we consider $\mathcal{W} = S^1 \vee I$ as the space obtained by identifying the points 0 and 1 in the closed interval $[0, 2]$. Thus, $\mathcal{W} \times \mathcal{W}$ can be considered has the space obtained from the square $[0, 2] \times [0, 2]$ by identifying the vertical edge $\{0\} \times [0, 2]$ with $\{1\} \times [0, 2]$ and the horizontal edge $[0, 2] \times \{0\}$ with $[0, 2] \times \{1\}$. When the diagonal Δ is deleted, we can realize a retraction of $\mathcal{W} \times \mathcal{W} \setminus \Delta$ onto a space $\mathcal{W}^{\Delta}$ homotopy equivalent the the bouquet $\vee^3 S^1$ of three circles as showed in Figure 2.

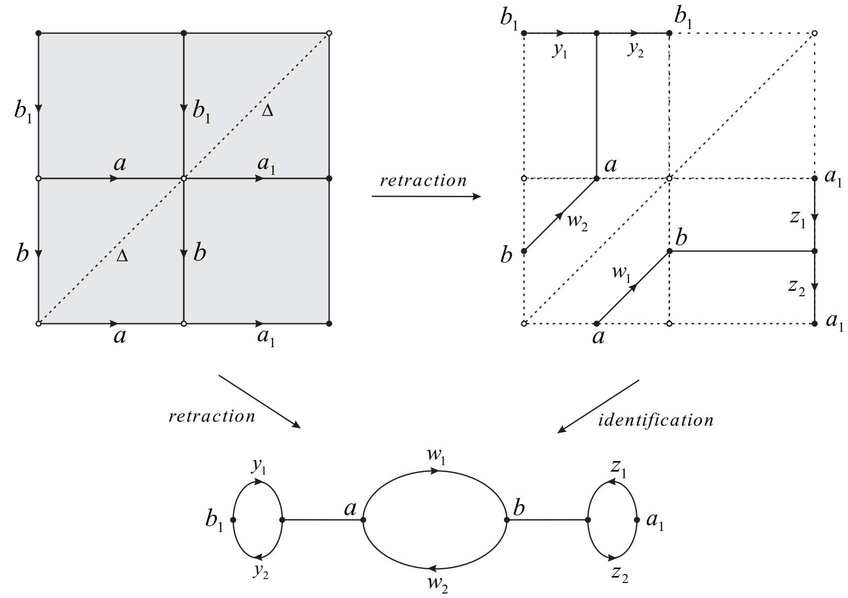

FIGURE 2. The retraction of $\mathcal{W} \times \mathcal{W} \setminus \Delta$ onto $\mathcal{W}^{\Delta} \simeq \vee^3 S^1$

As we have seen, the group $\pi_1(\mathcal{W}\times\mathcal{W})$ can be naturally identified with the fundamental group of the torus $S^1\times S^1$ obtained by restricting the identifications on the square $[0,2]\times[0,2]$ to the square $[0,1]\times[0,1]$. Thus, $\pi_1(\mathcal{W}\times\mathcal{W})$ is the abelian group $\mathbb{Z}\oplus\mathbb{Z}$ with its natural presentation $\mathcal{P}=\langle a,b \mid [a,b]\rangle$. Remember that the natural quotient homomorphism

$$F(a,b)\to\frac{F(a,b)}{\langle[a,b]\rangle}\approx\mathbb{Z}\oplus\mathbb{Z}$$

maps a to $(1,0)$ and b to $(0,1)$. More general, it maps the generic word $a^{\delta_1}b^{\varepsilon_1}$ $a^{\delta_2}b^{\delta_2}\dots a^{\delta_n}b^{\varepsilon_n}$ to the pair $(\delta,\varepsilon)\in\mathbb{Z}\oplus\mathbb{Z}$, where $\delta=\delta_1+\delta_2+\ldots+\delta_n$ and $\varepsilon=\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n$.

Now, if we consider $y=y_1y_2$, $z=z_1z_2$ and $w=w_1w_2$, with the notations of Figure 2, then we see that $\pi_1(\mathcal{W}^\Delta)\approx\pi_1(\mathcal{W}\times\mathcal{W}\setminus\Delta)$ is the free group $F(y,z,w)$ generated by the letters y, z and w. Moreover, it is easy to see that the homomorphism

$$l_\#\colon F(y,z,w)\approx\pi_1(\mathcal{W}^\Delta)\to\pi_1(\mathcal{W}\times\mathcal{W})\equiv\langle a,b \mid [a,b]\rangle\equiv\mathbb{Z}\oplus\mathbb{Z}$$

induced by the natural inclusion $l\colon\mathcal{W}^\Delta\hookrightarrow\mathcal{W}\times\mathcal{W}$ on fundamental groups is given by

$$l_\#(y)=[a]\equiv(1,0),\qquad l_\#(z)=[b]\equiv(0,1),\qquad l_\#(w)=[ab]\equiv(1,1).$$

The reader needs to understand our recurring abuse of notation.

The following result is a particular case of the main theorem of [5].

Proposition 4.1. *Every pair of maps from a one-complex into the bouquet $S^1\vee I$ can be homotoped to be coincidence free.*

Proof. Let $f_1,f_2\colon K^1\to\mathcal{W}$ be two maps from a (finite and connected) one-complex into $\mathcal{W}=S^1\vee I$ and let consider the pair-map $F=(f_1,f_2)\colon K^1\to\mathcal{W}\times\mathcal{W}$. Certainly, K^1 is homotopy equivalent to a bouquet of circles, we say $n\geq 0$ circles, and so its fundamental group $\pi_1(K^1)$ is the free group $F(x_1,\dots,x_n)$ of rank n. By Theorem 2.1, we need to prove that exists a homomorphism $\Phi\colon F(x_1,\dots,x_n)\to F(y,z,w)$ making commutative the diagram below, where $l_\#:F(y,z,w)\to\mathbb{Z}\oplus\mathbb{Z}$ is the homomorphism described above.

$$\begin{tikzcd} & F(y,z,w) \arrow[d, "l_\#"] \\ F(x_1,\dots,x_n) \arrow[ur, dashed, "\Phi"] \arrow[r, "F_\#"'] & \mathbb{Z}\oplus\mathbb{Z} \end{tikzcd}$$

For each $1 \leq i \leq n$, $F_\#(x_i)$ is of the form $F_\#(x_i) = (c_i, d_i)$ for some integers c_i and d_i. We define $\Phi\colon F(x_1, \dots, x_n) \to F(y, z, w)$ to be the (unique) homomorphism such that $\Phi(x_i) = y^{c_i} z^{d_i}$. Then Φ satisfies $l_\# \circ \Phi = F_\#$, which proves the proposition. □

Theorem 3.1 and Proposition 4.1 together show that there are pairs of maps from one-complexes into S^1 that can not be homotoped to be coincidence free, but if we attach a "tail" on the circle S^1 and so we consider maps from one-complexes into the bouquet $S^1 \vee I$, then every pair of maps can be homotoped to be coincidence free. In Section 5, we show that this is not true for maps from a two-complex into $S^1 \vee I$. Specifically, we prove that for maps from the bi-dimensional torus into S^1 it is not sufficient, in general, to attach a "tail" on S^1 for that every pair of maps can be homotoped to be coincidence free. In Section 6 we show that for the Klein bottle instead of the torus the situation is similar to that of maps between graphs, that is, there are pairs of maps from the Klein bottle into S^1 that can not be homotoped to be coincidence free, but every pair of maps from the Klein bottle into the bouquet $S^1 \vee I$ can be homotoped to be coincidencefree.

5. Maps from the torus into the bouquet $S^1 \vee I$

As application of the main theorem (Theorem 2.1) we answer in this section when a pair of maps from the bi-dimensional torus $\mathbb{T} = S^1 \times S^1$ into the bouquet $\mathcal{W} = S^1 \vee I$ (we keep the notation of the previous section) can be homotoped to be coincidence free. If we consider maps from $\mathbb{T}$ into the circle S^1, then this problem is completely solved by Theorem 3.1. In fact, by that theorem, a pair (f_1, f_2) of maps from the torus $\mathbb{T}$ into S^1 can be homotoped to be coincidence free if and only if f_1 and f_2 are homotopic. We prove that for maps from $\mathbb{T}$ into $\mathcal{W}$ we have more flexibility, that is, if (f_1, f_2) is a pair of maps from $\mathbb{T}$ into $\mathcal{W}$, then the condition "f_1 is homotopic to f_2" is sufficient to $\mu(f_1, f_2) = 0$, but this condition is not necessary. In fact, we prove that $\mu(f_1, f_2) = 0$ if and only if f_1 and f_2 are *algebraically-pseudo-homotopic* (see Definition 5.1). Let $f\colon \mathbb{T} \to \mathcal{W}$ be a map from the torus $\mathbb{T}$ into $\mathcal{W} = S^1 \vee I$. Then the homomorphism

$$f_\#\colon \mathbb{Z} \oplus \mathbb{Z} \approx \pi_1(\mathbb{T}) \to \pi_1(\mathcal{W}) \approx \mathbb{Z}$$

is completely defined by its values on the canonical generators $(1,0)$ and $(0,1)$ of the abelian free group $\mathbb{Z} \oplus \mathbb{Z}$. If this values are $f_\#(1,0) = c$ and $f_\#(0,1) = d$, the pair $(c, d) \in \mathbb{Z} \oplus \mathbb{Z}$ is called the *bi-degree* of f and it is denoted be $\mathrm{bideg}(f)$.

If f_1 and f_2 are two maps from $\mathbb{T}$ into $S^1 \vee I$, then we have

$$f_1 \simeq f_2 \;\Leftrightarrow\; f_{1\#} = f_{2\#} \;\Leftrightarrow\; \mathrm{bideg}(f_1) = \mathrm{bideg}(f_2).$$

It is interesting for us the case in which $\mathrm{bideg}(f_1)$ and $\mathrm{bideg}(f_2)$ are not necessarily equal, but they satisfy a specific relation which we describe in the following definition.

DEFINITION 5.1. We say that two maps f_1 and f_2 from the torus $\mathbb{T}$ into $S^1 \vee I$ are *algebraically-pseudo-homotopic* if there are integers n_1 and n_2 and a pair $(p, q) \in \mathbb{Z} \oplus \mathbb{Z}$ such that

$$\mathrm{bideg}(f_1) = n_1(p, q) \quad \text{and} \quad \mathrm{bideg}(f_2) = n_2(p, q).$$

It is obvious that if f_1 and f_2 are homotopic then f_1 and f_2 are also algebraically-pseudo-homotopic, but the reciprocal is not true.

THEOREM 5.2. *A pair (f_1, f_2) of maps from the torus $\mathbb{T} = S^1 \times S^1$ into the bouquet $S^1 \vee I$ can be homotoped to be coincidence free if and only f_1 and f_2 are algebraically-pseudo-homotopic.*

PROOF. Let f_1 and f_2 be maps from the torus $\mathbb{T}$ into $\mathcal{W} = S^1 \vee I$ with bi-degree

$$\mathrm{bideg}(f_1) = (\delta_{11}, \delta_{12}) \quad \text{and} \quad \mathrm{bideg}(f_2) = (\delta_{21}, \delta_{22}).$$

Then the homomorphisms $f_{1\#}$ and $f_{2\#}$ from $\pi_1(\mathbb{T}) \approx \mathbb{Z} \oplus \mathbb{Z}$ into $\pi_1(\mathcal{W}) \approx \mathbb{Z}$ are given by

$$f_{1\#}(1,0) = \delta_{11}, \quad f_{1\#}(0,1) = \delta_{12} \quad \text{and} \quad f_{2\#}(1,0) = \delta_{21}, \quad f_{2\#}(0,1) = \delta_{22}.$$

Let us consider the pair-map $F = (f_1, f_2) \colon \mathbb{T} \to \mathcal{W} \times \mathcal{W}$. Then the induced homomorphism

$$F_\# \colon \mathbb{Z} \oplus \mathbb{Z} \approx \pi_1(\mathbb{T}) \to \pi_1(\mathcal{W} \times \mathcal{W}) \approx \pi_1(\mathcal{W}) \oplus \pi_1(\mathcal{W}) \approx \mathbb{Z} \oplus \mathbb{Z}$$

is given by $F_\# = f_{1\#} \oplus f_{2\#}$, and so it is completely defined by its values

$$F_\#(1,0) = (\delta_{11}, \delta_{21}) \quad \text{and} \quad F_\#(0,1) = (\delta_{12}, \delta_{22}).$$

By Theorem 2.1, the pair (f_1, f_2) can be homotoped to be coincidence free if and only if there exists a homomorphism $\Phi \colon \mathbb{Z} \oplus \mathbb{Z} \to F(y, z, w)$ making commutative the diagram below, where $l_\#$ is as in Section 4, that is, $l_\#(y) = (1,0)$, $l_\#(z) = (0,1)$ and $l_\#(w) = (1,1)$.

$$\begin{tikzcd}
 & F(y,z,w) \arrow[d, "l_\#"] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[ur, dashed, "\Phi"] \arrow[r, "F_\#"'] & \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}$$

In order to prove the "if" part of the theorem, suppose that f_1 and f_2 are algebraically-pseudo-homotopic, so that there are integers n_1 and n_2 and a pair $(p, q) \in \mathbb{Z} \oplus \mathbb{Z}$ such that

$$(\delta_{11}, \delta_{12}) = n_1(p, q) \quad \text{and} \quad (\delta_{21}, \delta_{22}) = n_2(p, q).$$

It follows that

$$F_{\#}(1, 0) = p(n_1, n_2) \quad \text{and} \quad F_{\#}(0, 1) = q(n_1, n_2).$$

Let us consider the homomorphism $\Phi\colon \mathbb{Z} \oplus \mathbb{Z} \to F(y, z, w)$ defined by $\Phi(a, b) = (y^{n_1} z^{n_2})^{ap+bq}$ for every $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$. In fact Φ is a well-defined homomorphism and we have

$$\Phi(1, 0) = (y^{n_1} z^{n_2})^p \quad \text{and} \quad \Phi(0, 1) = (y^{n_1} z^{n_2})^q.$$

Since $l_{\#}(y) = (1, 0)$ and $l_{\#}(z) = (0, 1)$, it follows that $l_{\#} \circ \Phi(1, 0) = p(n_1, n_2) = F_{\#}(1, 0)$ and $l_{\#} \circ \Phi(0, 1) = q(n_1, n_2) = F_{\#}(0, 1)$, what proves that Φ is a homomorphism making commutative the diagram above and so, by Theorem 2.1, the pair (f_1, f_2) can be homotoped to be coincidence free.

In order to prove the "only if" part of the theorem, suppose that $\Phi\colon \mathbb{Z} \oplus \mathbb{Z} \to F(y, z, w)$ is a homomorphism making commutative the diagram above, that is, $l_{\#} \circ \Phi = F_{\#}$. Let denote $\phi_1 = \Phi(1, 0)$ and $\phi_2 = \Phi(0, 1)$. Since $\mathbb{Z} \oplus \mathbb{Z}$ is abelian, the image of Φ is an abelian subgroup of the free group $F(y, z, w)$. Hence, ϕ_1 and ϕ_2 commute. By the results of Section 3 of Chapter III of [2], ϕ_1 and ϕ_2 are power of a same element, that is, there are $\phi \in F(y, z, w)$ and integers p and q such that $\phi_1 = \phi^p$ and $\phi_2 = \phi^q$. Let $(n_1, n_2) = l_{\#}(\phi)$. Since $\mathbb{Z} \oplus \mathbb{Z}$ is abelian, we have

$$l_{\#}(\phi_1) = p(n_1, n_2) \quad \text{and} \quad l_{\#}(\phi_2) = q(n_1, n_2).$$

It follows from the commutativity of the diagram above that $F_{\#}(1, 0) = p(n_1, n_2)$ and $F_{\#}(0, 1) = q(n_1, n_2)$, what implies that

$$\operatorname{bideg}(f_1) = n_1(p, q) \quad \text{and} \quad \operatorname{bideg}(f_2) = n_2(p, q).$$

Therefore, f_1 and f_2 are algebraically-pseudo-homotopic. □

6. Maps from the Klein bottle into the bouquet $S^1 \vee I$

As we have said in the end of Section 4, we show in this section that pairs of maps from the Klein bottle into S^1 or $S^1 \vee I$ behave, in a sense, like pairs of maps from a graph into S^1 or $S^1 \vee I$, that is, there are pairs of maps from the Klein bottle into S^1 that can not be homotoped to be coincidence free, but every pair of maps from the Klein bottle into the bouquet $S^1 \vee I$ can be homotoped to be coincidence free.

The Klein bottle is usually meant as the square with identification of reciprocal sides one of them twisted, therefore given by the relation $x_1tx_1^{-1}t$. However, by performing a cut on the diagonal of the square, which we indexed with the letter x_2, and pasting properly two of the sides of the square (exactly the sides corresponding to the letter t), we see that the Klein bottle can be given by the relation $x_1^2x_2^2$. Therefore, if we denote the Klein bottle by $\mathbb{K}$, then $\mathbb{K}$ is the model two-complex (see [1]) of the presentation $\mathcal{P} = \langle x_1, x_2 \mid x_1^2x_2^2\rangle$ and the fundamental group $\pi_1(\mathbb{K})$ of $\mathbb{K}$ is the group presented by $\mathcal{P}$, that is,

$$\pi_1(\mathbb{K}) \approx \frac{F(x_1, x_2)}{\langle x_1^2x_2^2\rangle}.$$

Let $\Omega\colon F(x_1, x_2) \to \pi_1(\mathbb{K})$ be the corresponding quotient homomorphism. Then, given an arbitrary group G, every homomorphism $\beta\colon \pi_1(\mathbb{K}) \to G$ corresponds to a homomorphism $\alpha\colon F(x_1, x_2) \to G$ such that $\beta \circ \Omega = \alpha$ and $\alpha(x_1^2x_2^2) = e_G$, the identity element of G.

We show that there exists a pair (f_1, f_2) of maps from the Klein bottle $\mathbb{K}$ into the circle S^1 that can not be homotoped to be coincidence free: Let $\alpha_1, \alpha_2\colon F(x_1, x_2) \to \mathbb{Z}$ be the homomorphism given by $\alpha_1(x_1) = 1$, $\alpha_1(x_2) = -1$, $\alpha_2(x_1) = 2$ and $\alpha_2(x_2) = -2$. Since $\alpha_1(x_1^2x_2^2) = 0$ and $\alpha_2(x_1^2x_2^2) = 0$, there are homomorphisms $\beta_1, \beta_2\colon \pi_1(\mathbb{K}) \to \mathbb{Z}$ such that $\beta_1 \circ \Omega = \alpha_1$ and $\beta_2 \circ \Omega = \alpha_2$. By Lemma 2.5, there are maps $f_1, f_2\colon \mathbb{K} \to S^1$ such that $\beta_i = f_{i\#} : \pi_1(\mathbb{K}) \to \pi_1(S^1) \approx \mathbb{Z}$ for $i = 1, 2$. Since $\alpha_1 \neq \alpha_2$, also $f_{1\#} \neq f_{2\#}$, what implies that f_1 and f_2 are not homotopic. By Theorem 3.1, the pair (f_1, f_2) can not be homotoped to be coincidence free.

Now, for maps from the Klein bottle into the bouquet $S^1 \vee I$ we have the following one:

THEOREM 6.1. *Every pair of maps from the Klein bottle $\mathbb{K}$ into the bouquet $S^1 \vee I$ can be homotoped to be coincidence free.*

PROOF. By Theorem 2.1, it is sufficient to prove that every homomorphism $\beta\colon \pi_1(\mathbb{K}) \to \mathbb{Z} \oplus \mathbb{Z}$ lifts to a homomorphism $\Phi\colon \pi_1(\mathbb{K}) \to F(y, z, w)$ through $l_\#\colon F(y, z, w) \to \mathbb{Z} \oplus \mathbb{Z}$, where $l_\#$ is as in Section 4, that is, $l_\#(y) = (1, 0)$, $l_\#(z) = (0, 1)$ and $l_\#(w) = (1, 1)$.

Given such a homomorphism $\beta\colon \pi_1(\mathbb{K}) \to \mathbb{Z} \oplus \mathbb{Z}$, let consider the homomorphism $\alpha = \beta \circ \Omega\colon F(x_1, x_2) \to \mathbb{Z} \oplus \mathbb{Z}$, where $\Omega\colon F(x_1, x_2) \to \pi_1(\mathbb{K})$ is as above. Let denote $\alpha(x_1) = (a_1, b_1)$ and $\alpha(x_2) = (a_2, b_2)$. Since Ω is the homomorphism corresponding to the quotient of $F(x_1, x_2)$ by its normal subgroup generated by the word $x_1^2x_2^2$, we have

$$(0, 0) = \alpha(x_1^2x_2^2) = 2(a_1 + a_2, b_1 + b_2) \ \Rightarrow\ (a_2, b_2) = -(a_1, b_1).$$

Therefore, we can consider simply $\alpha(x_1) = (a, b)$ and $\alpha(x_2) = -(a, b)$.

Let $\phi\colon F(x_1,x_2)\to F(y,z,w)$ be the (unique) homomorphism defined by $\phi(x_1)=y^az^b$ and $\phi(x_2)=z^{-b}y^{-a}$. Then $l_\#\circ\phi(x_1)=\alpha(x_1)$ and $l_\#\circ\phi(x_2)=\alpha(x_2)$, what proves that $l_\#\circ\phi=\alpha$. Now, since $\phi(x_1^2x_2^2)$ is the empty word in $F(y,z,w)$, it follows that there exists a homomorphism $\Phi\colon\pi_1(\mathbb{K})\to F(y,z,w)$ verifying $\Phi\circ\Omega=\phi$. Obviously, $\pi_1(\mathbb{K})$ is generated by $\Omega(x_1)$ and $\Omega(x_2)$. Moreover, it follows by the relationships $l_\#\circ\phi=\alpha$ and $\Phi\circ\Omega=\phi$ that

$$l_\#\circ\Phi(\Omega(x_i))=l_\#\circ\phi(x_i)=\alpha(x_i)=\beta(\Omega(x_i))\quad\text{for } i=1,2.$$

This proves that $l_\#\circ\Phi=\beta$, that is, the homomorphism Φ is a lifting of β through $l_\#$, as we wanted to prove. □

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