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ON UNIFORM ATTRACTORS FOR NON-AUTONOMOUS p-LAPLACIAN EQUATION WITH A DYNAMIC BOUNDARY CONDITION

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Abstract. In this paper, we consider the non-autonomous p-Laplacian equation with a dynamic boundary condition. The existence and structure of a compact uniform attractor in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ are established for the case of time-dependent internal force $h(t)$. While the nonlinearity f and the boundary nonlinearity g are dissipative for large values without restriction on the growth order of the polynomial.

1. Introduction

In this paper, we study the dynamical behavior of solutions of the following non-autonomous parabolic equation with nonlinear dynamic boundary condition:

$$
\begin{cases}
u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = h(x,t) & \text{in } \Omega,\\
u_t + |\nabla u|^{p-2}\partial_n u + g(u) = 0 & \text{on } \Gamma,\\
u(\tau) = u_\tau & \text{in } \overline{\Omega},
\end{cases}
\tag{1.1}
$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary Γ and $p \geq 2$. $h(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$. The functions f and $g \in C^1(\mathbb{R},\mathbb{R})$, satisfy the following

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conditions:

$$k_1'|s|^{q_1} - k_1 \le f(s)s \le k_2'|s|^{q_1} + k_2, \quad q_1 \ge p, \tag{1.2}$$

$$k_3'|s|^{q_2} - k_3 \le g(s)s \le k_4'|s|^{q_2} + k_4, \quad q_2 \ge 2, \tag{1.3}$$

$$f'(s) \ge -l \quad \text{and} \quad g'(s) \ge -m, \tag{1.4}$$

where $l, m > 0$, $k_i, k_i' > 0$, $i = 1, 2, 3, 4$.

The dynamic boundary condition arises in hydrodynamics and the heat transfer theory, it is very natural in many mathematical models, such as heat transfer in a solid in contact with a moving fluid, thermoelastic distortion, diffusion phenomena, heat transfer in two medium, problems in fluid dynamics etc. (see [7], [8], [10], [11] and references therein).

Recently, the reaction-diffusion equation with a dynamic boundary condition has been studied by many authors. For example, in [4], [15] considered the phase-field systems with coupling dynamic boundary conditions. Some estimates of convergence rate of the solutions has been obtained in [22], [23].

In this paper, the operator Δ_p in (1.1) denotes the $p-$Laplacian operator. It is obvious that for the case $p = 2$, the equation $(1.1)_1$ will become the reaction-diffusion equation. For the case of Dirichlet boundary condition, recently, Song *et al.* [21] obtained the existence of a uniform attractor in $H_0^1(\Omega)$, where the compactness in $H_0^1(\Omega)$ was verified by using of the compactness of $L^{q_1}(\Omega)$.

As for the reaction-diffusion equation with a dynamic boundary condition, Fan and Zhong [12] obtained the existence of a global attractor in $(H^1(\Omega) \cap L^{q_1}(\Omega)) \times L^{q_1}(\Gamma)$ under some additional conditions. For the non-autonomous case, in [1], the authors proved the existence of a weak solution, and established the existence of a pullback attractor. [24] proved the existence of a uniform attractor in $L^{q_1}(\Omega) \times L^{q_1}(\Gamma)$. The authors in [26], [25] considered the long-time behavior of the reaction-diffusion equation with nonlinear boundary condition and competing nonlinearities.

On the other hand, for the p-Laplacian equation, Carvalho, Cholewa and Dlotko gave a detailed discussion about Dirichlet boundary condition in [2], and then they proved the existence of $(L^2(\Omega), L^2(\Omega))$-global attractor, see [6]. In Carvalho and Gentile [3], the authors obtained that the corresponding semigroup has a $(L^2(\Omega), W_0^{1,p}(\Omega))$-global attractor.

However, the long time behavior about the p-Laplacian equation with dynamic boundary is less discussed, especially for the non-autonomous systems. In this case of autonomous systems, Gal *et al.* [13], [14] presented the general results about the well-posedness and the asymptotic behavior.

Our main goal of this paper is to study the long-time behavior of solutions of problem (1.1)–(1.4) by the theory of uniform attractors.

The existence and structure of a uniform attractor for the problem (1.1)–(1.4) in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ has been verified.

For the existence of a uniform attractor, as in the autonomous case, some kind of compactness of the family of processes is a key ingredient. In our paper, the growth orders of nonlinear terms $f(u)$ and $g(u)$ have no further restrictions and the solutions have not higher regularities, one can not obtain the compactness of the process in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ by embedding theorem. Furthermore, due to the dynamic boundary conditions, the compactness of the process in $L^{q_1}(\Omega) \times L^{q_2}(\Gamma)$ apparently can not be obtained, namely, it seems to be difficult to obtain the asymptotic compactness in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ through the compactness of $L^{q_1}(\Omega) \times L^{q_2}(\Gamma)$ (as that in [21]). Therefore, some new ideas and methods seem to be needed.

In this paper, we testify the uniform asymptotic compactness in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ only based on the compactness in $L^2(\Omega) \times L^2(\Gamma)$ and without any compactness in $L^{q_1}(\Omega) \times L^{q_2}(\Gamma)$, $q_1, q_2 > 2$.

At the same time, we use the closed process to obtain the structure of the uniform attractor, see more details in [24] (see Pata and Zelik [20] for autonomous case).

For convenience, in what follows, we use the notation $\|\cdot\|$ and $\|\cdot\|_\Gamma$ stand for the norm in $L^2(\Omega)$ and $L^2(\Gamma)$, $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_\Gamma$ stand for the inner product in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. $|e|$ denotes the Lebesgue measure of e, while C, C_i denote general positive constants, $i = 1, 2, \dots$, which will be different in different estimates.

Hereafter, we also assume $2 \le p < N$.

For the case $p \ge N$, the embeddings $W^{1,p}(\Omega) \hookrightarrow L^{s_1}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow W^{1-1/p,p}(\Gamma) \hookrightarrow L^{s_2}(\Gamma)$ hold for any $s_1, s_2 \in [1, \infty)$, which make the nonlinear terms $f(\cdot)$ and $g(\cdot)$ to be trivial terms.

This paper is organized as follows: in Section 2, we give some preparations for our consideration; in Section 3, the existence and structure of a uniform attractor in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ is obtained.

2. Preliminaries

In this section, we first recall some basic concepts about non-autonomous systems, we refer to [5] for more details.

Let X be a Banach space, and Σ be a parameter set.

The operators $\{U_\sigma(t,\tau), \sigma \in \Sigma\}$ are said to be a family of processes in X with symbol space Σ if for any $\sigma \in \Sigma$

$$\begin{aligned} U_\sigma(t,s) \circ U_\sigma(s,\tau) &= U_\sigma(t,\tau), \quad \text{for all } t \ge s \ge \tau,\ \tau \in \mathbb{R}, \\ U_\sigma(\tau,\tau) &= \mathrm{Id}, \qquad\quad\ \text{for all } \tau \in \mathbb{R}. \end{aligned}$$

Let $\{T(s)\}_{s\geq 0}$ be the translation semigroup on Σ, we say that a family of processes $\{U_\sigma(t,\tau),\sigma\in\Sigma\}$ satisfies the translation identity if

$$\begin{aligned} U_\sigma(t+s,\tau+s) &= U_{T(s)\sigma}(t,\tau), && \text{for all } \sigma\in\Sigma,\ t\geq\tau,\ \tau\in\mathbb{R},\ s\geq 0,\\ T(s)\Sigma &= \Sigma, && \text{for all } s\geq 0. \end{aligned}$$

By $\mathcal{B}(X)$ we denote the collection of all bounded sets of X and $\mathbb{R}_\tau=\{t\in\mathbb{R}, t\geq\tau\}$.

Definition 2.1 ([5]). A bounded set $B_0\in\mathcal{B}(X)$ is said to be a *bounded uniformly* (w.r.t. $\sigma\in\Sigma$) *absorbing set* for $\{U_\sigma(t,\tau),\sigma\in\Sigma\}$ if for any $\tau\in\mathbb{R}$ and $B\in\mathcal{B}(X)$ there exists $T_0=T_0(B,\tau)$ such that $\bigcup_{\sigma\in\Sigma}U_\sigma(t;\tau)B\subset B_0$ for all $t\geq T_0$.

Definition 2.2 ([5]). A set $\mathcal{A}\subset X$ is said to be *uniformly* (w.r.t. $\sigma\in\Sigma$) *attracting* for the family of processes $\{U_\sigma(t,\tau),\ \sigma\in\Sigma\}$ if for any fixed $\tau\in\mathbb{R}$ and any $B\in\mathcal{B}(X)$

$$\lim_{t\to+\infty}\Big(\sup_{\sigma\in\Sigma}\operatorname{dist}(U_\sigma(t;\tau)B;\mathcal{A})\Big)=0,$$

here $\operatorname{dist}(\,\cdot\,,\,\cdot\,)$ is the usual Hausdorff semidistance in X between two sets.

Definition 2.3 ([5]). A closed set $\mathcal{A}_\Sigma\subset X$ is said to be the *uniform* (w.r.t. $\sigma\in\Sigma$) *attractor* of the family of processes $\{U_\sigma(t,\tau),\sigma\in\Sigma\}$ if it is uniformly (w.r.t. $\sigma\in\Sigma$) attracting (attracting property) and contained in any closed uniformly (w.r.t. $\sigma\in\Sigma$) attracting set $\mathcal{A}'$ of the family of processes $\{U_\sigma(t,\tau),\sigma\in\Sigma\}$: $\mathcal{A}_\Sigma\subseteq\mathcal{A}'$ (minimality property).

Definition 2.4 ([5]). A function φ is said to be translation bounded in $L^2_{\rm loc}(\mathbb{R};X)$, if

$$\|\varphi\|_b^2=\sup_{t\in\mathbb{R}}\int_t^{t+1}\|\varphi\|_X^2\,ds<+\infty.$$

Denote by $L^2_b(\mathbb{R};X)$ the set of all translation bounded functions in $L^2_{\rm loc}(\mathbb{R};X)$.

The next is an estimate of the p-Laplacian operator, e.g. see [9] for the proof.

Lemma 2.5. *Let $p\geq 2$. Then there exists constant $K>0$ such that for any $a,b\in\mathbb{R}^n$,*

$$\langle |a|^{p-2}a-|b|^{p-2}b,a-b\rangle\geq K|a-b|^p,$$

where K depends only on p and n; $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product of $\mathbb{R}^n$.

3. Uniform attractor in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$

Since $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary Γ, we define the Sobolev spaces $W^{k,p}(\Omega)$ and $W^{k,p}(\Gamma)$ to be, respectively, the completion of $C^k(\overline{\Omega})$ and $C^k(\Gamma)$, with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{0\le|\alpha|\le k} \bigg(\int_\Omega |\nabla^\alpha u|^p\,dx\bigg)^{1/p}$$

and

$$\|u\|_{W^{k,p}(\Gamma)} := \sum_{j=0}^{k} \bigg(\int_\Gamma |\nabla_\Gamma^j u|^p\,dS\bigg)^{1/p}.$$

Here, dx denotes the Lebesgue measure on Ω and dS denotes the natural surface measure on Γ. For $p \in (1,\infty)$, we define the fractional order Sobolev space

$$W^{1-1/p,p}(\Gamma) := \bigg\{u \in L^p(\Gamma) : \int_\Gamma\int_\Gamma \bigg(\frac{|u(x)-u(y)|}{|x-y|^{1-1/p}}\bigg)^p \frac{1}{|x-y|^{N-1}}\,dS_x\,dS_y < \infty\bigg\}.$$

Moreover, since $W^{1,p}(\Omega) \hookrightarrow W^{1-1/p,p}(\Gamma)$, one has that the norms on $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ and $W^{1,p}(\Omega)$ are equivalent.

Next, as in [14], we introduce the following rigorous notion of weak solution to our problem.

Definition 3.1 ([1], [14]). The pair of functions $(u(t),v(t))$ is said to be a weak solution of (1.1), if $v(t) = u(t)_{|\Gamma}$ in the trace sense, for almost every $t \in (\tau,T)$, for any $\tau, T \in \mathbb{R}$, $T > \tau$, it satisfying:

$$\begin{cases} u(t) \in \mathcal{C}([\tau,\infty);\, L^2(\Omega)) \cap L^p_{\mathrm{loc}}(\tau,T;W^{1,p}(\Omega)),\\ v(t) \in \mathcal{C}([\tau,\infty);\, L^2(\Gamma)) \cap L^p_{\mathrm{loc}}(\tau,T;W^{1-1/p,p}(\Gamma)),\end{cases}$$

and for all $\sigma \in W^{1,p}(\Omega)$ (hence, $\sigma_{|\Gamma} \in W^{1-1/p,p}(\Gamma)$) and for almost every $t \in (\tau,T)$, the following relation holds:

$$\begin{aligned}(3.1)\quad \langle \partial_t u(t),\sigma\rangle + \langle \partial_t v(t),\sigma_{|\Gamma}\rangle_\Gamma + \langle |\nabla u|^{p-2}\nabla u(t),\nabla\sigma\rangle&\\ + \langle f(u(t)),\sigma\rangle + \langle g(v(t)),\sigma_{|\Gamma}\rangle_\Gamma = \langle h(t),\sigma\rangle&.\end{aligned}$$

Moreover, in the space $L^2(\Omega)\times L^2(\Gamma)$, we have $u(\tau) = u_\tau$, $v(\tau) = v_\tau$.

Follows the well posedness result in [1], [14], we have the following result and the time-dependent terms make no essential complications.

Theorem 3.2 ([1], [14]). *Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $h(t)$ is translation bounded in $L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$, f and g satisfy* (1.2)–(1.4). *Then for any initial data $(u_\tau,v_\tau) \in L^2(\Omega)\times L^2(\Gamma)$, the problem* (1.1) *has* *a unique solution* $(u(t), v(t))$*. Moreover,* $(u_\tau, v_\tau) \mapsto (u(t), v(t))$ *is continuous on* $L^2(\Omega) \times L^2(\Gamma)$*.*

We now define the symbol space Σ for (1.1). Taking a fixed symbol $\sigma_0 = h_0$, $h_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$. We denote by $L^{2,w}_{\rm loc}(\mathbb{R}; L^2(\Omega))$ the space $L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ endowed with local weak convergence topology.

Set $\Sigma_0 = \{h_0(s+h) \mid h \in \mathbb{R}\}$, and let Σ be the closure of Σ_0 in $L^{2,w}_{\rm loc}(\mathbb{R}; L^2(\Omega))$. Thus, from Theorem 3.2, we know that the problem (1.1)–(1.4) is well posed for all $\sigma(s) \in \Sigma$ and generates a family of processes $\{U_\sigma(t,\tau),\ \sigma \in \Sigma\}$ given by the formula:

$$U_\sigma(t,\tau)(u_\tau, v_\tau) = (u(t), v(t)),$$

where $(u(t), v(t))$ is the solution of (1.1)–(1.4) and $\{U_\sigma(t,\tau),\ \sigma \in \Sigma\}$ satisfies (2.1)–(2.2). At the same time, due to the unique solvability, we know $\{U_\sigma(t,\tau),\ \sigma \in \Sigma\}$ satisfies the translation identity (2.3)–(2.4).

Then, we prove the existence of an uniformly (w.r.t. $\sigma \in \Sigma$) bounded absorbing set in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$. The proof is basically same as in [24], and for the sake of completeness, we replicate it here.

Theorem 3.3. *Assume that* $h(t)$ *is translation bounded in* $L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$*,* f *and* g *satisfy* (1.2)–(1.3)*. Then the family of processes* $\{U_\sigma(t,\tau),\ \sigma \in \Sigma\}$ *corresponding to* (1.1) *has a bounded uniformly* (*w.r.t.* $\sigma \in \Sigma$) *absorbing set* B_0 *in* $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$*.*

Proof. The following estimates can be deduced by a formal argument, this can be justified by means of the approximation procedure devised in the [14, Theorem 2.6]. Taking $\sigma = u(t)$ and $\sigma_{|\Gamma} = v(t)$ in (3.1), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Gamma |v|^2\,dS + \frac{1}{2}\int_\Omega |\nabla u|^p\,dx \\ + k_1'\int_\Omega |u|^p\,dx + k_3'\int_\Gamma |v|^q\,dS \le C + \frac{1}{4\delta}\int_\Omega |h_0(t,x)|^2\,dx,\end{aligned} \tag{3.2}$$

this implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Gamma |v|^2\,dS + \frac{1}{2}\int_\Omega |\nabla u|^p\,dx + C\bigg(\int_\Omega |u|^2\,dx + \int_\Gamma |v|^2\,dS\bigg) \\ \le C + \frac{1}{4\delta}\int_\Omega |h_0(t,x)|^2\,dx.\end{aligned}$$

Using the Gronwall lemma, we know that there exist positive constants $T_0 > \tau$ and $\alpha > 0$, such that

$$\|u(t)\|^2 + \|u(t)\|_\Gamma^2 \le \alpha, \quad \text{for any } t \ge T_0,\ \sigma \in \Sigma. \tag{3.3}$$

Then let $F(s) = \int_0^s f(\tau)\, d\tau$, $G(s) = \int_0^s g(\tau) d\tau$. Using (1.2)–(1.3) again, from (3.2) we deduce that

$$\frac{d}{dt}\left(\int_\Omega |u|^2\, dx + \int_\Gamma |v|^2\, dS\right) + \int_\Omega |\nabla u|^p\, dx + C_f' \int_\Omega F(u)\, dx + C_g' \int_\Gamma G(v)\, dS \leq C + \frac{1}{2\delta}\int_\Omega |h_0(t,x)|^2\, dx.$$

Integrating the inequality above from t to $t+1$, and combining (3.3), it follows that for any $t \geq T_0$, we have

$$\int_t^{t+1} \left(\int_\Omega |\nabla u|^p\, dx + C_f' \int_\Omega F(u)\, dx + C_g' \int_\Gamma G(v)\, dS\right) ds \leq C + \frac{1}{2\delta}\int_t^{t+1} \|h_0(s)\|^2\, ds \leq M_1, \tag{3.4}$$

where the constant M_1 depends on $|\Omega|$, $S(\Gamma)$, α, $\|h(t)\|_b^2$.

On the other hand, taking $\sigma = \partial_t u(t)$ and $\sigma_{|\Gamma} = \partial_t v(t)$ in (3.1), we obtain

$$\int_\Omega |u_t|^2\, dx + \int_\Gamma |v_t|^2\, dS + \frac{1}{p}\frac{d}{dt}\|\nabla u\|^p + \frac{d}{dt}\left(\int_\Omega F(u)\, dx + \int_\Gamma G(v)\, dS\right) = \int_\Omega h_0(t) u_t\, dx \leq \frac{1}{2}\|h_0(t)\|^2 + \frac{1}{2}\|u_t\|^2, \tag{3.5}$$

so we obtain

$$\frac{d}{dt}\left(\|\nabla u\|^p + p\int_\Omega F(u)\, dx + p\int_\Gamma G(v)\, dS\right) \leq \frac{p}{2}\|h_0(t)\|^2. \tag{3.6}$$

Combining (3.4) and (3.6), by the uniformly Gronwall lemma, we have

$$\|\nabla u\|^p + p\int_\Omega F(u)\, dx + p\int_\Gamma G(v)\, dS \leq \rho_0, \quad \text{for any } t \geq T_0 + 1,\ \sigma \in \Sigma, \tag{3.7}$$

where ρ_0 depends on $|\Omega|$, $S(\Gamma)$, M_1, $\|h(t)\|_b^2$. From (3.7), we obtain that for any $t \geq T_0 + 1$, $\sigma \in \Sigma$, there exists a positive constant ρ depending on $|\Omega|$, $S(\Gamma)$, M_1, $\|h(t)\|_b^2$, such that

$$\|\nabla u(t)\|^p + \|u(t)\|_{L^{q_1}(\Omega)} + \|v(t)\|_{L^{q_2}(\Gamma)} \leq \rho, \quad \text{for any } t \geq T_0 + 1,\ \sigma \in \Sigma.$$

As mentioned in [14], since $W^{1,p}(\Omega) \hookrightarrow W^{1-1/p,p}(\Gamma)$, one has that the norms on $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ and $W^{1,p}(\Omega)$ are equivalent. The proof is complete. □

Note that, $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ is compactly embedded into $L^2(\Omega) \times L^2(\Gamma)$. From Theorem 3.3, the existence of a uniform attractor in $L^2(\Omega) \times L^2(\Gamma)$ can be obtained immediately.

Corollary 3.4. *Under the assumption of Theorem* 3.3*, the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ *corresponding to* (1.1) *has a compact uniform* (*w.r.t.* $\sigma \in \Sigma$) *attractor* $\mathcal{A}_{\Sigma 0}$ *in* $L^2(\Omega)\times L^2(\Gamma)$.

Then, we will give some a priori estimates about u_t. In what follows, we always denote the weak differential of $h(t)$ with respect to t by $h'(t)$.

Lemma 3.5. *Let* $h(t)$ *and* $h'(t)$ *be translation bounded in* $L^2_{\rm loc}(\mathbb{R};L^2(\Omega))$, f *and* g *satisfy* (1.2)–(1.4)*, then for any* $\tau \in \mathbb{R}$ *and any bounded subset* $B \subset L^2(\Omega)\times L^2(\Gamma)$*, there exist two positive constants* $T = T(B,\tau) > \tau$ *and* M_2*, such that*

$$\int_\Omega |u_t(s)|^2\,dx + \int_\Gamma |v_t(s)|^2\,dS \le M_2 \quad \text{for all } s \ge T,\ (u_\tau,v_\tau)\in B,\ \sigma\in\Sigma,$$

where

$$u_t(s) = \frac{d}{dt}(U_\sigma(t,\tau)u_\tau)\bigg|_{t=s} \quad \text{and} \quad v_t(s) = \frac{d}{dt}(U_\sigma(t,\tau)v_\tau)\bigg|_{t=s},$$

M_2 *is a positive constant which depends on* $|\Omega|$, $S(\Gamma)$, ρ, $\|h(t)\|_b^2$, $\|h'(t)\|_b^2$.

Proof. Our estimates can be justified by means of the approximation procedure, where we proceed formally. By differentiating (1.1) with external force $h_0(t)$ in the time and denoting $\theta = u_t$, $\varrho = v_t$, we have

$$\begin{aligned}\langle \partial_t\theta,\sigma\rangle &+ \langle \partial_t\varrho,\sigma_{|\Gamma}\rangle_\Gamma + \langle |\nabla u|^{p-2}\nabla\theta,\nabla\sigma\rangle \\ &+ (p-2)\langle |\nabla u|^{p-4}(\nabla u\cdot\nabla\theta)\nabla u,\nabla\sigma\rangle \\ &+ \langle f'(u)\theta,\sigma\rangle + \langle g'(v)\varrho,\sigma_{|\Gamma}\rangle_\Gamma = \langle h(t),\sigma\rangle,\end{aligned} \tag{3.8}$$

for all $\sigma \in W^{1,p}(\Omega)$ and $\sigma_{|\Gamma} \in W^{1-1/p,p}(\Gamma)$, almost everywhere in (τ,∞), where "$\cdot$" denotes the dot product in $\mathbb{R}^n$, $\varrho(t) := \theta(t)_{|\Gamma}$.

Taking $\sigma = \theta$ and $\sigma_{|\Gamma} = \varrho$ in (3.8), we obtain that

$$\begin{aligned}\frac12\frac{d}{dt}\int_\Omega |\theta|^2dx &+ \frac12\frac{d}{dt}\int_\Gamma |\varrho|^2\,dS + \int_\Omega |\nabla u|^{p-2}|\nabla\theta|^2\,dx \\ &+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u\cdot\nabla\theta)^2\,dx + \int_\Omega f'(u)\theta^2\,dx + \int_\Gamma g'(v)\varrho^2\,dS \\ &= \int_\Omega h_0'(t,x)\theta(x)\,dx.\end{aligned}$$

From (1.4), this yields

$$\begin{aligned}\frac12\frac{d}{dt}\int_\Omega |\theta|^2\,dx &+ \frac12\frac{d}{dt}\int_\Gamma |\varrho|^2\,dS \\ &+ \int_\Omega |\nabla u|^{p-2}|\nabla\theta|^2\,dx + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u\cdot\nabla\theta)^2\,dx \\ &\le l\int_\Omega |\theta|^2\,dx + m\int_\Gamma |\varrho|^2dS + \frac12\int_\Omega |\theta|^2\,dx + \frac12\|h_0'(t)\|^2,\end{aligned}$$

so we have

$$(3.9)\qquad \frac{d}{dt}\int_\Omega |\theta|^2\,dx + \frac{d}{dt}\int_\Gamma |\varrho|^2\,dS \le C\bigg(\int_\Omega |\theta|^2 dx + \int_\Gamma |\varrho|^2\,dS\bigg) + \|h_0'(t)\|^2.$$

On the other hand, integrating (3.5) from t to $t+1$, and using (3.7), we have

$$(3.10)\qquad \int_t^{t+1}\bigg(\int_\Omega |\theta|^2 dx + \int_\Gamma |\varrho|^2 dS\bigg) \le \widetilde{C},$$

where $\widetilde{C}$ depends on $|\Omega|$, $S(\Gamma)$, M, $\|h(t)\|_b^2$. Combining (3.9)–(3.10), and using the uniform Gronwall lemma, we get

$$\int_\Omega |u_t(s)|^2\,dx + \int_\Gamma |v_t(s)|^2\,dS \le M_2 \quad \text{for all } s \ge T,\ (u_\tau, v_\tau) \in B,\ \sigma \in \Sigma,$$

where M_2 depends on $|\Omega|$, $S(\Gamma)$, M, $\|h(t)\|_b^2$, $\|h'(t)\|_b^2$. □

Finally, the following theorem gives the existence and structure of an uniform attractor in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$:

Theorem 3.6. *Assume that* $h(t) \in L^\infty(\mathbb{R}; L^2(\Omega))$ *and* $h'(t)$ *is translation bounded in* $L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$*,* f *and* g *satisfy* (1.2)–(1.4)*. Then the family of processes* $\{U_\sigma(t,\tau)\}$*,* $\sigma \in \Sigma$ *corresponding to* (1.1) *has a compact uniform* (*w.r.t.* $\sigma \in \Sigma$) *attractor* $\mathcal{A}_{\Sigma 1}$ *in* $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ *and* $\mathcal{A}_{\Sigma 1}$ *satisfies:*

$$\mathcal{A}_{\Sigma 1} = \omega_{0,\Sigma}(B_0) = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(s), \quad \text{for all } s \in \mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ *is the section at* $t = s$ *of the kernel* $\mathcal{K}_\sigma$ *of the process* $\{U_\sigma(t,\tau)\}$ *with symbol* σ*.*

Proof. Let B_0 be a $(W^{1,p}(\Omega) \cap L^{q_1}(\Omega) \times W^{1-1/p,p}(\Gamma) \cap L^{q_2}(\Gamma))$-bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set obtained in Theorem 3.3, then we need only to show that:

(3.11) for any $\{(u_{\tau_n}, v_{\tau_n})\} \subset B_0$, $\{\sigma_n\} \subset \Sigma$ and $t_n \to \infty$,

$$\{(U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}, U_{\sigma_n}(t_n,\tau_n)v_{\tau_n})\}_{n=1}^\infty$$

is precompact in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$.

Thanks to Corollary 3.4, we know that $\{(U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}, U_{\sigma_n}(t_n,\tau_n)v_{\tau_n})\}_{n=1}^\infty$ is precompact in $L^2(\Omega) \times L^2(\Gamma)$. Without loss of generality, we assume that $\{(U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}, U_{\sigma_n}(t_n,\tau_n)v_{\tau_n})\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\Omega) \times L^2(\Gamma)$.

Next, we prove that $\{(U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}, U_{\sigma_n}(t_n,\tau_n)v_{\tau_n})\}_{n=1}^\infty$ is a Cauchy sequence in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$.

Denote by $u_n^{\sigma_n}(t_n) := U_{\sigma_n}(t_n, \tau_n)u_{\tau_n}$, $v_n^{\sigma_n}(t_n) := U_{\sigma_n}(t_n, \tau_n)v_{\tau_n}$, from Lemma 2.5, which is the property of p-Laplacian operator when $p \geq 2$, and using (1.4) again, we know that there exists a constant $c > 0$, such that

$$\begin{aligned} c(\|u_n^{\sigma_n}(t_n) &- u_m^{\sigma_m}(t_m)\|^p_{W^{1,p}(\Omega)} + \|v_n^{\sigma_n}(t_n) - v_m^{\sigma_m}(t_m)\|^p_{W^{1-1/p,p}(\Gamma)}) \\ &\leq \int_\Omega \left|\frac{d}{dt}u_n^{\sigma_n}(t_n) - \frac{d}{dt}u_m^{\sigma_m}(t_m)\right| |u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)| \\ &\quad + \int_\Gamma \left|\frac{d}{dt}v_n^{\sigma_n}(t_n) - \frac{d}{dt}v_m^{\sigma_m}(t_m)\right| |v_n^{\sigma_n}(t_n) - v_m^{\sigma_m}(t_m)| \\ &\quad + \int_\Omega |\sigma_n - \sigma_m|\, |u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)| \\ &\quad + l\|u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)\|^2 + m\|v_n^{\sigma_n}(t_n) - v_m^{\sigma_m}(t_m)\|^2_\Gamma, \end{aligned}$$

which implies that

$$\begin{aligned} c(\|u_n^{\sigma_n}(t_n) &- u_m^{\sigma_m}(t_m)\|^p_{W^{1,p}(\Omega)} + \|v_n^{\sigma_n}(t_n) - v_m^{\sigma_m}(t_m)\|^p_{W^{1-1/p,p}(\Gamma)}) \\ &\leq \|\frac{d}{dt}u_n^{\sigma_n}(t_n) - \frac{d}{dt}u_m^{\sigma_m}(t_m)\|\, \|u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)\| \\ &\quad + \left\|\frac{d}{dt}v_n^{\sigma_n}(t_n) - \frac{d}{dt}v_m^{\sigma_m}(t_m)\right\|_\Gamma \|v_n^{\sigma_n}(t_n) - v_m^{\sigma_m}(t_m)\|_\Gamma \\ &\quad + \|\sigma_n - \sigma_m\|\, \|u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)\| \\ &\quad + l\|u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)\|^2 + m\|v_n^{\sigma_n}(t_n) - v_m^{\sigma_m}(t_m)\|^2_\Gamma, \end{aligned}$$

which, combining with Theorem 3.3 and Lemma 3.5, and since the norms on $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ and $W^{1,p}(\Omega)$ are equivalent, we have (3.11) immediately. Then, we use the closed process to obtain the structure of $\mathcal{A}_{\Sigma 1}$ in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$, see more details in [24] (see Pata and Zelik [20] for autonomous case). □

Remark 3.7. Note that, the growth orders of nonlinear terms $f(u)$ and $g(u)$ have no further restrictions and the solutions have not higher regularities, one can not obtain the compactness in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ by an embedding theorem. Furthermore, it seems difficult to obtain the compactness in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ through the compactness of $L^{q_1}(\Omega) \times L^{q_2}(\Gamma)$ (as that in [12], [21]).

Remark 3.8. In this paper, the compactness in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ was verified only by using of the compactness in $L^2(\Omega) \times L^2(\Gamma)$ and without any compactness in $L^{q_1}(\Omega) \times L^{q_2}(\Gamma)$, $q_1, q_2 > 2$. This implies that the compactness of the process in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ did not depend on the compactness of the process in $L^{q_1}(\Omega) \times L^{q_2}(\Gamma)$, $q_1, q_2 > 2$, i.e. did not depend on the growth orders of nonlinear terms f and g only if the nonlinear terms f and g satisfy a very weak condition that $f' \geq -l$, $g' \geq -m$.

REMARK 3.9. Using the argument of the closed process (see more details in Pata and Zelik [20]), we can easily obtain the structure of the uniform attractors.

REMARK 3.10. In Theorem 3.6, the assumption $h(x,t) \in L^\infty(\mathbb{R}; L^2(\Omega))$ is only needed to guarantee the uniform asymptotic compactness in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$. In fact, if we are only concerned with the existence of the uniform attractor in $L^{q_1}(\Omega) \times L^{q_1}(\Gamma)$, then we only assume that $h(x,t) \in L^2_n(\mathbb{R}; L^2(\Omega))$ (i.e. normal, see [24] for more details).

REMARK 3.11. As for the autonomous case of (1.1), that is $h(x,t) = h(x)$, under the assumption that $h(x) \in L^2(\Omega)$, the method in Section 3 also is valid, and the main result – Theorem 3.6 also holds.

REMARK 3.12. In this paper, we study the asymptotic behavior of the solutions of problem (1.1) by the concept of uniform attractors. For the non-autonomous dynamical systems, the theory of pullback attractors is also a good tool to describe the long time behavior of the solutions, see more detail in [16], etc. When considered for pullback attractors, the external forces $h(x,t)$ usually only satisfy some weaker condition than $h(x,t)$ of this paper (see, e.g. [19]), and it seems difficult to directly apply the method of this paper for obtaining the $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$-compactness, especially, we can not perform as that in Lemma 3.5 to derive the estimates of u_t and v_t.

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