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ABSTRACT CAUCHY PROBLEM FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Yong Zhou — Feng Jiao — Josip Pečarić

Abstract. In this paper, the existence and continuation of solutions for the Cauchy initial value problem of fractional functional differential equations in an arbitrary Banach space is discussed under hypotheses based on Carathéodory condition and the measure of noncompactness. In addition, an example is given to show that the criteria on existence of solutions for the initial value problem of fractional differential equations in finite-dimensional spaces may not be true in infinite-dimensional cases.

1. Introduction

Fractional differential equations are generalization of classical differential equations with integer order derivatives. Based on the wide application in engineering and sciences such as physics, mechanics, chemistry, economics and biology, research of fractional differential equations is active and extensive around the world. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [9], Miller and Ross [6], Podlubny [20], Lakshmikantham et al. [11] and the papers [1], [3]–[6], [8], [12]–[14], [17], [21]–[25] and

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the references therein. In [12], Lakshmikantham and Vasundhara Devi investigated the theory of fractional differential equations in a Banach space. In [17], N'Guerekata discussed some fractional abstract differential equations with nonlocal conditions. In [21], Salem gave some existence results of solutions to a class of nonlinear integral equations in Banach spaces and apply these results to the boundary value problem of fractional order.

In this paper, we assume that E is a Banach space with the norm $\|\cdot\|$. Let $J\subset\mathbb{R}$. Denote $C(J,E)$ be the Banach space of continuous functions from J into E.

Let $r>0$ and $\mathcal{C}=C([-r,0],E)$ be the space of continuous functions from $[-r,0]$ into E. For any element $z\in\mathcal{C}$, define the norm $\|z\|_*=\sup_{\theta\in[-r,0]}\|z(\theta)\|$.

Consider the initial value problem (IVP) for fractional functional differential equation given by

$$
\begin{cases} {}^cD^q x(t)=f(t,x_t), & t\in(0,a),\\ x_0=\varphi\in\mathcal{C}, \end{cases}
\tag{1.1}
$$

where ${}^cD^q$ is Caputo fractional derivative of order $0<q<1$, $f\colon[0,a)\times\mathcal{C}\to E$ is a given function satisfying some assumptions and define x_t by $x_t(\theta)=x(t+\theta)$, for $\theta\in[-r,0]$.

In this paper, we shall start with an example to illustrate that the existence result of nonlocal Cauchy problem for fractional abstract differential equations which have been obtained in [17, Theorem 2.3] is not true. We then discuss the existence and continuation of the solutions for IVP (1.1) under assumptions that f satisfies Carathéodory condition and the condition on measure of noncompactness. Finally, we give an example to illustrate the application of our abstract results.

2. Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Definition 2.1 ([9], [20]). The fractional integral of order μ with the lower limit 0 for a function f is defined as

$$
I^\mu f(t)=\frac{1}{\Gamma(\mu)}\int_0^t\frac{f(s)}{(t-s)^{1-\mu}}\,ds,\quad t>0,\ \mu>0,
$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.2 ([9], [20]). Riemann–Liouville derivative of order μ with the lower limit 0 for a function $f\colon [0,\infty) \to \mathbb{R}^n$ can be written as

$$D^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\mu+1-n}}\, ds, \quad t > 0,\ n-1 < \mu < n. \tag{2.1}$$

Definition 2.3 ([9], [20]). Caputo derivative of order μ with the lower limit 0 for a function $f\colon [0,\infty) \to \mathbb{R}^n$ can be written as

$$^cD^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\mu+1-n}}\, ds = I^{n-\mu} f^{(n)}(t),$$

for $t > 0$, $0 \le n-1 < \mu < n$. Obviously, Caputo's derivative of a constant is equal to zero.

Remark 2.4. We need to mention that there exits a link between Riemann–Liouville and Caputo's fractional derivative of order μ (see [9]). Namely,

$$\begin{aligned} ^cD^\mu f(t) =& \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\mu+1-n}}\, ds \\ =& D^\mu f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\mu+1)}\, t^{k-\mu} = D^\mu \bigg[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\, t^k \bigg], \end{aligned}$$

for $t > 0$, $n-1 < \mu < n$.

If f is an abstract function with values in E, then the integrals which appear in Definitions 2.1–2.3 and Remark 2.4 are taken in Bochner's sense.

Definition 2.5. A function $x \in C([-r,T],E)$ is a solution for IVP (1.1) on $[-r,T]$ for $T \in (0,a)$ if:

(a) the function $x(t)$ is absolutely continuous on $[0,T]$,
(b) $x_0 = \varphi$, and
(c) x satisfies the equation in (1.1).

Let A be a bounded subset in a Banach space E. The diameter of A is defined by

$$\operatorname{diam}(A) = \sup\{\|x-y\| : x, y \in A\}.$$

Clearly, $0 \le \operatorname{diam}(A) < \infty$.

Kuratowski's measure of noncompactness of A is defined by

$$\alpha(A) = \inf\{d > 0 : A \text{ is covered by a finite number of sets with diameter} \le d\}.$$

We have $\alpha(A) \le \operatorname{diam}(A)$ and $\alpha(A) \le 2d$ if $\sup\limits_{x\in A} \|x\| \le d$. We recall some properties for α (see [10]).

Let A, B be bounded subsets of E. Then:

(1) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact, where $\overline{A}$ denotes the closure of A,

(2) $\alpha(A) = \alpha(\overline{A}) = \alpha(\overline{\mathrm{co}}(A))$,
(3) $\alpha(\lambda A) = |\lambda|\alpha(A)$ for every $\lambda \in \mathbb{R}$, where $\lambda A = \{\lambda x : x \in A\}$,
(4) $\alpha(A) \le \alpha(B)$ if $A \subset B$,
(5) $\alpha(A + B) \le \alpha(A) + \alpha(B)$, where $A + B = \{x + y : x \in A, y \in B\}$,
(6) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

Assume that $J \subset \mathbb{R}$ and $1 \le p \le \infty$. For measurable functions $m\colon J \to \mathbb{R}$, define the norm

$$\|m\|_{L^p J} = \begin{cases} \bigg(\displaystyle\int_J |m(t)|^p \, dt\bigg)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\inf_{\mu(\overline{J})=0} \Big\{ \sup_{t \in J - \overline{J}} |m(t)| \Big\}, & p = \infty, \end{cases}$$

where $\mu(\overline{J})$ is the Lebesgue measure on $\overline{J}$. Let $L^p(J, \mathbb{R})$ be the Banach space of all Lebesgue measurable functions $m\colon J \to \mathbb{R}$ with $\|m\|_{L^p J} < \infty$.

Lemma 2.6 (Hölder inequality). *Assume that $\sigma, p \ge 1$, and $1/\sigma + 1/p = 1$. If $l \in L^\sigma(J, \mathbb{R})$, $m \in L^p(J, \mathbb{R})$, then for $1 \le p \le \infty$, $lm \in L^1(J, \mathbb{R})$ and*

$$\|lm\|_{L^1 J} \le \|l\|_{L^\sigma J} \|m\|_{L^p J}.$$

Lemma 2.7 (Bochner's theorem). *A measurable function $Q\colon (0, a) \to E$ is Bochner integrable if $\|Q\|$ is Lebesgue integrable.*

Lemma 2.8 ([2], [19]). *Let E be a Banach space and $A \in C([a_1, a_2], E)$ bounded and equicontinuous. Then $t \to \alpha(A(t))$ is continuous on $[a_1, a_2]$, and*

$$\alpha(A) = \max_{t \in [a_1, a_2]} \alpha(A(t)).$$

Lemma 2.9 ([19]). *If E is a Banach space and $\{u_n\}_{n \ge 1}$ is a sequence of Bochner integrable functions from $[0, b]$ into E with $\|u_n(t)\| \le h(t)$ for almost all $t \in [0, b]$ and every $n \ge 1$, where $h \in L^1([0, b], \mathbb{R})$, then the function $\psi(t) = \alpha(\{u_n(t) : n \ge 1\})$ belongs to $L^1([0, b], \mathbb{R})$ and satisfies*

$$\alpha\bigg(\bigg\{ \int_0^b u_n(s) \, ds : n \ge 1 \bigg\}\bigg) \le 2 \int_0^b \psi(s) \, ds.$$

We note that the factor 2 in the above inequality can be dropped if E is a separable Banach space and α is the Hausdorff measure of noncompactness (see [19] and [15]).

3. Main results

It is well know that Peano's theorem of integer order ordinary differential equations is not true in infinite-dimensional Banach spaces. The first result in this direction was obtained by Dieudonne [7]. He produced an example which showed that Peano's theorem is not true in the space c_0 of sequences which converge to zero. In fact, Peano's theorem of fractional differential equations is also not true in infinite-dimensional Banach spaces. In the following, we shall show that the existence result of nonlocal Cauchy problem for fractional abstract differential equations which has been obtained in [17] is not true in the space c_0.

Example 3.1. Let $E = c_0 = \{z = (z_1, z_2, \dots) : z_n \to 0 \text{ as } n \to \infty\}$ with the norm $\|z\| = \sup_{n\geq 1} |z_n|$ and $f(z) = 2(\sqrt{|z_1|}, \sqrt{|z_2|}, \dots)$ with $z = (z_1, z_2, \dots) \in c_0$. Consider the nonlocal Cauchy problem for fractional differential equations given by

$$(3.1) \qquad {}^cD^q x(t) = f(x(t)), \quad x(0) = \xi, \ t \in (0, t_0]$$

where ${}^cD^q$ is Caputo fractional derivative of order $0 < q < 1$, $\xi = (1, 1/2^2, \dots) \in c_0$, $t_0 < \min\{1, (\Gamma(1+q)/2)^{1/q}\}$.

It is obvious that $f\colon c_0 \to c_0$ is continuous. According to [17], there exists a constant $k^* = \Gamma(1+q)/(\Gamma(1+q) - 2t_0^q)$, such that IVP (3.1) possesses at least one continuous solution $x \in C([0, t_0], c_0)$ and $x(t) = (x_1(t), x_2(t), \dots) \in c_0$ on $[0, t_0]$ with $\sup_{t\in[0,t_0]} \|x(t)\| \leq k^*$. According to the definition of the norm of c_0, we can conclude that

$$(3.2) \qquad {}^cD^q x_n(t) = 2\sqrt{|x_n(t)|}, \quad x_n(0) = \frac{1}{n^2}, \quad t \in (0, t_0], \ n = 1, 2, \dots,$$

where x_n satisfies that $x_n \in C([0, t_0], \mathbb{R})$ with $\sup_{t\in[0,t_0]} |x_n(t)| \leq k^*$.

Let us consider equation (3.2) which can be written as the following equivalent form

$$(3.3) \qquad x_n(t) = \frac{1}{n^2} + 2I^q\sqrt{|x_n(t)|} = \frac{1}{n^2} + \frac{2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sqrt{|x_n(s)|}\, ds,$$

for $t \in [0, t_0]$. Since $(t-s)^{q-1} > 1$ with $s \in [0, t)$ for $t \in (0, t_0]$, we have by (3.3)

$$(3.4) \qquad x_n(t) \geq \frac{1}{n^2} + \frac{2}{\Gamma(q)} \int_0^t \sqrt{|x_n(s)|}\, ds, \quad t \in [0, t_0], \ n = 1, 2, \dots$$

Assume that $y_n \in C([0, t_0], \mathbb{R})$ is a solution of the following integral equation

$$(3.5) \qquad y_n(t) = \frac{1}{4n^2} + \frac{2}{\Gamma(q)} \int_0^t \sqrt{|y_n(s)|}\, ds, \quad t \in [0, t_0], \ n = 1, 2, \dots$$

We can get

$$(3.6) \qquad x_n(t) \geq y_n(t), \quad t \in [0, t_0], \ n = 1, 2, \dots$$

In fact, suppose (for contraction) that the conclusion (3.6) is not true. Then, because of the continuity of x and y, and that $x_n(0) > y_n(0)$, it follows that there exists a $t_1 \in (0, t_0]$ such that

$$(3.7) \qquad x_n(t_1) = y_n(t_1), \quad x_n(t) > y_n(t), \quad t \in [0, t_1), \ n = 1, 2, \dots$$

Then using (3.4) and (3.7), for $n = 1, 2, \ldots$, we get

$$y_n(t_1) = \frac{1}{4n^2} + \frac{2}{\Gamma(q)} \int_0^{t_1} \sqrt{|y_n(s)|}\, ds < \frac{1}{n^2} + \frac{2}{\Gamma(q)} \int_0^{t_1} \sqrt{|x_n(s)|}\, ds \leq x_n(t_1),$$

which is a contraction in view of (3.7). Hence the conclusion (3.6) is valid.

Since the integral (3.5) is equivalent to the following IVP

$$y_n'(t) = \frac{2}{\Gamma(q)} \sqrt{|y_n(t)|}, \quad y_n(0) = \frac{1}{4n^2}, \quad t \in [0, t_0],\ n = 1, 2, \ldots, \tag{3.8}$$

and noting $y_n(t) > 0$, $t \in [0, t_0]$, we can conclude that IVP (3.8) has a continuous solution

$$y_n(t) = \left(\frac{t}{\Gamma(q)} + \frac{1}{2n} \right)^2, \quad t \in [0, t_0],\ n = 1, 2, \ldots,$$

which means that

$$x_n(t) \geq y_n(t) = \left(\frac{t}{\Gamma(q)} + \frac{1}{2n} \right)^2, \quad t \in [0, t_0],\ n = 1, 2, \ldots \tag{3.9}$$

Therefore, for $t \in (0, t_0]$, $\lim_{n\to\infty} x_n(t) \neq 0$ by (3.9), contracting $x(t) \in c_0$. Hence IVP (3.1) has no nonlocal solution in c_0.

We are now ready to prove the existence and continuation of the solutions for IVP (1.1) under the following hypotheses:

- (H_1) For almost all $t \in [0, a)$, the function $f(t, \cdot)\colon \mathcal{C} \to E$ is continuous and for each $z \in \mathcal{C}$, the function $f(\cdot, z)\colon [0, a) \to E$ is strongly measurable,
- (H_2) for each $\tau > 0$, there exist a constant $q_1 \in [0, q)$ and $m_1 \in L^{1/q_1}([0, a), \mathbb{R}^+)$ such that $\|f(t, z)\| \leq m_1(t)$ for all $z \in \mathcal{C}$ with $\|z\|_* \leq \tau$ and almost all $t \in [0, a)$,
- (H_3) there exist a constant $q_2 \in (0, q)$ and $m_2 \in L^{1/q_2}([0, a), \mathbb{R}^+)$ such that $\alpha(f(t, B)) \leq m_2(t)\alpha(B)$ for almost all $t \in [0, a)$ and B a bounded set in $\mathcal{C}$.

In order to prove our main results, we need the following lemma.

Lemma 3.2. *Assume that the hypotheses* (H_1)–(H_2) *hold.* $x \in C([-r, T], E)$ *is a solution for* IVP (1.1) *on* $[-r, T]$ *for* $T \in (0, a)$ *if and only if* x *satisfies the following relation*

$$\begin{cases} x(\theta) = \varphi(\theta) & \text{for } \theta \in [-r, 0], \\ x(t) = \varphi(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t - s)^{q-1} f(t, x_s)\, ds & \text{for } t \in [0, T]. \end{cases} \tag{3.10}$$

Proof. Since x_t is continuous in $t \in [0, a)$, according to (H_1), $f(t, x_t)$ is a measurable function in $[0, a)$. Direct calculation gives that $(t - s)^{q-1} \in L^{1/(1-q_1)}[0, t]$ for $t \in (0, a)$ and $q_1 \in [0, q)$. Let

$$b_1 = \frac{q - 1}{1 - q_1} \in (-1, 0), \quad M = \|m_1\|_{L^{1/q_1}[0, a)}.$$

By using Lemma 2.6 (Hölder inequality) and (H_2), for $t \in (0,a)$, we obtain that

$$(3.11)\quad \int_0^t \|(t-s)^{q-1} f(s,x_s)\|\, ds \le \left(\int_0^t (t-s)^{(q-1)/(1-q_1)}\, ds\right)^{1-q_1} \|m_1\|_{L^{1/q_1}[0,t]} \le \frac{M}{(1+b_1)^{1-q_1}}\, a^{(1+b_1)(1-q_1)}.$$

Thus, $\|(t-s)^{q-1} f(s,x_s)\|$ is Lebesgue integrable with respect to $s \in [0,t)$ for all $t \in (0,a)$. From Lemma 2.7 (Bochner's theorem), it follows that $(t-s)^{q-1} f(s,x_s)$ is Bochner integrable with respect to $s \in [0,t)$ for all $t \in (0,a)$.

Let $L(\tau,s) = (t-\tau)^{-q}|\tau - s|^{q-1} m_1(s)$. Since $L(\tau,s)$ is a nonnegative, measurable function on $D = [0,t]\times[0,t]$, then we have

$$\int_0^t \left[\int_0^t L(\tau,s)\, ds\right] d\tau = \int_D L(\tau,s)\, ds\, d\tau = \int_0^t \left[\int_0^t L(\tau,s)\, d\tau\right] ds$$

and

$$\begin{aligned}
\int_D L(\tau,s)\, ds\, d\tau &= \int_0^t \left[\int_0^t L(\tau,s)\, ds\right] d\tau \\
&= \int_0^t (t-\tau)^{-q}\left[\int_0^t |\tau-s|^{q-1} m_1(s)\, ds\right] d\tau \\
&= \int_0^t (t-\tau)^{-q}\left[\int_0^\tau (\tau-s)^{q-1} m_1(s)\, ds\right] d\tau \\
&\quad + \int_0^t (t-\tau)^{-q}\left[\int_\tau^t (s-\tau)^{q-1} m_1(s)\, ds\right] d\tau \\
&\le \frac{2M}{(1+b_1)^{1-q_1}}\, a^{(1+b_1)(1-q_1)} \int_0^t (t-\tau)^{-q}\, d\tau \\
&\le \frac{2M}{(1-q)(1+b_1)^{1-q_1}}\, a^{(1+b_1)(1-q_1)+1-q}.
\end{aligned}$$

Therefore, $L_1(\tau,s) = (t-\tau)^{-q}(\tau-s)^{q-1} f(s,x_s)$ is a Bochner integrable function on $D = [0,t]\times[0,t]$, then we have

$$\int_0^t d\tau \int_0^\tau L_1(\tau,s)\, ds = \int_0^t ds \int_s^t L_1(\tau,s)\, d\tau.$$

We now prove that $D^q I^q f(t,x_t) = f(t,x_t)$, for $t \in (0,T]$, where D^q is Riemann−Liouville fractional derivative. Indeed, for $t \in (0,T]$, we have

$$\begin{aligned}
D^q I^q f(t,x_t) &= \frac{1}{\Gamma(1-q)\Gamma(q)} \frac{d}{dt} \int_0^t (t-\tau)^{-q}\left[\int_0^\tau (\tau-s)^{q-1} f(s,x_s)\, ds\right] d\tau \\
&= \frac{1}{\Gamma(1-q)\Gamma(q)} \frac{d}{dt} \int_0^t d\tau \int_0^\tau L_1(\tau,s)\, ds \\
&= \frac{1}{\Gamma(1-q)\Gamma(q)} \frac{d}{dt} \int_0^t ds \int_s^t L_1(\tau,s)\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-q)\Gamma(q)} \frac{d}{dt} \int_0^t f(s,x_s)\,ds \int_s^t (t-\tau)^{-q}(\tau-s)^{q-1}\,d\tau \\
&= \frac{d}{dt} \int_0^t f(s,x_s)\,ds = f(t,x_t).
\end{aligned}
$$

If x satisfies the relation (3.10), then we can get that $x(t)$ is absolutely continuous on $[0,T]$. In fact, for any disjoint family of open intervals $\{(c_i,d_i)\}_{1\le i\le n}$ in $[0,T]$ with $\sum_{i=1}^n (d_i-c_i) \to 0$, we have

$$
\begin{aligned}
&\sum_{i=1}^n \|x(d_i)-x(c_i)\| \\
&\quad = \sum_{i=1}^n \frac{1}{\Gamma(q)} \bigg\| \int_0^{d_i} (d_i-s)^{q-1} f(s,x_s)\,ds - \int_0^{c_i} (c_i-s)^{q-1} f(s,x_s)\,ds \bigg\| \\
&\quad \le \sum_{i=1}^n \frac{1}{\Gamma(q)} \bigg\| \int_{c_i}^{d_i} (d_i-s)^{q-1} f(s,x_s)\,ds \bigg\| \\
&\qquad + \sum_{i=1}^n \frac{1}{\Gamma(q)} \bigg\| \int_0^{c_i} (d_i-s)^{q-1} f(s,x_s)\,ds - \int_0^{c_i} (c_i-s)^{q-1} f(s,x_s)\,ds \bigg\| \\
&\quad \le \sum_{i=1}^n \frac{1}{\Gamma(q)} \int_{c_i}^{d_i} (d_i-s)^{q-1} m_1(s)\,ds \\
&\qquad + \sum_{i=1}^n \frac{1}{\Gamma(q)} \int_0^{c_i} ((c_i-s)^{q-1} - (d_i-s)^{q-1}) m_1(s)\,ds \\
&\quad \le \sum_{i=1}^n \frac{1}{\Gamma(q)} \bigg(\int_{c_i}^{d_i} (d_i-s)^{(q-1)/(1-q_1)}\,ds \bigg)^{1-q_1} \|m_1\|_{L^{1/q_1}[0,T]} \\
&\qquad + \sum_{i=1}^n \frac{1}{\Gamma(q)} \bigg(\int_0^{c_i} (c_i-s)^{(q-1)/(1-q_1)} \\
&\qquad - (d_i-s)^{(q-1)/(1-q_1)}\,ds \bigg)^{1-q_1} \|m_1\|_{L^{1/q_1}[0,T]} \\
&\quad = \sum_{i=1}^n \frac{(d_i-c_i)^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{1/q_1}[0,T]} \\
&\qquad + \sum_{i=1}^n \frac{(c_i^{1+b_1} - d_i^{1+b_1} + (d_i-c_i)^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{1/q_1}[0,T]} \\
&\quad \le 2\sum_{i=1}^n \frac{(d_i-c_i)^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{1/q_1}[0,T]} \to 0.
\end{aligned}
$$

Therefore, $x(t)$ is absolutely continuous on $[0,T]$, which implies that $x(t)$ is differentiable almost everywhere on $[0,T]$. According to the argument above and Remark 2.4, for $t \in (0,T]$, we have

$$\begin{aligned}
{}^cD^q x(t) &= {}^cD^q\left[\varphi(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x_s)\,ds\right] \\
&= {}^cD^q\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x_s)\,ds\right] = {}^cD^q(I^q f(t,x_t)) \\
&= D^q(I^q f(t,x_t)) - [I^q f(t,x_t)]_{t=0}\,\frac{t^{-q}}{\Gamma(1-q)} \\
&= f(t,x_t) - [I^q f(t,x_t)]_{t=0}\,\frac{t^{-q}}{\Gamma(1-q)}.
\end{aligned}$$

Since $(t-s)^{q-1} f(s,x_s)$ is Lebesgue integrable with respect to $s \in [0,t)$ for all $t \in (0,T]$, we know that $[I^q f(t,x_t)]_{t=0} = 0$, which means that ${}^cD^q x(t) = f(t,x_t)$, for $t \in (0,T]$. Hence, $x \in C([-r,T],E)$ is a solution of IVP (1.1). On the other hand, it is obvious that if $x \in C([-r,T],E)$ is a solution of IVP (1.1), then x satisfies the relation (3.10), and this completes the proof. □

Theorem 3.3 (Existence). *Assume that hypotheses* (H_1)–(H_3) *hold. Then, for every* $\varphi \in \mathcal{C}$, *there exists a solution* $x \in C([-r,T],E)$ *for* IVP (1.1) *with some* $T \in (0,a)$.

Proof. Let $k > 0$ be any number and we can choose $T \in (0,a)$ such that

$$\frac{T^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}}\,\|m_1\|_{L^{1/q_1}[0,T]} \le k, \tag{3.12}$$

$$\frac{T^{(1+b_2)(1-q_2)}}{\Gamma(q)(1+b_2)^{1-q_2}}\,\|m_2\|_{L^{1/q_2}[0,T]} < 1, \tag{3.13}$$

where $b_i = (q-1)/(1-q_i) \in (-1,0)$, $i = 1,2$.

Consider the set B_k defined as follows

$$B_k = \left\{x \in C([-r,T],E) : x_0 = \varphi,\ \sup_{s\in[0,T]} \|x(s) - \varphi(0)\| \le k\right\}.$$

Define the operator F on B_k as follows

$$\begin{cases} Fx(\theta) = \varphi(\theta) & \text{for } \theta \in [-r,0], \\ Fx(t) = \varphi(0) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1} f(t,x_s)\,ds, & \text{for } t \in [0,T], \end{cases}$$

where $x \in B_k$. We prove that the operator equation $x = Fx$ has a solution $x \in B_k$, which means that x is a solution of IVP (1.1).

First, we observe that for every $y \in B_k$, $(Fy)(t)$ is continuous on $t \in [-r,T]$ and for $t \in [0,T]$, by (3.12) and Lemma 2.6 (Hölder inequality), we have

$$
\begin{aligned}
\|(Fy)(t) - \varphi(0)\| &\leq \frac{1}{\Gamma(q)} \int_0^t \|(t-s)^{q-1} f(s,y_s)\|\, ds \\
&\leq \frac{1}{\Gamma(q)} \bigg(\int_0^t (t-s)^{(q-1)/(1-q_1)}\, ds \bigg)^{1-q_1} \|m_1\|_{L^{1/q_1}[0,T]} \\
&\leq \frac{T^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{1/q_1}[0,T]} \leq k,
\end{aligned} \tag{3.14}
$$

where $b_1 = (q-1)/(1-q_1) \in (-1,0)$. Thus, $\sup\limits_{t\in[0,T]} \|(Fy)(t) - \varphi(0)\| \leq k$, which implies that $F\colon B_k \to B_k$.

Further, we prove that F is a continuous operator on B_k. Let $\{y^n\} \subseteq B_k$ with $y^n \to y$ on B_k. Then by (H_1) and the fact that $y_t^n \to y_t$, $t \in [0,T]$, we have

$$
f(s,y_s^n) \to f(s,y_s), \quad \text{a.e. } t \in [0,T] \text{ as } n \to \infty.
$$

Noting that $\|f(s,y_s^n) - f(s,y_s)\| \leq 2m_1(s)$, by the dominated convergence theorem, as $n \to \infty$, we have

$$
\begin{aligned}
\sup_{t\in[0,T]} \|(Fy^n)(t) - (Fy)(t)\| &= \sup_{t\in[0,T]} \bigg\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s,y_s^n) - f(s,y_s)]\, ds \bigg\| \\
&\leq \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \|f(s,y_s^n) - f(s,y_s)\|\, ds \to 0,
\end{aligned}
$$

which implies that F is continuous.

For each $n \geq 1$, we define a sequence $\{x^n : n \geq 1\}$ in the following way

$$
x^n(t) = \begin{cases} \varphi^0(t) & \text{for } t \in [-r, T/n], \\ \varphi(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^{t-T/n} (t-s)^{q-1} f(t,x_s^n)\, ds & \text{for } t \in [T/n, T], \end{cases}
$$

where $\varphi^0 \in C([-r,a),E)$ denotes the function defined by

$$
\varphi^0(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r,0], \\ \varphi(0) & \text{for } t \in [0,a). \end{cases}
$$

Using the similar method as we did in (3.14), we get that $x^n \in B_k$ for all $n \geq 1$.

Let $A = \{x^n : n \geq 1\}$. It follows that the set A is uniformly bounded. Further, we show that the set A is equicontinuous on $[-r,T]$.

If $-r \leq t_1 < t_2 \leq T/n$, then for each $x^n \in A$, we have

$$
\lim_{t_1\to t_2} \|x^n(t_2) - x^n(t_1)\| = \lim_{t_1\to t_2} \|\varphi^0(t_2) - \varphi^0(t_1)\| = 0
$$

independently of $x^n \in A$.

Next, if $-r \le t_1 \le T/n < t_2 \le T$, then for each $x^n \in A$, by using Lemma 2.6, we have

$$\begin{aligned}
&\|x^n(t_2) - x^n(t_1)\| \le \|\varphi(0) - \varphi^0(t_1)\| + \left\| \frac{1}{\Gamma(q)} \int_0^{t_2 - T/n} (t_2 - s)^{q-1} f(s, x_s^n)\, ds \right\| \\
&\le \|\varphi(0) - \varphi^0(t_1)\| + \frac{1}{\Gamma(q)} \int_0^{t_2 - T/n} (t_2 - s)^{q-1} m_1(s)\, ds \\
&\le \|\varphi(0) - \varphi^0(t_1)\| + \frac{1}{\Gamma(q)} \left(\int_0^{t_2 - T/n} (t_2 - s)^{(q-1)/(1-q_1)}\, ds \right)^{1-q_1} \|m_1\|_{L^{1/q_1}[0,T]} \\
&= \|\varphi(0) - \varphi^0(t_1)\| + \frac{(t_2^{1+b_1} - (T/n)^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{1/q_1}[0,T]}.
\end{aligned}$$

According to the definition of φ^0, and using the last inequality, we obtain that

$$\|x^n(t_2) - x^n(t_1)\| \to 0$$

independently of $x^n \in A$, as $t_1 \to t_2$.

Finally, if $T/n \le t_1 < t_2 \le T$, then for each $x^n \in A$, by using Lemma 2.6, we have

$$\begin{aligned}
&\|x^n(t_2) - x^n(t_1)\| \\
&= \left\| \frac{1}{\Gamma(q)} \int_0^{t_2 - T/n} (t_2 - s)^{q-1} f(s, x_s^n)\, ds - \frac{1}{\Gamma(q)} \int_0^{t_1 - T/n} (t_1 - s)^{q-1} f(s, x_s^n)\, ds \right\| \\
&\le \left\| \frac{1}{\Gamma(q)} \int_{t_1 - T/n}^{t_2 - T/n} (t_2 - s)^{q-1} f(s, x_s^n)\, ds \right\| \\
&\quad + \left\| \frac{1}{\Gamma(q)} \int_0^{t_1 - T/n} (t_2 - s)^{q-1} f(s, x_s^n)\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1 - T/n} (t_1 - s)^{q-1} f(s, x_s^n) ds \right\| \\
&\le \frac{1}{\Gamma(q)} \int_{t_1 - T/n}^{t_2 - T/n} (t_2 - s)^{q-1} m_1(s)\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{t_1 - T/n} ((t_1 - s)^{q-1} - (t_2 - s)^{q-1}) m_1(s)\, ds \\
&\le \frac{1}{\Gamma(q)} \left(\int_{t_1 - T/n}^{t_2 - T/n} (t_2 - s)^{(q-1)/(1-q_1)}\, ds \right)^{1-q_1} \|m_1\|_{L^{1/q_1}[0,T]} \\
&\quad + \frac{1}{\Gamma(q)} \left(\int_0^{t_1 - T/n} (t_1 - s)^{(q-1)/(1-q_1)} - (t_2 - s)^{(q-1)/(1-q_1)}\, ds \right)^{1-q_1} \\
&\quad \cdot \|m_1\|_{L^{1/q_1}[0,T]} \\
&= \frac{((t_2 - t_1 + T/n)^{1+b_1} - (T/n)^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{1/q_1}[0,T]}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(t_1^{1+b_1}-(T/n)^{1+b_1}-t_2^{1+b_1}+(t_2-t_1+T/n)^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}}\,\|m_1\|_{L^{1/q_1}[0,T]}\\
&\leq 2\frac{((t_2-t_1+T/n)^{1+b_1}-(T/n)^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}}\,\|m_1\|_{L^{1/q_1}[0,T]}.
\end{aligned}
$$

It is easy to see that the last inequality tends to zero independently of $x^n\in A$, as $t_1\to t_2$, which means that the set A is equicontinuous.

Set $A(t)=\{x^n(t):n\geq 1\}$ and $A_t=\{x_t^n:n\geq 1\}$ for any $t\in[0,T]$. By the properties (3.2) and (3.4) of the measure of noncompactness, for any fixed $t\in(0,T]$ and $\delta\in(0,t)$, we have

$$
\begin{aligned}
\alpha(A(t))\leq\,&\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_0^{t-\delta}(t-s)^{q-1}f(s,x_s^n)\,ds:n\geq 1\bigg\}\bigg)\\
&+\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-\delta}^{t}(t-s)^{q-1}f(s,x_s^n)\,ds:n\geq 1\bigg\}\bigg)\\
&+\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-T/n}^{t}(t-s)^{q-1}f(s,x_s^n)\,ds:n\geq 1\bigg\}\bigg),
\end{aligned}
$$

for all $\varepsilon>0$, we can find δ sufficiently small such that

$$
\frac{\delta^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}}\,\|m_1\|_{L^{1/q_1}[0,T]}<\frac{\varepsilon}{4}.
$$

Therefore, for each $t\in(0,T]$, we have that

$$
\begin{aligned}
&\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-\delta}^{t}(t-s)^{q-1}f(s,x_s^n)\,ds:n\geq 1\bigg\}\bigg)\\
&\qquad\qquad\leq\frac{2}{\Gamma(q)}\int_{t-\delta}^{t}(t-s)^{q-1}m_1(s)\,ds<\frac{\varepsilon}{2}.
\end{aligned}
$$

Moreover, we can choose $N_\delta\geq 1$ such that $T/n\leq\delta$ for $n\geq N_\delta$. Then we obtain that

$$
\begin{aligned}
&\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-T/n}^{t}(t-s)^{q-1}f(s,x_s^n)\,ds:n\geq N_\delta\bigg\}\bigg)\\
&\qquad\qquad\leq\frac{2}{\Gamma(q)}\sup_{n\geq N_\delta}\int_{t-T/n}^{t}(t-s)^{q-1}m_1(s)\,ds<\frac{\varepsilon}{2},
\end{aligned}
$$

for each $t\in(0,T]$. Hence, by the properties (1.1) and (3.5) of the measure of noncompactness, it follows that

$$
\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-T/n}^{t}(t-s)^{q-1}f(s,x_s^n)\,ds:n\geq 1\bigg\}\bigg)<\frac{\varepsilon}{2}.
$$

Then, we obtain that

$$
\alpha(A(t))\leq\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_0^{t-\delta}(t-s)^{q-1}f(s,x_s^n)\,ds:n\geq 1\bigg\}\bigg)+\varepsilon,
$$

for $t \in (0,T]$. By Lemma 2.9 and (H_3), we have that

$$\begin{aligned}\alpha(A(t)) &\le \frac{2}{\Gamma(q)}\int_0^{t-\delta}\alpha((t-s)^{q-1}f(s,A_s))\,ds+\varepsilon\\ &=\frac{2}{\Gamma(q)}\int_0^{t-\delta}(t-s)^{q-1}\alpha(f(s,A_s))\,ds+\varepsilon\\ &\le\frac{2}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}m_2(s)\alpha(A_s)\,ds+\varepsilon,\end{aligned}$$

where $t \in (0,T]$. Since $x^n(\theta)=\varphi(\theta)$, $\theta\in[-r,0]$, we have $\alpha(\{x^n(\theta):n\ge 1\})=0$ for $\theta\in[-r,0]$. Moreover, by Lemma 2.8, for $s\in[0,t]$ with $t\in(0,T]$, we deduce that

$$\alpha(A_s)=\max_{\theta\in[-r,0]}\alpha(\{x^n_s(\theta):n\ge1\})\le\sup_{s\in[0,t]}\alpha(\{x^n(s):n\ge1\})=\sup_{s\in[0,t]}\alpha(A(s)).$$

Since ε is arbitrary, we have that

$$\alpha(A(t))\le\frac{2T^{(1+b_2)(1-q_2)}}{\Gamma(q)(1+b_2)^{1-q_2}}\,\|m_2\|_{L^{1/q_2}[0,T]}\sup_{s\in[0,t]}\alpha(A(s)),$$

where $t\in(0,T]$ and $b_2=(q-1)/(1-q_2)\in(-1,0)$.

Since (3.13) and $x^n_0=\varphi$, we must have that $\alpha(A(t))=0$ for every $t\in[-r,T]$. Then, by Lemma 2.8, we have that $\alpha(A)=\sup\limits_{t\in[-r,T]}\alpha(A(t))=0$. Therefore, A is a relatively compact subset of B_k. Then, there exists a subsequence if necessary, we may assume that the sequence $\{x^n\}_{n\ge1}$ converges uniformly on $[-r,T]$ to a continuous function $x\in B_k$ with $x(\theta)=\varphi(\theta)$, $\theta\in[-r,0]$.

Moreover, for $t\in[0,T/n]$, we have

$$\begin{aligned}\|(Fx^n)(t)-x^n(t)\|&\le\frac{1}{\Gamma(q)}\int_0^{T/n}(t-s)^{q-1}\|f(t,x^n_s)\|\,ds\\ &\le\frac{1}{\Gamma(q)}\int_0^{T/n}(t-s)^{q-1}m_1(s)\,ds\end{aligned}$$

and for $t\in[T/n,T]$, we have

$$\begin{aligned}&\|(Fx^n)(t)-x^n(t)\|\\ &\quad=\frac{1}{\Gamma(q)}\bigg\|\int_0^t(t-s)^{q-1}f(t,x^n_s)\,ds-\int_0^{t-T/n}(t-s)^{q-1}f(t,x^n_s)\,ds\bigg\|\\ &\quad=\frac{1}{\Gamma(q)}\bigg\|\int_{t-T/n}^t(t-s)^{q-1}f(t,x^n_s)\,ds\bigg\|\le\frac{1}{\Gamma(q)}\bigg\|\int_{t-T/n}^t(t-s)^{q-1}m_1(s)\,ds\bigg\|.\end{aligned}$$

Therefore, it follows that

$$\sup_{t\in[0,T]}\|(Fx^n)(t)-x^n(t)\|\to0\quad\text{as } n\to\infty. \tag{3.15}$$

Since

$$\sup_{t\in[0,T]}\|(Fx)(t)-x(t)\| \le \sup_{t\in[0,T]}\|(Fx)(t)-(Fx^n)(t)\| + \sup_{t\in[0,T]}\|(Fx^n)(t)-x^n(t)\| + \sup_{t\in[0,T]}\|x^n(t)-x(t)\|,$$

then, by (3.15) and the fact that F is a continuous operator, we obtain that $\sup_{t\in[0,T]}\|(Fx)(t)-x(t)\| = 0$. It follows that $x(t) = (Fx)(t)$ for every $t \in [0,T]$. Hence

$$x(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r,0], \\ \varphi(0) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1} f(t,x_s)\,ds & \text{for } t \in [0,T], \end{cases}$$

solve IVP (1.1), and this completes the proof. □

Corollary 3.4. *Assume that hypotheses* (H_1)–(H_3) *hold. Then, for every* $\varphi \in \mathcal{C}$, *there exists* $T \in (0,a)$ *and a sequence of continuous functions* $x^n\colon [-r,T] \to E$, *such that*:

(a) x^n *are absolutely continuous on* $[0,T]$,
(b) $x_0^n = \varphi$, *for every* $n \ge 1$, *and*
(c) *extracting a subsequence which is labeled in the same way such that* $x^n(t) \to x(t)$ *uniformly on* $[-r,T]$ *and* $x\colon[-r,T] \to E$ *is a solution for* IVP (1.1).

Theorem 3.5 (Continuation). *Assume that hypotheses* (H_1)–(H_3) *hold. Then the largest interval of existence of any bounded solution of* IVP (1.1) *is* $[0,a)$.

Proof. Let $x\colon[-r,\beta) \to E$ be a solution of IVP (1.1) existing on the interval $[-r,\beta)$, where $\beta \in (0,a)$. Suppose (for contraction) that the value of β cannot be increased. For $0 < t_1 < t_2 < \beta$, we have

$$\begin{aligned}
\|x(t_2) &- x(t_1)\| \\
&= \bigg\|\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}f(s,x_s)\,ds - \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}f(s,x_s)\,ds\bigg\| \\
&\le \bigg\|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x_s)\,ds\bigg\| \\
&\quad + \bigg\|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_2-s)^{q-1}f(s,x_s)\,ds - \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}f(s,x_s)\,ds\bigg\| \\
&\le \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}m_1(s)\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{t_1}((t_1-s)^{q-1}-(t_2-s)^{q-1})m_1(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(q)}\bigg(\int_{t_1}^{t_2}(t_2-s)^{(q-1)/(1-q_1)}\,ds\bigg)^{1-q_1}\|m_1\|_{L^{1/q_1}[0,\beta]}\\
&\quad+\frac{1}{\Gamma(q)}\bigg(\int_0^{t_1}(t_1-s)^{(q-1)/(1-q_1)}-(t_2-s)^{(q-1)/(1-q_1)}\,ds\bigg)^{1-q_1}\\
&\quad\cdot\|m_1\|_{L^{1/q_1}[0,\beta]}\\
&\leq\frac{(t_2-t_1)^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}}\,\|m_1\|_{L^{1/q_1}[0,\beta]}\\
&\quad+\frac{(t_1^{1+b_1}-t_2^{1+b_1}+(t_2-t_1)^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}}\,\|m_1\|_{L^{1/q_1}[0,\beta]}\\
&\leq 2\,\frac{(t_2-t_1)^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}}\,\|m_1\|_{L^{1/q_1}[0,\beta]}.
\end{aligned}
$$

Letting $t_1,t_2\to\beta^-$ and using Cauchy criterion, it follows that $\lim\limits_{t\to\beta^-}x(t)$ exists. We denote $x(\beta)=\lim\limits_{t\to\beta^-}x(t)$, then the function x can be extended by continuity on $[0,\beta]$.

Further, let $g(t,z)=f(t+\beta,z)$ for $t\in[0,a-\beta)$ and $z\in\mathcal{C}$. Then, for each $\tau>0$ and almost all $t\in[0,a-\beta)$, we have $\|g(t,z)\|\leq m_1^*(t)=m_1(t+\beta)$ for all $z\in\mathcal{C}$ with $\|z\|_*\leq\tau$ and $\alpha(f(t,B))\leq m_2^*(t)\alpha(B)=m_2(t+\beta)\alpha(B)$ for B a bounded set in $\mathcal{C}$.

Consider the new IVP

$$
\begin{cases}
{}^cD^qy(t)=g(t,y_t) & \text{for a.e. } t\in(0,a-\beta),\\
y_0=\phi,
\end{cases}
\tag{3.16}
$$

where $\phi\in C([-r,0],E)=\mathcal{C}$ is defined by $\phi(\theta)=x(\theta+\beta)$, for all $\theta\in[-r,0]$.

By Theorem 3.3, there exists a solution $y\colon[-r,\tau]\to E$ of IVP (3.16), where $\tau\in(0,a-\beta)$. It follows that $v\colon[-r,\beta+\tau]\to E$, given by

$$
v(t)=\begin{cases}
x(t) & \text{for } t\in[-r,\beta],\\
y(t-\beta) & \text{for } t\in[\beta,\beta+\tau],
\end{cases}
$$

is a solution of IVP (1.1) because, for almost all $t\in[\beta,\beta+\tau]$, we have that

$$
{}^cD^qv(t)={}^cD^qy(t-\beta)=g(t-\beta,y_{t-\beta})=f(t,y_{t-\beta})=f(t,v_t).
$$

Therefore, the solution x can be continued beyond β, contracting our assumption. Hence every solution x of IVP (1.1) exists on $[-r,a)$ and the proof is complete.□

We now give an example to illustrate the application of our abstract results.

Example 3.6. Consider the infinite system of fractional functional differential equations

$$
\begin{cases}
{}^cD^{1/2}x_n(t)=\dfrac{1}{nt^{1/3}}\,x_n^2(t-r) & \text{for } t\in(0,a),\\
x_n(\theta)=\varphi(\theta)=\dfrac{\theta}{n} & \text{for } \theta\in[-r,0],\ n=1,2,\dots
\end{cases}
\tag{3.17}
$$

Let $E = c_0 = \{x = (x_1, x_2, \dots)\colon x_n \to 0\}$ with norm $\|x\| = \sup\limits_{n\geq 1} |x_n|$. Then the infinite system (3.17) can be regarded as a IVP of form (1.1) in E. In this situation,

$$\begin{aligned} q &= 1/2, \\ x &= (x_1, \dots, x_n, \dots), \\ x_t &= x(t-r) = (x_1(t-r), \dots, x_n(t-r), \dots), \\ \varphi(\theta) &= (\theta, \theta/2, \dots, \theta/n, \dots) \quad \text{for } \theta \in [-r, 0] \end{aligned}$$

and $f = (f_1, \dots, f_n, \dots)$, in which

$$f_n(t, x_t) = \frac{1}{nt^{1/3}} x_n^2(t-r). \tag{3.18}$$

It is obviously that conditions (H_1) and (H_2) are satisfied. Now, we check the condition (H_3) and the argument is similar to [25]. Let $t \in (0, a)$, $R > 0$ be given and $\{w^{(m)}\}$ be any sequence in $f(t, B)$, where $w^{(m)} = (w_1^{(m)}, \dots, w_n^{(m)}, \dots)$ and $B = \{z \in \mathcal{C} : \|z\|_* \leq R\}$ is a bounded set in $\mathcal{C}$. By (3.18), we have

$$0 \leq w_n^{(m)} \leq \frac{R^2}{nt^{1/3}}, \quad n, m = 1, 2, \dots \tag{3.19}$$

So, $\{w_n^{(m)}\}$ is bounded and, by the diagonal method, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$w_n^{(m_i)} \to w_n \quad \text{as } i \to \infty,\ n = 1, 2, \dots, \tag{3.20}$$

which implies by virtue of (3.19) that

$$0 \leq w_n \leq \frac{R^2}{nt^{1/3}}, \quad n = 1, 2, \dots \tag{3.21}$$

Hence $w = (w_1, \dots, w_n, \dots) \in c_0$. It is easy to see from (3.19)–(3.21) that

$$\|w^{(m_i)} - w\| = \sup_n |w_n^{(m_i)} - w_n| \to 0 \quad \text{as } i \to \infty.$$

Thus, we have proved that $f(t, B)$ is relatively compact in c_0 for $t \in (0, a)$, which means that $f(t, B) = 0$ for almost all $t \in [0, a)$ and B a bounded set in $\mathcal{C}$. Hence, the condition (H_3) is satisfied. Finally, from Theorems 3.3 and 3.5, we can conclude that the infinite system (3.17) has a continuous solution and the largest interval of existence of the solution is $[-r, a)$.

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Yong Zhou
School of Mathematics and Computational Science
Xiangtan University
Hunan 411105, P.R. CHINA

E-mail address: yzhou@xtu.edu.cn

Feng Jiao
School of Mathematics and Information Sciences
Guangzhou University
Guangdong 510006, P.R. CHINA

E-mail address: jf1704@163.com

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb, CROATIA

E-mail address: pecaric@mahazu.hazu.hr