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VIABILITY FOR UPPER SEMICONTINUOUS DIFFERENTIAL INCLUSIONS WITHOUT CONVEXITY

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Abstract. The aim of this paper is to prove the existence result of viable solutions for the differential inclusion

$$\dot{x}(t) \in F(x(t)), \qquad x(t) \in K \quad \text{on } [0, T],$$

where F is an upper semicontinuous set-valued map with compact values.

1. Introduction

The aim of this paper is to prove the existence of solutions for the following nonconvex differential inclusions:

$$\begin{cases} \dot{x}(t) \in F(x(t)) & \text{a.e. on } [0, T], \\ x(0) = x_0 \in K, \\ x(t) \in K, \end{cases} \tag{1.1}$$

where F is an upper semicontinuous set-valued map with compact values and K is a subset of a real separable Hilbert space H.

Existence result of local solution, in finite dimensional space, for nonconvex differential inclusions with upper semicontinuous right hand-side, was first established by Bressan, Cellina and Colombo (see [8]). The authors assumed that the values of the set-valued map is contained in the subdifferential (in the

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sense of analysis convex) of convex lower semicontinuous function. Ancona and Colombo (see [2]), under the same hypotheses, extend this result to the perturbed problem

$$\dot{x}(t) \in f(t, x(t)) + F(x(t))$$

where $f(\cdot\,,\,\cdot)$ is a Carathéodory function.

In this context, Yarou (see [19]) extend the perturbed problem in [2] to infinite dimensional space. However, the values of F always contained in the Clarke subdifferential and under very strong assumptions on F and f. In this framework, consult [5]–[7], [11], [15], [17] for other related results concerning the extension of the main result in [8].

Recently, Aitalioubrahim and Sajid have proved (see [1]) an exact viability version of the work of Ancona and Colombo assuming the following hypotheses: F is upper semicontinuous, the set $\{f(s,\cdot);\ s \in \mathbb{R}\}$ is equicontinuous, where for each $x \in K$, $x \mapsto f(s,x)$ is measurable, $F(x) \cap T_K^f(t,x) \neq \emptyset$ and $F(x) \subset \partial_c V(x)$ for all $(t,x) \in \mathbb{R} \times K$, where V is uniformly regular function and

$$T_K^f(t,x) = \left\{ v \in H,\ \liminf_{h \mapsto 0^+} \frac{1}{h}\, d_K\left(x + hv + \int_t^{t+h} f(s,x)\, ds \right) = 0 \right\}.$$

Moreover, in all the above works, the values of the set-valued map is contained in the subdifferential (in the sense of analysis convex or in the sense of Clarke), and the convexity or the uniformly regularity assumption of V were widely used in the proof.

On the other hand, Kannai and Tallos [16] and Cernea [10] proved the existence of solutions to the following differential inclusion $\dot{x}(t) \in F(t, x(t))$, $x(t) \in K$, where K is a convex subset and F is measurable with respect to the first argument and upper semicontinuous with respect to the second argument. The proof in [10], [16] bases on Scorza–Dragoni type results for upper semicontinuous maps and the results are obtained under the following assumption $F(t,x) \cap T_K(x) \cap \partial_c V(x) \neq \emptyset$, where V is lower regular in [16] and is convex in [10]. $T_K(x)$ is the Bouligand tangent cone of K at x.

This paper is devoted to establish a viable solutions of the problem of Bressan, Cellina and Colombo, but with weaker hypotheses, namely, F is upper semicontinuous such that

$$F(x) \cap \partial_c V(x) \cap T_K^f(x) \neq \emptyset \quad \text{for all } x \in K, \tag{1.2}$$

where $\partial_c V$ denotes the Clarke subdifferential of a regular function V. More specifically, we should point out that the class of regular functions is so large, it contains the class of convex functions and the class of uniformly regular functions (see [19]), and that the condition (1.2) is weaker than the all such conditions supposed in the above works. These signify that our result generalizes the previous works and all of the results in the literature concerning this topic of problems.

2. Notations, definitions and the main result

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$. For $x \in H$ and $r > 0$, let $B(x,r)$ be the open ball centered at x with radius r and $\overline{B}(x,r)$ be its closure. Put $B = B(0,1)$.

We shortly review some notions used in this paper (see [12], [13], [18] as general references).

Let $V\colon H \to \mathbb{R}$ be a lower semicontinuous function and x be any point where V is finite. The generalized Rockafellar directional derivative $V^{\uparrow}(x,\,\cdot\,)$ is

$$V^{\uparrow}(x,v) := \limsup_{\substack{x'\to x\\ V(x')\to V(x)\\ t\to 0^+}} \inf_{v'\to v} \frac{V(x'+tv')-V(x')}{t}.$$

The upper generalized Clarke directional derivative $V^o(x,\,\cdot\,)$ is

$$V^o(x,v) := \limsup_{\substack{h\to 0^+\\ y\to x}} \frac{V(y+hv)-V(y)}{h}.$$

Analogously the lower generalized Clarke directional derivative $V_o(x,\,\cdot\,)$ is

$$V_o(x,v) := \liminf_{\substack{h\to 0^+\\ y\to x}} \frac{V(y+hv)-V(y)}{h}.$$

If V is Lipschitz around x, then $V^{\uparrow}(x,v)$ coincides with $V^o(x,v)$ for all $v \in H$. We also recall that the Clarke subdifferential of V at x is defined by

$$\partial_c V(x) := \{y \in H : \langle y, v\rangle \le V^{\uparrow}(x,v),\ \text{for all } v \in H\}.$$

In the following proposition we summarize some useful properties of Clarke generalized directional derivatives.

Proposition 2.1 ([12], [13]). *Let $V\colon H \to \mathbb{R}$ be locally Lipschitz. Then the following conditions holds*:

(a) $\partial_c V(x) = \{p \in H : V^o(x,v) \ge \langle p,v\rangle,\ \textit{for all}\ v \in H\} = \{p \in H : V_o(x,v) \le \langle p,v\rangle,\ \textit{for all}\ v \in H\}$,

(b) $V^o(x,v) = \max\{\langle p,v\rangle,\ p \in \partial_c V(x)\}$ *and* $V_o(x,v) = \min\{\langle p,v\rangle,\ p \in \partial_c V(x)\} = -V^o(x,-v)$.

Let us recall the definition of the concept of the regularity (in the sense of Clarke).

Definition 2.2 ([12]). Let $V\colon H \to \mathbb{R}$ be a locally Lipschitz function. We say that V is *regular at x* if for all $v \in H$, the usual directional derivative $V'(x,v)$ exists and $V'(x,v) = V^o(x,v)$. We say that V is regular over a set S if it is regular at any point in S.

If S is a bounded set of H, then the Kuratowski's measure of noncompactness of S, $\beta(S)$, is defined by

$$\beta(S) = \inf\{d > 0 : S \text{ can be covered by a finite number of sets with diameter less than } d\}.$$

In the following lemma we recall some useful properties for the measure of noncompactness β. For instance see Proposition 9.1 in [14].

Lemma 2.3. *Let X be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of X.*

(a) $\beta(D_1) = 0$ *if and only if D_1 is relatively compact.*
(b) $\beta(\lambda D_1) = |\lambda|\beta(D_1)$; $\lambda \in \mathbb{R}$.
(c) *If $D_1 \subseteq D_2$ then $\beta(D_1) \leq \beta(D_2)$.*
(d) $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$.
(e) *If $x_0 \in X$ and r is a positive real number then $\beta(B(x_0, r)) = 2r$.*

Now let us introduce the following hypotheses which we shall use throughout this paper.

Hypothesis (H). $V : H \to \mathbb{R}$ is a locally Lipschitz function, and regular over a locally compact subset K in H, and $F : K \to 2^H$ is an upper semicontinuous set-valued map with compact values satisfying:

$$F(x) \cap T_K(x) \cap \partial_c V(x) \neq \emptyset \quad \text{for all } x \in K.$$

We are now ready to state the main result of this paper.

Theorem 2.4. *If assumptions* (H) *are satisfied, then there exist $T > 0$ and an absolutely continuous function $x(\,\cdot\,) : [0, T] \to H$ such that $x(\,\cdot\,)$ is a solution of* (1.1).

Remark 2.5. It is interesting to note that there is no relation between Theorem 2.4 in this paper and Theorem 2 in [16] (which is an extension of Theorem 3.1 in [10]). Furthermore, in [16], the convexity of K is widely used in the proof, then in spite of the weaker hypothesis on V, the result in [16] is not an extension of Theorem 2.4. On the other hand, the principal hypothesis in [16], if F is not depending of times, becomes $F(x) \cap T_K(x) \cap \partial_c V(x) \neq \emptyset$ with K is convex and V is lower regular but in Theorem 2.4 K is nonconvex and V is regular.

3. Preliminary results

First, let us introduce the following notations which we shall use throughout this paper. Let $x_0 \in K$ and choose $r > 0$ such that $K_0 = K \cap (x_0 + (r/2)\overline{B})$

is compact and V is Lipschitz continuous on $x_0 + r\overline{B}$ with Lipschitz constant $\lambda > 0$. Then $\partial_c V(x) \subset \lambda\overline{B}$ for every $x \in K_0$. Consider $T > 0$ such that

$$\int_0^T (\lambda + 1)\, d\tau \leq \frac{r}{2}. \tag{3.1}$$

For $\varepsilon > 0$, set

$$\mu(\varepsilon) := \sup\left\{\rho \in \,]0,\varepsilon] : \left| \int_{t_1}^{t_2} (\lambda + 1)^2\, d\tau \right| < \varepsilon, \text{ if } |t_1 - t_2| \leq \rho\right\}. \tag{3.2}$$

In the sequel, we will use the following important lemma. It will play a crucial role in the proof of the main result.

Lemma 3.1. *If assumptions* (H) *are satisfied, then for all* $0 < \varepsilon < \inf(T, 1)$, *there exists* $\eta > 0$ $(\eta < \varepsilon)$ *such that for all* $x \in K_0$, *there exists* $h_x \in [\eta, \mu(\varepsilon)]$, $y_x \in K_0$, $u \in (F(x) + \varepsilon B/T) \cap (\partial_c V(y_x) + \varepsilon B/T)$ *and* $b_x \in B$ *such that:*

(a) $\|x - y_x\| \leq \varepsilon h_x$,
(b) $(x + h_x u) \in K$,
(c) $V(x + h_x u) - V(x) \geq \langle h_x u, u - \varepsilon b_x/T\rangle - \alpha\varepsilon h_x$ *where* $\alpha = 4\lambda + 1$.

Proof. Let $x \in K_0$ be fixed and let $0 < \varepsilon < \inf(T, 1)$. Since F is u.s.c. on x, there exist $\delta_x > 0$ such that $F(y) \subset F(x) + (\varepsilon^2/2T)B$, for all $y \in B(x, \delta_x)$. Now, let $y \in K_0$ and select $v \in F(y) \cap T_K(y) \cap \partial_c V(y)$. There exists $0 < \rho < 1$ such that, for all $0 < h < \rho$,

$$V(y + hv) - V(y) \geq hV'(y, v) - \varepsilon h.$$

By the regularity of V, we rewrite this last inequality as

$$V(y + hv) - V(y) \geq h\langle v, w\rangle - \varepsilon h \quad \text{for all } w \in \partial_c V(y). \tag{3.3}$$

Moreover, since $v \in F(y) \cap T_K(y)$, there exists $h_y \in \,]0, \inf\{\rho, \mu(\varepsilon)\}]$ satisfying

$$d_K(y + h_y v) < h_y \frac{\varepsilon^2}{4T}.$$

Next, consider the subset

$$N(y) = \left\{z \in B(x_0, r) : d_K(z + h_y v) < h_y \frac{\varepsilon^2}{4T}\right\}.$$

The function $z \mapsto d_K(z + h_y v)$ is continuous and consequently $N(y)$ is open. Moreover, since y belongs to $N(y)$, there exists a ball $B(y, \eta_y)$ of radius $\eta_y < \inf\{\varepsilon h_y, \delta_x\}$ contained in $N(y)$, therefore, the compact subset K_0 can be covered by q such balls $B(y_i, \eta_{y_i})$. For simplicity, we set $h_i := h_{y_i}$ and $\eta_i := \eta_{y_i}$ for all $i = 1, \ldots, q$. Put $\eta = \min\{h_i : 1 \leq i \leq q\}$ and let $i \in \{1, \ldots, q\}$ such that $x \in B(y_i, \eta_i)$, hence $x \in N(y_i)$. Then

$$d_K(x + h_i v_i) < h_i \frac{\varepsilon^2}{4T},$$

where $v_i \in F(y_i) \cap \partial_c V(y_i)$. Thus there exists $x_i \in K$ such that

$$\frac{1}{h_i}\|x_i - (x + h_i v_i)\| \le \frac{1}{h_i}\, d_K(x + h_i v_i) + \frac{\varepsilon^2}{4T}.$$

Obviously, we have

$$\left\|\frac{x_i - x}{h_i} - v_i\right\| < \frac{\varepsilon^2}{2T}$$

and if we set $u = (x_i - x)/h_i$ we get

$$x_i = (x + h_i u) \in K, \quad u \in F(y_i) + \frac{\varepsilon}{2T}\, B \quad \text{and} \quad u \in \partial_c V(y_i) + \frac{\varepsilon}{T}\, B.$$

By construction one has $\|x - y_i\| < \eta_i < \delta_x$, then $F(y_i) \subset F(x) + (\varepsilon/2T)B$, which implies that $u \in F(x) + (\varepsilon/T)B$. So the first part of Lemma 3.1 is proved.

Now, choose $b_i \in B$ such that $(u - (\varepsilon/T)b_i) \in \partial_c V(y_i)$. Taking into account inequation (3.3), we have

$$V(y_i + h_i v_i) - V(y_i) \ge h_i\left\langle v_i, u - \frac{\varepsilon}{T}\, b_i\right\rangle - \varepsilon h_i. \tag{3.4}$$

To complete the proof of Lemma 3.1, we need the following claim:

Claim 3.2. *We have*:

(C1) $V(x + h_i u) - V(y_i + h_i v_i) \ge -2\lambda\varepsilon h_i$;
(C2) $V(x) - V(y_i) \ge -\lambda\varepsilon h_i$;
(C3) $\langle v_i, u - (\varepsilon/T)b_i\rangle \ge -\lambda\varepsilon + \langle u, u - (\varepsilon/T)b_i\rangle$.

Proof. From the inequalities

$$\|x + h_i u - x_0\| \le \frac{r}{2} + \int_0^T (\lambda + 1)\, d\tau \le r$$

and

$$\|y_i + h_i v_i - x_0\| \le \frac{r}{2} + T\lambda \le r,$$

we get $(x + h_i u) \in \overline{B}(x_0, r)$ and $(y_i + h_i v_i) \in \overline{B}(x_0, r)$. Since V is λ-Lipschitz over $\overline{B}(x_0, r)$, we conclude that

$$\begin{aligned}|V(x + h_i u) - V(y_i + h_i v_i)| &\le \lambda(\|x - y_i\| + h_i\|u - v_i\|)\\ &\le \lambda\left(\eta_i + h_i\frac{\varepsilon^2}{2T}\right) \le \lambda(\varepsilon h_i + \varepsilon h_i) \le 2\lambda\varepsilon h_i.\end{aligned}$$

So (C1) is checked.

(C2) follows from $|V(x) - V(y_i)| \le \lambda\|x - y_i\| \le \lambda\eta_i \le \lambda\varepsilon h_i$.

In order to prove (C3) we observe that

$$\left\langle v_i, u - \frac{\varepsilon}{T}\, b_i\right\rangle = \left\langle v_i - u, u - \frac{\varepsilon}{T}\, b_i\right\rangle + \left\langle u, u - \frac{\varepsilon}{T}\, b_i\right\rangle.$$

Since

$$\left|\left\langle v_i - u, u - \frac{\varepsilon}{T}\, b_i\right\rangle\right| \le \|v_i - u\|\left\|u - \frac{\varepsilon}{T}\, b_i\right\| \le \frac{\varepsilon^2}{2T}\,\lambda \le \lambda\varepsilon$$

we have

$$\left\langle v_i - u, u - \frac{\varepsilon}{T}\, b_i\right\rangle \ge -\lambda\varepsilon.$$

Consequently, we have

$$\left\langle v_i, u - \frac{\varepsilon}{T}\, b_i\right\rangle \ge -\lambda\varepsilon + \left\langle u, u - \frac{\varepsilon}{T}\, b_i\right\rangle.$$

Thus (C3) is verified. □

Next, using Claim 3.2 and relation (3.4) we obtain

$$\begin{aligned} V(x + h_i u)\, &- V(x) \\ &= V(x + h_i u) - V(y_i + h_i v_i) + V(y_i + h_i v_i) - V(y_i) + V(y_i) - V(x) \\ &\ge -2\lambda\varepsilon h_i + h_i\left\langle v_i, u - \frac{\varepsilon}{T}\, b_i\right\rangle - \varepsilon h_i - \lambda\varepsilon h_i \\ &\ge -3\lambda\varepsilon h_i - \lambda\varepsilon h_i + h_i\left\langle u, u - \frac{\varepsilon}{T}\, b_i\right\rangle - \varepsilon h_i \\ &\ge h_i\left\langle u, u - \frac{\varepsilon}{T}\, b_i\right\rangle - \varepsilon h_i(4\lambda + 1) \ge \left\langle h_i u, u - \frac{\varepsilon}{T}\, b_i\right\rangle - \alpha\varepsilon h_i. \end{aligned}$$

The proof of lemma is complete. □

In order to construct a sequence of approximate solutions, we need the following proposition.

Proposition 3.3. *If assumptions* (H) *are satisfied, then for all* $0 < \varepsilon < \inf(T, 1)$, *there exist* $\eta > 0$, $(\eta < \varepsilon)$, $s(\varepsilon) \in \mathbb{N}^*$, $(h_p)_p \subset [\eta, \mu(\varepsilon)]$, $(x_p)_p \subset H$, $(y_p)_p \subset K_0$ *and* $((u_p)_p, (b_p)_p) \subset H \times B$ *such that, for all* $p = 0, \dots, s$,

(a) $x_{p+1} = x_p + h_p u_p$;

(b) $x_p \in K_0$ *and* $\|x_p - y_p\| \le \varepsilon$;

(c) $u_p \in \left(F(x_p) + \dfrac{\varepsilon}{T} B\right) \cap \left(\partial_c V(y_p) + \dfrac{\varepsilon}{T} B\right)$;

(d) $V(x_{p+1}) - V(x_p) \ge \left\langle h_p, u_p - \dfrac{\varepsilon}{T}\, b_p\right\rangle - \varepsilon\alpha h_p$;

(e) $\displaystyle\sum_{i=0}^{s-1} h_i < T \le \sum_{i=0}^{s} h_i$.

Proof. Let $0 < \varepsilon < \inf(T, 1)$. In view of Lemma 3.1, there exist $\eta > 0$, $h_0 \in [\eta, \mu(\varepsilon)]$, $y_0 \in K_0$, $u_0 \in (F(x_0) + (\varepsilon/T)B) \cap (\partial_c V(y_0) + (\varepsilon/T)B)$ and $b_0 \in B$ such that $\|x_0 - y_0\| \le \varepsilon$, $x_1 = (x_0 + h_0 u_0) \in K$, and

$$V(x_0 + h_0 u_0) - V(x_0) \ge \left\langle h_0 u_0, u_0 - \frac{\varepsilon}{T}\, b_0\right\rangle - \varepsilon\alpha h_0.$$

Then taking account of (H) and (3.1), we have

$$\|x_1 - x_0\| = \|h_0 u_0\| \le \int_0^{h_0} (\lambda + 1)\, d\tau \le \frac{r}{2}$$

from which we deduce that $x_1 \in K_0$. Hence the assertions (a)–(d) are fulfilled for $p = 0$. Let now $p \ge 1$. Assume that (a)–(d) are satisfied for any $p = 1, \dots, q$. If $\sum_{i=0}^{q-1} h_i < T \le \sum_{i=0}^{q} h_i$, then we stop this process of iterations and we get (a)–(d) satisfied with $s = q$. In the other case: $\sum_{i=0}^{q} h_i < T$, we can apply on $\left(\sum_{i=0}^{q-1} h_i, x_q\right)$ the same technics applied on $(0, x_0)$, at the beginning of this proof, and we get (a)–(d) satisfied for $p = q+1$. It remains to prove that $x_{q+1} \in K_0$. By induction, we have

$$x_{q+1} = x_0 + \sum_{i=0}^{q} h_i u_i.$$

Thus by (H), (3.1) and because $\sum_{i=0}^{q} h_i < T$, we get

$$\|x_{q+1} - x_0\| = \sum_{i=0}^{q} h_i \|u_i\| \le \int_0^{T} (\lambda + 1)\, ds \le \frac{r}{2},$$

hence $x_{q+1} \in K_0$. Thus the conditions (a)–(d) are satisfied for $q + 1$. On the other hand, since $h_i \ge \eta > 0$, there exists an integer s such that

$$\sum_{i=0}^{s-1} h_i < T \le \sum_{i=0}^{s} h_i.$$

Therefore, there is an integer $s \ge 1$ for which the assertions (a)–(e) are fulfilled.□

4. Proof of the main result 2.4

In view of Proposition 3.3, for any integer $k > \sup\{1/T, 1\}$, we can define inductively sequences $(h_q^k)_q \subset [\eta_k, \mu(1/k)]$, $(x_q^k)_q \subset K_0$, $(y_q^k)_q \subset K_0$ and $((u_q^k)_q, (b_q^k)_q) \subset H \times B$ such that for all $q = 0, \dots, s_k$,

(a) $x_{q+1}^k = x_q^k + h_q^k u_q^k$;

(b) $\|x_q^k - y_q^k\| \le 1/k$;

(c) $u_q^k \in \left(F(x_q^k) + \dfrac{1}{kT} B\right) \cap \left(\partial_c V(y_q^k) + \dfrac{1}{kT} B\right)$;

(d) $V(x_{q+1}^k) - V(x_q^k) \ge \left\langle h_q^k u_q^k, u_q^k - \dfrac{1}{kT}\, b_q^k \right\rangle - \dfrac{\alpha h_q^k}{k}$;

(e) $\sum_{i=0}^{s_k - 1} h_i^k < T \le \sum_{i=0}^{s_k} h_i^k$.

Consider the sequence $(\tau_k^q)_k$ defined as the following:

$$\begin{cases} \tau_k^0 = 0, \quad \tau_k^{s_k+1} = T; \\ \tau_k^q = h_0^k + \ldots + h_{q-1}^k \quad \text{if } 1 \le q \le s_k, \end{cases}$$

and define on $[0,T]$ the sequence of functions $(x_k(\,\cdot\,))_k$ by

$$x_k(t) = x_{q-1}^k + (t - \tau_k^{q-1})\, u_{q-1}^k, \quad \text{for all } t \in [\tau_k^{q-1}, \tau_k^q].$$

So, it is easily seen that $\dot{x}_k(t) = u_{q-1}^k$ for almost every $t \in [\tau_k^{q-1}, \tau_k^q]$. Taking into account (H), for almost every $t \in [0,T]$, we get

$$\|\dot{x}_k(t)\| \le \lambda + 1. \tag{4.1}$$

Hence the sequence $(x_k(\,\cdot\,))_k$ is equicontinuous. In order to apply Ascoli–Arzela theorem, we are going to show that for every $t \in [0,T]$, the set $S(t) = \{x_k(t) : k \ge k_0\}$, where $k_0 > \sup\{1/T, 1\}$, is relatively compact in H. So, for every $k \ge k_0$ let $\theta_k\colon [0,T] \to [0,T]$ defined by

$$\theta_k(0) = 0, \quad \theta_k(t) = \tau_k^{q-1}, \quad \text{for all } t \in [\tau_k^{q-1}, \tau_k^q[.$$

By construction, for all $t \in [0,T]$, $x_k(\theta_k(t)) \in K_0$. Thus for all $t \in [0,T]$, the set $\{x_k(\theta_k(t)) : k \ge k_0\}$ is relatively compact in H, hence by Lemma 2.3, $\beta(\{x_k(\theta_k(t)) : k \ge k_0\}) = 0$. Next, for all $t \in [0,T]$,

$$\beta(S(t)) = \beta(\{x_k(t) : k \ge k_0\}) = \beta(\{x_k(t) - x_k(\theta_k(t)) + x_k(\theta_k(t)) : k \ge k_0\}).$$

Then by Lemma 2.3 and relation (4.1), we obtain:

$$\begin{aligned} \beta(S(t)) &\le \beta(\{x_k(t) - x_k(\theta_k(t)) : k \ge k_0\}) + \beta(\{x_k(\theta_k(t)) : k \ge k_0\}) \\ &\le \beta(\{x_k(t) - x_k(\theta_k(t)) : k \ge k_0\}) = \beta\bigg(\bigg\{ \int_{\theta_k(t)}^t \dot{x}_k(s)\, ds : k \ge k_0 \bigg\}\bigg) \\ &\le \beta\bigg(B\bigg(0, \int_{\theta_k(t)}^t (\lambda+1)\, ds\bigg)\bigg) = 2\int_{\theta_k(t)}^t (\lambda+1)\, ds. \end{aligned}$$

Since $\int_{\theta_k(t)}^t (\lambda+1)\, ds$ converges to 0 as $k \to \infty$, we get $\beta(S(t)) = 0$. Hence $S(t)$ is relatively compact in H. Therefore, by Arzelà–Ascoli's theorem (see [3]), we can select a subsequence, again denoted by $(x_k(\,\cdot\,))_k$ which converges uniformly to an absolutely continuous function $x(\,\cdot\,)$ on $[0,T]$, moreover $\dot{x}_k(\,\cdot\,)$ converges weakly to $\dot{x}(\,\cdot\,)$ in $L^2([0,T], H)$. Now, let $t \in [0,T]$, there exists $q \in \{1, \ldots, s_k+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and $\lim_{k \to +\infty} \tau_k^{q-1} = t$. By the fact that $x_k(\tau_k^{q-1})$ converges to $x(t)$ as $k \to \infty$, $x_k(\tau_k^{q-1}) \in K_0$ and K_0 is closed, we conclude that $x(t) \in K_0 \subset K$.

The function $x(\,\cdot\,)$ has the following property:

Proposition 4.1. *For almost every $t \in [0,T]$, we have $\dot{x}(t) \in \partial_c V(x(t))$.*

Proof. The weak convergence of $\dot{x}_k(\,\cdot\,)$ to $\dot{x}(\,\cdot\,)$ in $L^2([0,T],H)$ and the Mazur's Lemma entail $\dot{x}(t) \in \bigcap_k \overline{\text{co}}\{\dot{x}_m(t) : m \geq k\}$, for almost every $t \in [0,T]$. Fix any $t \in [0,T]$ such that $t \neq \tau_k^q$ for all integer $k > \sup\{1/T, 1\}$ and all $q \in \{0,\dots,s_k+1\}$. Now, for all integer $k > \sup\{1/T,1\}$, there exists $q \in \{1,\dots,s_k+1\}$ such that $t \in]\tau_k^{q-1}, \tau_k^q[$. Since $\lim_{k\to+\infty} \tau_k^q - \tau_k^{q-1} = 0$, we have $\lim_{k\to+\infty} \tau_k^{q-1} = t$. Then, for all $y \in H$, $\langle y, \dot{x}(t)\rangle \leq \inf_m \sup_{k\geq m} \langle y, \dot{x}_k(t)\rangle$ which together with $\dot{x}_k(t) \in \partial_c V(y_{q-1}^k) + (1/kT)B$ gives, for all m,

$$\langle y, \dot{x}(t)\rangle \leq \sup_{k\geq m} \sigma\bigg(y, \partial_c V(y_{q-1}^k) + \frac{1}{kT}B\bigg),$$

from which we deduce that

$$\langle y, \dot{x}(t)\rangle \leq \limsup_{k\to+\infty} \sigma\bigg(y, \partial_c V(y_{q-1}^k) + \frac{1}{kT}\,B\bigg).$$

On the other hand, by construction, one has

$$\|x(t) - y_{q-1}^k\| \leq \|x(t) - x_{q-1}^k\| + \|x_{q-1}^k - y_{q-1}^k\| \leq \|x(t) - x_k(\tau_k^{q-1})\| + \frac{1}{k}.$$

Since $x_k(\,\cdot\,)$ converges to $x(\,\cdot\,)$, the second member of the above inequality converges to 0, hence y_{q-1}^k converges to $x(t)$.

Next, by Proposition 6.4.9 in [4], the function $x \mapsto \sigma(y, \partial_c V(x))$ is u.s.c. and hence we get $\langle y, \dot{x}(t)\rangle \leq \sigma(y, \partial_c V(x(t)))$. So, the convexity and the closedness of the set $\partial_c V(x(t))$ ensure $\dot{x}(t) \in \partial_c V(x(t))$. □

Now, we use the regularity of the function V to prove the following proposition:

Proposition 4.2. *The set $\{\langle p, \dot{x}(t)\rangle, p \in \partial_c V(x(t))\}$ is reduced to the singleton $\left\{\frac{d}{dt}V(x(t))\right\}$ for almost every $t \in [0,T]$.*

Proof. Since $x(\,\cdot\,)$ is absolutely continuous function and V is locally Lipschitz continuous. The function $\text{Vox}(\,\cdot\,)$ is absolutely continuous and then for almost all t there exists $\frac{d}{dt}V(x(t))$. Let $t \in [0,T]$ be such that there exists both $\dot{x}(t)$ and $\frac{d}{dt}V(x(t))$. There is $\delta > 0$ such that, for every $|h| < \delta$,

$$x(t+h) \in B(x_0, r), \quad (x(t) + h\dot{x}(t)) \in B(x_0, r)$$

and

$$x(t+h) - x(t) - h\dot{x}(t) = r(h) \quad \text{where } \lim_{h\to 0} \|r(h)\|/h = 0.$$

Since V is Lipschitz continuous on $B(x_0, r)$ with Lipschitz constant $\lambda > 0$, we have

$$|V(x(t+h)) - V(x(t) + h\dot{x}(t))| \leq \lambda \|r(h)\|,$$

whenever $|h| < \delta$. Consequently, the function $h \to V(x(t) + h\dot{x}(t))$ is differentiable at $h = 0$, and its derivative is the same as the derivative of $h \to V(x(t+h))$ at $h = 0$. Hence

$$\frac{d}{dt}V(x(t)) = \lim_{h\to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}. \tag{4.2}$$

Since V is regular over K and $x(t) \in K$, we obtain

$$V^o(x(t), \dot{x}(t)) = \lim_{h\to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}. \tag{4.3}$$

In addition, one has

$$\begin{aligned} V^o(x(t), -\dot{x}(t)) &= \lim_{h\to 0} \frac{V(x(t) + h(-\dot{x}(t))) - V(x(t))}{h} \\ &= -\lim_{h\to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}. \end{aligned}$$

By Proposition 2.1, $V^o(x(t), -\dot{x}(t)) = -V_o(x(t), \dot{x}(t))$, then

$$V_o(x(t), \dot{x}(t)) = \lim_{h\to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}. \tag{4.4}$$

By (4.2)–(4.4), we deduce that

$$V^o(x(t), \dot{x}(t)) = \frac{d}{dt}V(x(t)) = V_o(x(t), \dot{x}(t)).$$

This means, by Proposition 2.1, that for almost all t the set $\{\langle p, \dot{x}(t)\rangle,\ p \in \partial_c V(x(t))\}$ reduces to the singleton $\left\{\frac{d}{dt}V(x(t))\right\}$. □

Proposition 4.3. *The application $x(\cdot)$ is a solution of the problem* (1.1).

Proof. First, by using Propositions 4.1 and 4.2, we obtain

$$\frac{d}{dt}V(x(t)) = \langle \dot{x}(t), \dot{x}(t)\rangle, \quad \text{a.e. on } [0,T].$$

Therefore, by integrating on $[0,T]$, we get

$$V(x(T)) - V(x_0) = \int_0^T \|\dot{x}(s)\|^2\, ds. \tag{4.5}$$

On the other hand, by construction, for all $q = 1, \dots, s_k$, we have

$$\begin{aligned}
V(x_k(\tau_k^q)) &- V(x_k(\tau_k^{q-1}))\\
&\geq \left\langle h_{q-1}^k u_{q-1}^k, u_{q-1}^k - \frac{1}{kT}\, b_{q-1}^k \right\rangle - \frac{\alpha h_{q-1}^k}{k}\\
&\geq \left\langle x_k(\tau_k^q) - x_k(\tau_k^{q-1}), \dot{x}_k(t) - \frac{1}{kT}\, b_{q-1}^k \right\rangle - \frac{\alpha h_{q-1}^k}{k}\\
&\geq \left\langle \int_{\tau_k^{q-1}}^{\tau_k^q} \dot{x}_k(s)\, ds, \dot{x}_k(t) \right\rangle - \left\langle \int_{\tau_k^{q-1}}^{\tau_k^q} \dot{x}_k(s)\, ds, \frac{1}{kT}\, b_{q-1}^k \right\rangle - \frac{\alpha h_{q-1}^k}{k}\\
&\geq \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), \dot{x}_k(s) \rangle\, ds - \int_{\tau_k^{q-1}}^{\tau_k^q} \left\langle \dot{x}_k(s), \frac{1}{kT}\, b_{q-1}^k \right\rangle ds - \frac{\alpha h_{q-1}^k}{k}\\
&\geq \int_{\tau_k^{q-1}}^{\tau_k^q} \|\dot{x}_k(s)\|^2\, ds - \int_{\tau_k^{q-1}}^{\tau_k^q} \left\langle \dot{x}_k(s), \frac{1}{kT} b_{q-1}^k \right\rangle ds - \frac{\alpha h_{q-1}^k}{k}.
\end{aligned}$$

By adding, one has

$$\begin{aligned}
V(x_k(\tau_k^{s_k})) - V(x_0)&\\
\geq \int_0^{\tau_k^{s_k}} \|\dot{x}_k(s)\|^2\, ds &- \sum_{q=1}^{s_k} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\langle \dot{x}_k(s), \frac{1}{kT}\, b_{q-1}^k \right\rangle ds - \frac{\alpha}{k} \sum_{q=1}^{s_k} h_{q-1}^k.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\tau_k^{s_k}}^{T} \|\dot{x}_k(s)\|^2\, ds + V(x_k(T)) - V(x_0) + V(x_k(\tau_k^{s_k})) - V(x_k(T))&\\
\geq \int_0^T \|\dot{x}_k(s)\|^2\, ds - \sum_{q=1}^{s_k} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\langle \dot{x}_k(s), \frac{1}{kT}\, b_{q-1}^k \right\rangle ds - \frac{\alpha T}{k}&.
\end{aligned}$$

By (4.1) and since $T - \tau_k^{s_k} \leq \mu(1/k)$, we get

$$\int_{\tau_k^{s_k}}^{T} \|\dot{x}_k(s)\|^2\, ds \leq \int_{\tau_k^{s_k}}^{T} (1+\lambda)^2\, ds \leq \frac{1}{k},$$

and, by the fact that V is Lipschitz on $\overline{B}(x_0, r)$, we obtain

$$\begin{aligned}
|V(x_k(\tau_k^{s_k})) - V(x_k(T))| &\leq \lambda \|x_k(\tau_k^{s_k}) - x_k(T)\| \leq \lambda \int_{\tau_k^{s_k}}^{T} \|\dot{x}_k(s)\|\, ds\\
&\leq \lambda \int_{\tau_k^{s_k}}^{T} (1+\lambda)\, ds \leq \lambda \int_{\tau_k^{s_k}}^{T} (1+\lambda)^2\, ds \leq \frac{\lambda}{k}.
\end{aligned}$$

So, the above relation becomes

$$\begin{aligned}
&V(x_k(T)) - V(x_0)\\
&\quad\geq \int_0^T \|\dot{x}_k(s)\|^2\, ds - \sum_{q=1}^{s_k} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\langle \dot{x}_k(s), \frac{1}{kT}\, b_{q-1}^k \right\rangle ds - \frac{\alpha T}{k} - \frac{1}{k} - \frac{\lambda}{k}.
\end{aligned} \tag{4.6}$$

On the other hand, we have

$$\left|\sum_{q=1}^{s_k}\int_{\tau_k^{q-1}}^{\tau_k^q}\left\langle \dot{x}_k(s), \frac{1}{kT}\, b_{q-1}^k\right\rangle ds\right| \le \frac{1}{kT}\int_0^T \|\dot{x}_k(s)\|\, ds \le \frac{1}{kT}\int_0^T (\lambda+1)\, ds.$$

The last term converges to 0, then

$$\lim_{k\to+\infty}\sum_{q=1}^{s_k}\int_{\tau_k^{q-1}}^{\tau_k^q}\left\langle \dot{x}_k(s), \frac{1}{kT}\, b_{q-1}^k\right\rangle ds = 0.$$

Now, by passing to the limit for $k\to\infty$ in (4.6) and using the continuity of the function V on the ball $\overline{B}(x_0, r)$, we obtain

$$V(x(T)) - V(x_0) \ge \limsup_{k\to+\infty}\int_0^T \|\dot{x}_k(s)\|^2\, ds.$$

Moreover, by (4.5), we have $\|\dot{x}\|_2^2 \ge \limsup_{k\to+\infty}\|\dot{x}_k\|_2^2$ and by the weak l.s.c. of the norm ensures $\|\dot{x}\|_2^2 \le \liminf_{k\to+\infty}\|\dot{x}_k\|_2^2$. Hence we get $\|\dot{x}\|_2^2 = \lim_{k\to+\infty}\|\dot{x}_k\|_2^2$. Finally, there exists a subsequence of $(\dot{x}_k(\,\cdot\,))_k$ (still denoted $(\dot{x}_k(\,\cdot\,))_k$) converges pointwisely to $\dot{x}(\,\cdot\,)$. Now, let $t\in[0,T]\setminus\{\tau_k^0;\dots;\tau_k^{s_k+1}\}$, there exists $q\in\{1,\dots,s_k+1\}$ such that $t\in\,]\tau_k^{q-1},\tau_k^q[$ and $\lim_{k\to+\infty}\tau_k^{q-1}=t$. Since $(\dot{x}_k(t))\in F(x_{q-1}^k)+(1/kT)B$, we have

$$d_{grF}(x_k(t),\dot{x}_k(t)) \le \|x_k(t) - x_k(\tau_k^{q-1})\| + \frac{1}{kT},$$

hence

$$\lim_{k\to+\infty} d_{grF}(x_k(t),\dot{x}_k(t)) = 0,$$

from which we conclude that $d_{grF}((x(t),\dot{x}(t)) = 0$ and so, as F has a closed graph, we obtain $\dot{x}(t)\in F(x(t))$ for almost every $t\in[0,T]$. The proof is complete. □

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