Topological **M**ethods in **N**onlinear **A**nalysis
Volume 41, No. 2, 2013, 323–334

GENERIC PROPERTIES OF CRITICAL POINTS OF THE BOUNDARY MEAN CURVATURE

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Abstract. Given a bounded domain $\Omega \subset \mathbb{R}^N$ of class C^k with $k \geq 3$, we prove that for a generic deformation $I + \psi$, with ψ small enough, all the critical points of the mean curvature of the boundary of the domain $(I + \psi)\Omega$ are non degenerate.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain of class C^k with $k \geq 3$ and $N \geq 2$. We consider the domain $\Omega_\psi := (I + \psi)\Omega$ given by the deformation $I + \psi$. We are interested in studying the non degeneracy of the critical points of the mean curvature of the boundary of the domain Ω_ψ with respect to the parameter ψ.

Let $\mathfrak{C}^k$ be the vector space of all the C^k applications $\psi\colon \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\|\psi\|_k := \sup_{x \in \mathbb{R}^N} \max_{0 \leq |\alpha| \leq k} \left| \frac{\partial^\alpha \psi_i(x)}{\partial {x_1}^{\alpha_1} \ldots \partial {x_N}^{\alpha_N}} \right| < +\infty. \tag{1.1}$$

$\mathfrak{C}^k$ is a Banach space equipped with the norm $\|\cdot\|_k$. Let $\mathfrak{B}_\rho := \{\psi \in \mathfrak{C}^k : \|\psi\|_k \leq \rho\}$ be the ball in $\mathfrak{C}^k$ centered at 0 with radius ρ.

More precisely, we will prove the following result.

2010 *Mathematics Subject Classification.* 58G03, 58E30.

Key words and phrases. Mean curvature, non degenerate critical points, generic property.

Theorem 1.1. *The set $\mathfrak{A} := \{\psi \in B_\rho$: all the critical points of the mean curvature of the boundary of the domain Ω_ψ are non degenerate$\}$ is a residual (hence dense) subset of $\mathfrak{B}_\rho$, provided ρ is small enough.*

The result of Theorem 1.1 can be applied to the study of the following problem:

$$-\varepsilon^2\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where ε is a small positive parameter and ν denotes the unit outward normal to $\partial\Omega$ and $1 < p < (N+2)/(N-2) <$ if $N \geq 3$ or $p > 1$ if $N = 2$.

This problem arises from different mathematical models: for example, it appears in the study of stationary solutions for the Keller–Segal system in chemotaxis and the Gierer–Meinhardt system in biological pattern formation. Problem (1.2) has been widely studied in many aspects: a large number of papers have been devoted in investigating the existence, multiplicity and asymptotic behaviour of positive solutions in the semiclassical limit $\varepsilon \to 0^+$.

The analysis reveals that the solutions seem to exhibit a "point condensation phenomena", i.e. they tend to zero as $\varepsilon \to 0^+$ except at a finite number of points. In the pioneering papers [11], [13], [14], Lin, Ni and Takagi first proved that for ε sufficiently small there is a least energy solution u_ε with the property that u_ε has exactly one maximum point ξ_ε in $\overline{\Omega}$, and ξ_ε must be located on $\partial\Omega$ and near the *most curved* part of the $\partial\Omega$, i.e. $H(\xi_\varepsilon) \to \max\limits_{\xi\in\partial\Omega} H(P)$, where $H(\xi)$ denotes the mean curvature of the boundary $\partial\Omega$. Since then, there have been many papers looking for higher energy solutions. More specifically, solutions with multiple boundary peaks as well as multiple interior peaks have been established, with each peak concentrating at a different point whose location depends on the geometry of the domain (see [1], [4]–[8], [10], [13], [14], [19], [20] and the references therein). In particular, it turns out that if $\xi_1, \ldots, \xi_k$ are k different C^1-stable critical points of the boundary mean curvature H then problem (1.2) has a solution whose k boundary peaks approach $\xi_1, \ldots, \xi_k$ as ε goes to zero.

In the papers [2] and [9] the authors first construct solutions exhibiting a *cluster*, i.e. given $k \geq 1$ and ξ_0 a strict local minimum of H, there exists a solution with k boundary peaks concentrating at ξ_0 as ε goes to zero. As far as it concerns the existence of sign-changing solutions, the first result was due to Noussair and Wei in [15], where it is proved that for ε sufficiently small (1.2) has a least energy nodal solution with one positive boundary peak and one negative boundary peak; moreover such peaks approach the global minimum points of the mean curvature. In the particular case when the set of global minima consists of a single point, then the peaks concentrates at the same point giving rise to a *cluster*. Successively, in [12] the authors glued the single bump solutions and obtained nodal solutions with multiple boundary peaks concentrating at different critical points of the mean curvature.

The first result providing a multiplicity result for sign-changing peak solutions is due to Wei and Weth [21]: they consider the problem (1.2) in a two-dimensional domain and prove that, given $k \geq 1$ and given ξ_0 a local strict minimum of the mean curvature, there exists a *clustered*solution with k positive boundary peaks and k negative boundary peaks concentrating at ξ_0.

Very recently, in [3] the authors it proved that given a non degenerate maximum ξ_0 of the mean curvature H of $\partial\Omega$ and given two positive integers h, k with $h + k \leq 6$, for ε sufficiently small (1.2) possesses a *cluster* with h positive boundary peaks and k negative boundary peaks approaching ξ_0.

All the previous results require a sort of non degeneracy of critical points of the mean curvature of $\partial\Omega$. Theorem 1.1 allows to claim that for a generic deformation $I + \psi$ of the domain Ω, all the critical points of the mean curvature of the boundary of the domain $(I + \psi)\Omega$ are non degenerate and all the previous results hold.

The paper is organized as follows. In Section 2 we set the problem and in Section 3 we prove the main result, using some technical lemmas proved in Section 4.

2. Setting of the problem

Let us fix a bounded domain Ω in $\mathbb{R}^N$ of class C^k with $N \geq 2$ and $k \geq 3$. Then there exists ρ positive and small enough such that if $\psi \in \mathfrak{B}_\rho$ then the map $I + \psi \colon \overline{\Omega} \to (I + \psi)\overline{\Omega}$ is a diffeomorphism. We set $\Omega_\psi := (I + \psi)\Omega$. For any point $\xi \in \partial\Omega$ we have a local system of coordinates. Without loss of generality, we can assume $\xi = 0$. We can choose a neighbourhood U of 0 in $\mathbb{R}^N$, a ball $B(0, R)$ in $\mathbb{R}^{N-1}$ centered at 0 with radius R and a map $h \colon B(0, R) \to U \cap \partial\Omega$ defined by

$$h(y) := (y, f(y)), \quad y := (y_1, \ldots, y_{N-1}), \tag{2.1}$$

where $f \colon B(0, R) \to \mathbb{R}$ is a C^k-map with $f(0) = 0$, $\nabla f(0) = 0$ and

$$f(y) = \frac{1}{2} \sum_{i=1}^{N-1} \lambda_i y_i^2 + O(|y|^3).$$

Here λ_i are the principal curvatures at ξ and the mean curvature of the boundary $\partial\Omega$ at ξ is

$$H(\xi) := \frac{1}{N-1} \sum_{i=1}^{N-1} \lambda_i.$$

If $\xi \in \partial\Omega$ and $\psi \in \mathfrak{B}_\rho$ we consider the mean curvature $H(\xi, \psi) := H_\psi(\xi)$ of the boundary $\partial\Omega_\psi$ of the domain $\Omega_\psi := (I + \psi)\Omega$ at the point $(I + \psi)(\xi)$. In particular, using the local system of coordinates given by (2.1), we set $\widetilde{H}(y, \psi) =$

$H(h(y),\psi)$. $\widetilde{H}$ is nothing but the expression in local coordinates of the mean curvature of the boundary $\partial\Omega_\psi$ at the point $(I+\psi)(y,f(y))\in\partial\Omega_\psi$. Now we introduce the C^1-map

$$F\colon B(0,R)\times\mathfrak{B}_\rho\subset\mathbb{R}^{N-1}\times\mathfrak{E}^k\to\mathbb{R}^{N-1},\quad F(y,\psi):=\nabla_y\widetilde{H}(y,\psi). \tag{2.2}$$

We shall apply the following abstract transversality theorem to the map F (see [16], [17], [18]).

Theorem 2.1. *Let X, Y, Z be three Banach spaces and $U\subset X$, $V\subset Y$ open subsets. Let $F\colon U\times V\to Z$ be a $C^\alpha-$map with $\alpha\geq 1$. Assume that*

(a) *for any $y\in V$, $F(\,\cdot\,,y)\colon U\to Z$ is a Fredholm map of index l with $l\leq\alpha$;*
(b) *0 is a regular value of F, i.e. the operator $F'(x_0,y_0)\colon X\times Y\to Z$ is onto at any point (x_0,y_0) such that $F(x_0,y_0)=0$;*
(c) *the map $\pi\circ i\colon F^{-1}(0)\to Y$ is proper, where $i\colon F^{-1}(0)\to Y$ is the canonical embedding and $\pi\colon X\times Y\to Y$ is the projection.*

Then the set

$$\Theta:=\{y\in V: 0 \textit{ is a regular value of } F(\,\cdot\,,y)\}$$

is a dense open subset of V. If F satisfies (a), (b) *and*

(d) *the map $\pi\circ i\colon F^{-1}(0)\to Y$ is $\sigma-$proper, i.e. $F^{-1}(0)=\bigcup_{s=1}^{+\infty}C_s$ where C_s is a closed set and the restriction $\pi\circ i_{|C_s}$ is proper for any s*

then the set Θ is a residual subset of V, i.e. $V\setminus\Theta$ is a countable union of close subsets without interior points.

3. Proof of the main result

We are going to apply the transversality Theorem 2.1 to the map F defined by (2.2). In this case we have $X=Z=\mathbb{R}^{N-1}$, $Y=\mathfrak{E}^k$, $U=B(0,R)\subset\mathbb{R}^{N-1}$ and $V=\mathfrak{B}_\rho\subset\mathfrak{E}^k$, where R and ρ are small enough. Since X is a finite dimensional space, it is easy to check that for any $\psi\in\mathfrak{B}_\rho$, the map $y\to F(y,\psi)$ is a Fredholm map of index 0 and then assumption (a) holds.

Assumption (b) is verified in Lemma 4.7. As far as it concerns assumption (d), we have that

$$F^{-1}(0)=\bigcup_{s=1}^{+\infty}C_s,\quad\text{where } C_s:=\{\overline{B(0,R-1/s)}\times\overline{\mathfrak{B}_{\rho-1/s}}\}\cap F^{-1}(0).$$

By the compactness of $\overline{B(0,R-1/s)}\subset\mathbb{R}^{N-1}$ it follows that the restriction $\pi\circ i_{|C_s}$ is proper, namely if the sequence $(\psi_n)\subset\overline{\mathfrak{B}_{\rho-1/s}}$ converges to ψ_0 and the sequence $(x_n)\subset\overline{B(0,R-1/s)}$ is such that $F(x_n,\psi_n)=0$ then there exists a subsequence of (x_n) which converges to $x_0\in\overline{B(0,R-1/s)}$ and $F(x_0,\psi_0)=0$.

Finally, we can apply the transversality Theorem 2.1 and we get that given a point $\xi_0 \in \partial\Omega$ the set

$$\begin{aligned}\Theta(\xi_0) := \{&\psi \in \mathfrak{B}_\rho : F'_y(y,\psi)\colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1} \text{ is invertible}\\ &\quad \text{at any point } (y,\psi) \in B(0,R) \times \mathfrak{B}_\rho \text{ such that } F(y,\psi) = 0\}\\ = \{&\psi \in \mathfrak{B}_\rho : \text{the critical point in a suitable neighbourhood}\\ &\quad \text{of the point } (I+\psi)(\xi_0) \in \partial\Omega_\psi \text{ of the mean curvature}\\ &\quad \text{of the boundary } \partial\Omega_\psi \text{ are nondegenerate}\}\end{aligned}$$

is a residual subset of $\mathfrak{B}_\rho$.

At this point it holds that for any $\xi_0 \in \partial\Omega$ there exist a positive number $\rho = \rho(\xi_0)$ and a neighbourhood $\mathcal{I}(\xi_0, R) \subset \partial\Omega$ of the point ξ_0 with $R = R(\xi_0)$ such that the set

$$\begin{aligned}\Theta(\xi_0) := \{&\psi \in \mathfrak{B}_\rho : \text{any critical point } \xi \in (I+\psi)(\mathcal{I}(\xi_0,R))\\ &\quad \text{of the mean curvature of the boundary } \partial\Omega_\psi \text{ is nondegenerate}\}\end{aligned} \tag{3.1}$$

is a residual subset of $\mathfrak{B}_\rho$.

Since $\partial\Omega$ is compact there exist a finite number ν of points $\xi_i \in \partial\Omega$ and positive numbers R_i, ρ_i such that $\bigcup_{i=1}^{\nu} \mathcal{I}(\xi_i, R_i/2) = \partial\Omega$ and the set $\Theta(\xi_i)$ defined in (3.1) are residual subsets of $\mathfrak{B}_\rho$. It is clear that the set $\mathfrak{A} := \bigcap_{i=1}^{\nu} \Theta(\xi_i)$ is a residual subset of $\mathfrak{B}_\rho$ and

$$\begin{aligned}\mathfrak{A} = \{\psi \in \mathfrak{B}_\rho : &\text{ any critical point of the mean curvature}\\ &\qquad\qquad \text{of the boundary } \partial\Omega_\psi \text{ is nondegenerate}\}.\end{aligned}$$

Here ρ is small enough. By the following lemma we also deduce that $\mathfrak{A}$ is open. That concludes the proof of Theorem 1.1.

In the following lemma we prove that the set $\mathfrak{A}$ is open.

Lemma 3.1. *If the domain Ω of class C^k with $k \geq 3$ is such that all the critical points of the mean curvature of the boundary $\partial\Omega$ are non degenerate, then all the critical points of the mean curvature of the boundary $(I+\psi)\Omega$ are non degenerate for $\|\psi\|_k$ small enough.*

Proof. By the assumption, the critical points of the mean curvature of $\partial\Omega$ are in a finite number. Let $\xi_1, \ldots, \xi_\nu$ be the critical points. Fix a critical point ξ_1 and use a local system of coordinates as in (2.1), so that we can consider the function $\widetilde{H}(y,\psi)$ and the C^1-map $F(y,\psi) = \nabla_y \widetilde{H}(y,\psi)$ with $(y,\psi) \in B(0,R) \times \mathfrak{B}_\rho \subset \mathbb{R}^{N-1} \times \mathfrak{E}^k$. Since $F(0,0) = 0$ and $F'_y(0,0)\colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ is an isomorphism, by the implicit function theorem, there exist $y(\psi)$ and ρ_1 such that $F(y(\psi),\psi) = 0$ if $\|\psi\|_k \leq \rho_1$ and $y(0) = 0$. Therefore, the mean curvature of $\partial\Omega_\psi$ has a unique non degenerate critical point $\eta_\psi \in \partial\Omega_\psi$ if $\|\psi\|_k \le \rho_1$ with $\eta_0 = \xi_1$. The same argument holds for all the other critical points of the mean curvature of $\partial\Omega$.

Now, let us prove that $\mathfrak{A}$ is open. By contradiction, we assume that there exist sequences $(\psi_n) \subset \mathfrak{B}_\rho$ convergent to 0 and $(\xi_n) \subset \partial\Omega$ convergent to $\xi_0 \in \partial\Omega$ such that the points $(I+\psi_n)(\xi_n)$ are degenerate critical points of the mean curvature of the boundary $\partial\Omega_{\psi_n}$. Using the local system of coordinates (2.1) at the point $\xi_0 \in \partial\Omega$ we have $F(y_n, \psi_n) = \nabla_y \widetilde{H}(y_n, \psi_n) = 0$. Then we get $\nabla_y \widetilde{H}(0,0) = 0$, namely ξ_0 is a critical point of the mean curvature of $\partial\Omega$. Therefore $\xi_0 \in \{\xi_1, \ldots, \xi_\nu\}$. Using the above argument, if n is large enough $(I+\psi_n)(\xi_n)$ is a non degenerate critical point of the mean curvature of the boundary $\partial\Omega_{\psi_n}$ and a contradiction arises. $\square$

4. Some technical results

Given a point $\xi_0 \in \partial\Omega$ and chosen the local system of coordinates defined in (2.1), we are going to calculate $\widetilde{H}(y,\psi)$ when $\psi \in \mathfrak{B}_\rho \subset \mathfrak{C}^k$.

For the point $(I+\psi)(\xi_0) \in \partial\Omega_\psi$ we have a local system of coordinates defined by

$$h^\psi(y) := (y_1 + \widetilde{\psi}_1(y), \ldots, y_{N-1} + \widetilde{\psi}_{N-1}(y), f(y) + \widetilde{\psi}_N(y)) \in \partial\Omega_\psi \tag{4.1}$$

where $y \in B(0,R) \subset \mathbb{R}^{N-1}$ and $\widetilde{\psi}\colon B(0,R) \to \mathbb{R}^N$ is defined by

$$\widetilde{\psi}(y) := \psi(y, f(y)). \tag{4.2}$$

Using this coordinate system, the components of the Riemannian metric $g^\psi_{ij}(y)$ on the manifold $\partial\Omega_\psi$ can be expressed as follows

$$\begin{aligned} g^\psi_{ij}(y) = {} & \partial_{y_i} f \partial_{y_j} f + \partial_{y_i} \widetilde{\psi}_j + \partial_{y_j} \widetilde{\psi}_i \\ & + \partial_{y_i} f \partial_{y_j} \widetilde{\psi}_N + \partial_{y_j} f \partial_{y_i} \widetilde{\psi}_N + \sum_{k=1}^N \partial_{y_i} \widetilde{\psi}_k \partial_{y_j} \widetilde{\psi}_k, \end{aligned} \tag{4.3}$$

$$g^\psi_{ii}(y) = 1 + (\partial_{y_i} f)^2 + 2\partial_{y_i} \widetilde{\psi}_i + 2\partial_{y_i} f \partial_{y_i} \widetilde{\psi}_N + \sum_{k=1}^N (\partial_{y_i} \widetilde{\psi}_k)^2, \tag{4.4}$$

$$|g^\psi(y)| = \det g^\psi(y) = \sum_{\substack{\sigma \text{permutation} \\ \text{of } \{1,\ldots,N-1\}}} (-1)^\sigma g^\psi_{1,\sigma_1}(y) \ldots g_{N-1,\sigma_{N-1}}(y). \tag{4.5}$$

The tangent space of the manifold $\partial\Omega_\psi$ at the point $y + \widetilde{\psi}(y)$ is the vector space generated by the vectors

$$\tau^\psi_i(y) := \tau_i(y) + \partial_{y_i} \widetilde{\psi}(y), \quad \tau_i := (0, \ldots, \underbrace{1}_{i\text{-th}}, \ldots, 0, \partial_{y_i} f), \tag{4.6}$$

for $i = 1, \ldots, N-1$.

The normal vector $\nu^\psi(y) \in \mathbb{R}^N$ to the boundary $\partial\Omega_\psi$ is given by

$$\nu^\psi(y) = \big(\partial_{y_1} f, \dots, \partial_{y_{N-1}} f, -1\big) \frac{1}{\sqrt{1+\|\nabla f\|^2}} + \sigma(y,\psi(y)), \tag{4.7}$$

where $\sigma(y,\psi(y)) \to 0$ as $\psi \to 0$ uniformly with respect to y. We set for any $i,j = 1,\dots,N-1$

$$\gamma^\psi_{ij}(y) = (\partial^2_{y_i y_j}\widetilde{\psi}_1(y), \dots, \partial^2_{y_i y_j}\widetilde{\psi}_{N-1}(y), \partial^2_{y_i y_j} f(y) + \partial^2_{y_i y_j}\widetilde{\psi}_N(y)), \tag{4.8}$$

$$\Gamma^\psi_{ij}(y) = (\gamma^\psi_{ij}(y), \nu^\psi(y))_{\mathbb{R}^N}. \tag{4.9}$$

Finally, we have

$$\widetilde{H}(y,\psi) = \frac{1}{N-1}\sum_{i=1}^{N-1} \lambda_i(y,\psi),$$

where $\lambda_1,\dots,\lambda_{N-1}$ are the solutions of the equation

$$\det(\Gamma^\psi_{ij}(y) - \lambda g^\psi_{ij}(y)) = 0, \quad i,j = 1,\dots,N-1. \tag{4.10}$$

We remark that the $\nu^0(0) = (0,\dots,0,-1)$, $\gamma^0_{ij}(0) = (0,\dots,0,\partial^2_{y_i y_j} f(0))$, $\Gamma^0_{ij}(0) = -\partial^2_{y_i y_j} f(0)$ and $\partial^2_{y_i y_i} f(0) = \lambda_i$, $\partial^2_{y_i y_j} f(0) = 0$ if $i \neq j$.

Let $B^s_{ij}(y,\psi)$ be the matrix obtained by replacing the s-row of the matrix $g^\psi_{ij}(y)$ with $(\Gamma^\psi_{s\ 1}(y), \dots, \Gamma^\psi_{s\ N-1}(y))$. Then the determinant of the matrix $B^s_{ij}(y,\psi)$ is

$$|B^s(y,\psi)| = \sum_\sigma (-1)^\sigma g^\psi_{1,\sigma_1}(y)\dots g_{s-1,\sigma_{s-1}}(y)\Gamma^\psi_{s,\sigma_s}(y) g_{s+1,\sigma_{s+1}}(y)\dots g_{N-1,\sigma_{N-1}}(y).$$

Then

$$\sum_{i=1}^{N-1}\lambda_i(y,\psi) = \frac{1}{|g^\psi(y)|}\sum_{s=1}^{N-1}|B^s(y,\psi)| = (N-1)\widetilde{H}(y,\psi). \tag{4.11}$$

Moreover, we have for any $k = 1,\dots,N-1$

$$\begin{aligned}
(N-1)&D_\psi \widetilde{H}'_{y_k}(y_0,\psi_0)[\varphi] \\
&= |g^{\psi_0}(y_0)|^{-4}\Bigg\{\Bigg[D_\psi|g^{\psi_0}(y_0)|[\varphi]\sum_{s=1}^{N-1}\partial_{y_k}|B^s(y_0,\psi_0)| \\
&\quad + |g^{\psi_0}(y_0)|\sum_{s=1}^{N-1} D_\psi\partial_{y_k}|B^s(y_0,\psi_0)|[\varphi] \\
&\quad - D_\psi\partial_{y_k}|g^{\psi_0}(y_0)|[\varphi]\sum_{s=1}^{N-1}|B^s(y_0,\psi_0)| \\
&\quad - \partial_{y_k}|g^{\psi_0}(y_0)|\sum_{s=1}^{N-1} D_\psi|B^s(y_0,\psi_0)|[\varphi]\Bigg]|g^{\psi_0}(y_0)|^2
\end{aligned} \tag{4.12}$$

$$- 2|g^{\psi_0}(y_0)|\bigg[|g^{\psi_0}(y_0)|\sum_{s=1}^{N-1}\partial_{y_k}|B^s(y_0,\psi_0)|$$
$$- \partial_{y_k}|g^{\psi_0}(y_0)|\sum_{s=1}^{N-1}|B^s(y_0,\psi_0)|\bigg]D_\psi|g^{\psi_0}(y_0)|[\varphi]\bigg\}.$$

We just remark that if (y_0,ψ_0) is such that $\nabla_y H(y_0,\psi_0)=0$ then the last term in (4.12) is zero.

Given a point $y_0 \in B(0,R)$ we define a subset $\mathfrak{A}_{y_0}$ of the Banach space $\mathfrak{C}^k$ given by the functions φ whose first and second derivatives at the point $(y_0,f(y_0))$ are zero, namely

$$\mathfrak{A}_{y_0} := \bigg\{\varphi\in\mathfrak{C}^k : \frac{\partial^\alpha\varphi_i}{\partial_{x_1}^{\alpha_1}\ldots\partial {x_N}^{\alpha_N}}(y_0,f(y_0))=0, \quad i=1,\ldots,N,\ 1\le|\alpha|\le 2\bigg\}. \tag{4.13}$$

We compute the derivatives of $|g^\psi(y)|$ with respect to ψ and y.

Lemma 4.1. *For any $i,j,k=1,\ldots,N-1$ we have*

$$\begin{aligned}
D_\psi g_{ij}^\psi(y_0)[\varphi] &= \partial_{y_i}\widetilde{\varphi}_j(y_0)+\partial_{y_j}\widetilde{\varphi}_i(y_0)\\
&\quad+\partial_{y_i}f(y_0)\partial_{y_j}\widetilde{\varphi}_N(y_0)+\partial_{y_j}f(y_0)\partial_{y_i}\widetilde{\varphi}_N(y_0)\\
&\quad+\sum_{s=1}^N\Big(\partial_{y_i}\widetilde{\varphi}_s(y_0)\partial_{y_j}\widetilde{\psi}_s(y_0)+\partial_{y_i}\widetilde{\psi}_s(y_0)\partial_{y_j}\widetilde{\varphi}_s(y_0)\Big)\\
D_\psi\partial_{y_k}g_{ij}^\psi(y_0)[\varphi] &= \partial^2_{y_ky_i}\widetilde{\varphi}_j(y_0)+\partial_{y_ky_j}\widetilde{\varphi}_i(y_0)\\
&\quad+\partial^2_{y_ky_i}f(y_0)\partial_{y_j}\widetilde{\varphi}_N(y_0)+\partial_{y_i}f(y_0)\partial^2_{y_ky_j}\widetilde{\varphi}_N(y_0)\\
&\quad+\partial^2_{y_ky_j}f(y_0)\partial_{y_i}\widetilde{\varphi}_N(y_0)+\partial_{y_j}f(y_0)\partial^2_{y_ky_i}\widetilde{\varphi}_N(y_0)\\
&\quad+\sum_{s=1}^N(\partial^2_{y_ky_i}\widetilde{\varphi}_s(y_0)\partial_{y_j}\widetilde{\psi}_s(y_0)+\partial_{y_i}\widetilde{\varphi}_s(y_0)\partial^2_{y_ky_j}\widetilde{\psi}_s(y_0)\\
&\quad+\partial^2_{y_ky_i}\widetilde{\psi}_s(y_0)\partial_{y_j}\widetilde{\varphi}_s(y_0)+\partial_{y_i}\widetilde{\psi}_s(y_0)\partial^2_{y_ky_j}\widetilde{\varphi}_s(y_0)).
\end{aligned}$$

Here $\widetilde{\varphi}:=\varphi(y,f(y))$.

Lemma 4.2. *Given $y_0\in B(0,R)$ for any $\varphi\in\mathfrak{A}_{y_0}$ we get*

$$D_\psi|g^{\psi_0}(y_0)|[\varphi]=0 \quad \textit{and} \quad D_\psi\partial_{y_k}|g^{\psi_0}(y_0)|[\varphi]=0.$$

Proof. By (4.13) and Lemma 4.1 it follows immediately that for any i,j,k we have

$$D_\psi g_{ij}^{\psi_0}(y_0)[\varphi]=0 \quad \text{and} \quad D_\psi\partial_{y_k}g_{ij}^{\psi_0}(y_0)[\varphi]=0.$$

The claim follows. $\square$

LEMMA 4.3. *For any* $\varphi \in \mathfrak{A}_{y_0}$ *we have:*

(a) $D_\psi \gamma^\psi_{ij}(y_0)[\varphi] = (\partial^2_{y_i y_j}\widetilde{\varphi}_1(y_0), \ldots, \partial^2_{y_i y_j}\widetilde{\varphi}_N(y_0)) = 0$,
(b) $D_\psi \nu^\psi(y_0)[\varphi] = 0$, $D_\psi \partial_{y_k}\nu^\psi(y_0)[\varphi] = 0$, $k = 1, \ldots, N-1$,
(c) $D_\psi \Gamma^\psi_{ij}(y_0) = (D_\psi \gamma^\psi_{ij}(y_0)[\varphi], \nu^\psi(y_0)) + (\gamma^\psi_{ij}(y_0), D_\psi \nu^\psi(y_0[\varphi])) = 0$,
(d) $D_\psi |B^s(y_0, \psi_0)|[\varphi] = 0$, $s = 1, \ldots, N-1$.

PROOF. (b) follows by the fact that $\nu^\psi(y)$ can be expressed as a suitable $C^\infty-$function of the first derivatives $\partial_{y_i}\widetilde{\psi}_j$. (d) follows by the definition of $|B^s(y,\psi)|$, Lemma 4.2 and (c). $\square$

LEMMA 4.4. *For any* $\varphi \in \mathfrak{A}_{y_0}$ *we have*

$$D_\psi \partial_{y_k}\Gamma^{\psi_0}_{ij}(y_0)[\varphi] = (\partial^3_{y_k y_i y_j}\widetilde{\varphi}(y_0), \nu^{\psi_0}(y_0)),$$

where $\partial^3_{y_k y_i y_j}\widetilde{\varphi}(y_0) := (\partial^3_{y_k y_i y_j}\widetilde{\varphi}_1(y_0), \ldots, \partial^3_{y_k y_i y_j}\widetilde{\varphi}_N(y_0))$.

PROOF. By (a) and (b) of Lemma 4.3 we get

$$\begin{aligned} D_\psi \partial_{y_k}\Gamma^{\psi_0}_{ij}(y_0)[\varphi] &= (D_\psi \partial_{y_k}\gamma^{\psi_0}_{ij}(y_0)[\varphi], \nu^{\psi_0}(y_0)) + (D_\psi \gamma^{\psi_0}_{ij}(y_0)[\varphi], \partial_{y_k}\nu^{\psi_0}(y_0)) \\ &\quad + (\partial_{y_k}\gamma^{\psi_0}_{ij}, D_\psi \nu^{\psi_0}(y_0)[\varphi]) + (\gamma^{\psi_0}_{ij}(y_0), D_\psi \partial_{y_k}\nu^{\psi_0}(y_0)[\varphi]) \\ &= (D_\psi \partial_{y_k}\gamma^{\psi_0}_{ij}(y_0)[\varphi], \nu^{\psi_0}(y_0)) = (\partial^3_{y_k y_i y_j}\widetilde{\varphi}(y_0), \nu^{\psi_0}(y_0)). \quad \square \end{aligned}$$

REMARK 4.5. For any $\varphi \in \mathfrak{A}_{y_0}$ we have that $D_\psi \partial_{y_k}|B^s(y_0,\psi_0)|[\varphi]$ is the determinant of the matrix obtained by the matrix $g^{\psi_0}_{ij}(y_0)$ replacing the s-th row with

$$(D_\psi \partial_{y_k}\Gamma^{\psi_0}_{s,1}(y_0)[\varphi], \ldots, D_\psi \partial_{y_k}\Gamma^{\psi_0}_{s,N-1}(y_0)[\varphi])$$

and in virtue of Lemma 4.4 it coincides with the determinant of the matrix obtained by the matrix $g^{\psi_0}_{ij}(y_0)$ replacing the s-th row with

$$((\partial^3_{y_k y_s y_1}\widetilde{\varphi}(y_0), \nu^{\psi_0}(y_0)), \ldots, (\partial^3_{y_k y_s y_{N-1}}\widetilde{\varphi}(y_0), \nu^{\psi_0}(y_0))).$$

In the following we choose $(y_0, \psi_0) \in B(0,R) \times \mathfrak{B}_\rho$ such that $F(y_0,\psi_0) = \nabla_y H(y_0,\psi_0) = 0$, namely for any $k = 1, \ldots, N-1$

$$\begin{aligned} 0 &= \partial_{y_k} H(y_0,\psi_0) \\ &= \frac{|g^{\psi_0}(y_0)| \sum\limits_{s=1}^{N-1} \partial_{y_k}|B^s(y_0,\psi_0)| - \partial_{y_k}|g^{\psi_0}(y_0)| \sum\limits_{s=1}^{N-1} |B^s(y_0,\psi_0)|}{(N-1)|g^{\psi_0}(y_0)|^2}. \end{aligned} \tag{4.14}$$

By Lemmas 4.2, 4.3 and (4.14) for any $\varphi \in \mathfrak{A}_{y_0}$ and for any $k = 1, \ldots, N-1$ we get

$$\omega_k(\varphi) := D_\psi \partial_{y_k} H(y_0,\psi_0)[\varphi] = \frac{\sum\limits_{s=1}^{N-1} D_\psi \partial_{y_k}|B^s(y_0,\psi_0)|[\varphi]}{(N-1)|g^{\psi_0}(y_0)|^2}\, y_k. \tag{4.15}$$

Our aim is to verify that the $N-1$ functionals $\omega_1,\dots,\omega_{N-1}$ are linearly independent.

Given $\alpha \in \{1,\dots,N-1\}$ and $y_0 \in B(0,R) \subset \mathbb{R}^{N-1}$, let $\mathfrak{A}^\alpha_{y_0}$ be the subset of the Banach space $\mathfrak{E}^k$ defined by

$$\begin{aligned}\mathfrak{A}^\alpha_{y_0} :=\{\varphi \in \mathfrak{E}^k :\ & \text{the third derivatives of the functions}\\ & \widetilde{\varphi}_i(y) = \varphi_i(y, f(y)),\ i = 1,\dots,N\\ & \text{evaluated at the point } y_0 \text{ vanish except for } \partial^3_{y^3_\alpha}\widetilde{\varphi}_N(y_0) \neq 0\}.\end{aligned}$$

Lemma 4.6. *For any $\varphi \in \mathfrak{A}^\alpha_{y_0}$ we have*

(a) $\omega_t(\varphi) = 0$ *if* $t \neq \alpha$,

(b) $\omega_\alpha(\varphi) = \partial^3_{y^3_\alpha}\widetilde{\varphi}_N(y_0)\frac{\nu^{\psi_0}_N(y_0)M(y_0,\psi_0)}{(N-1)|g^{\psi_0}(y_0)|^2}$ *where M is the determinant of the matrix obtained by the matrix $g^{\psi_0}_{ij}(y_0)$ carrying out the α-th row and the α-th column and $\nu^{\psi_0}(y_0)$ is the $N-$component of the vector $\nu^{\psi_0}(y_0)$.*

Proof. By Lemma 4.4 and Remark 4.5 we have for any $\varphi \in \mathfrak{A}^\alpha_{y_0}$ and for any $s = 1,\dots,N-1$

$$\begin{aligned} D_\psi\partial_{y_\alpha}|B^s(y_0,\psi_0)|[\varphi] &= 0 \quad \text{if } s \neq \alpha,\\ D_\psi\partial_{y_t}|B^s(y_0,\psi_0)|[\varphi] &= 0 \quad \text{if } t \neq \alpha.\end{aligned} \tag{4.16}$$

By (4.15) and (4.16) we get $\omega_t(\varphi) = 0$ for any $\varphi \in \mathfrak{A}^\alpha_{y_0}$ and $t \neq \alpha$. By (4.15), Lemma 4.4 and Remark 4.5 we deduce

$$\begin{aligned}\omega_\alpha(\varphi)[(N-1)|g^{\psi_0}(y_0)|^2] &= D_\psi\partial_{y_\alpha}|B^\alpha(y_0,\psi_0)|[\varphi]\\ &= \text{is the determinant of the matrix obtained by the matrix } g^{\psi_0}_{ij}(y_0)\\ &\quad\ \text{replacing out the } s\text{-th row with}\\ &\quad\ \big(0,\dots,\underbrace{(\partial^3_{y^3_\alpha}\widetilde{\varphi}(y_0),\nu^{\psi_0}(y_0))}_{\alpha\text{-th}},\dots,0\big) = \partial^3_{y^3_\alpha}\widetilde{\varphi}(y_0)\nu^{\psi_0}_N(y_0)M(y_0,\psi_0). \qquad \square\end{aligned}$$

Lemma 4.7. *The map $(y,\varphi) \to F'_\psi(y_0,\psi_0)[\varphi] + F'_y(y_0,\psi_0)y$ is onto on $\mathbb{R}^{N-1}$ for any $(y_0,\psi_0) \in B(0,R) \times \mathfrak{B}_\rho$ such that $F(y_0,\psi_0) = 0$ when R and ρ are small enough.*

Proof. We will prove that the map $F'_\psi(y_0,\psi_0)\colon \mathfrak{E}^k \to \mathbb{R}^{N-1}$ is onto when $F(y_0,\psi_0) = 0$. More precisely we are going to show that given $e_1,\dots,e_{N-1}$ the canonical base of $\mathbb{R}^{N-1}$, for any $s = 1,\dots,N-1$ there exists $\varphi \in \mathfrak{E}^k$ such that $F'_\psi(y_0,\psi_0)[\varphi] = e_s$. We recall that

$$\begin{aligned}F'_\psi(y_0,\psi_0)[\varphi] &= (D_\psi\partial_{y_1}H(y_0,\psi_0)[\varphi],\dots,D_\psi\partial_{y_{N-1}}H(y_0,\psi_0)[\varphi])\\ &= (\omega_1(\varphi),\dots,\omega_{N-1}(\varphi)).\end{aligned}$$

Given s we choose $\varphi \in \mathfrak{A}^\alpha_{y_0}$ and we have

$$F'_\psi(y_0,\psi_0)[\varphi] = (0,\dots,\underbrace{\omega_s(\varphi)}_{s\text{-th}},\dots,0)$$

and $\omega_s(\varphi) \neq 0$ provided $y_0 \in B(0,R)$, $\psi_0 \in \mathfrak{B}_\rho$ with R and ρ small enough. $\square$

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Manuscript received October 6, 2011

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