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WHAT AN INFRA-NILMANIFOLD ENDOMORPHISM REALLY SHOULD BE...

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Abstract. Infra-nilmanifold endomorphisms were introduced in the late sixties. They play a very crucial role in dynamics, especially when studying expanding maps and Anosov diffeomorphisms. However, in this note we will explain that the two main results in this area are based on a false result and that although we can repair one of these two theorems, there remains doubt on the correctness of the other one. Moreover, we will also show that the notion of an infra-nilmanifold endomorphism itself has not always been interpreted in the same way.

Finally, we define a slightly more general concept of the notion of an infra-nilmanifold endomorphism and explain why this is really the right concept to work with.

1. Introduction

The notion of an infra-nilmanifold endomorphism appears for the first time in the proceedings of the symposium in pure mathematics of the American Mathematical Society held in 1968 in Berkeley (see [7], [12], [24]).

Nowadays, when using the term infra-nilmanifold endomorphism, most people refer to the paper of J. Franks [7], although J. Franks himself in that same paper (and also M. Shub in [24]), attributes this terminology to M.W. Hirsch [12].

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However, it is immediately clear and this will also be explained in the next section, that the definition used by J. Franks and M. Shub is not equivalent to the one used by M.W. Hirsch.

The results of [7] have played an important role in dynamics, especially in the study of Anosov diffeomorphisms and expanding maps. On the one hand, it was a crucial ingredient in the result of M. Gromov (see geometric corollary on page 55 of [9]) stating that an expanding map on an arbitrary compact manifold is topologically conjugate to an infra-nilmanifold endomorphism.

On the other hand, A. Manning [18], using the concept of an infra-nilmanifold endomorphism as introduced in [7], showed that any Anosov diffeomorphism of an infra-nilmanifold M is topologically conjugate to a so called hyperbolic infra-nilmanifold automorphism.

Unfortunately, some results of [7] depend on a theorem of L. Auslander [1, Theorem 2] which is not correct (not only the proof, but the statement of Auslander's theorem is wrong). We will explain this in the next section. Moreover, although most of the arguments in [7] which are based on Auslander's wrong result can be restored using a modified version of it (see Corollary 2.3 below), there remain some subtle problems with the definition of the concept of an infra-nilmanifold endomorphism as given in [7].

The aim of this note is to show that, even with a correct definition of an infra-nilmanifold endomorphism, both the proofs of the result of A. Manning and of M. Gromov are not correct, because they are heavily based on a wrong result [7, Proposition 3.5] of the paper of Franks. As both of these results are often referred to, I will point out as detailed as possible, where the problems in the work of L. Auslander and of J. Franks are situated and how this has it's influence in the work of M. Gromov and A. Manning. Moreover, I will give an example of an expanding map and of an Anosov diffeomorphism on a given infra-nilmanifold which are not topologically conjugate to an infra-nilmanifold endomorphism of that infra-nilmanifold. Fortunately, by the work of K.B. Lee and F. Raymond [15], who were, up to my knowledge, the first to discover the problems in the work of L. Auslander, it is rather easy to define a slightly broader concept of the notion of infra-nilmanifold endomorphism, namely the class of affine endomorphisms, which is more suited to study self maps of infra-nilmanifolds. We will show that using this broader concept the result of M. Gromov on expanding maps can be repaired, but one has to be very careful with the precise interpretation of the statement. On the other hand, although it is also to be expected that A. Manning's result might be repaired, I haven't been able to prove this in its full generality yet.

2. Infra-nilmanifolds and endomorphisms of their fundamental groups

Let N be a connected and simply connected nilpotent Lie group and let $\mathrm{Aut}(N)$ be the group of continuous automorphisms of N. Then $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ acts on N in the following way:

$$(n, \alpha) \cdot x = n\alpha(x) \quad \text{for all } (n, \alpha) \in \mathrm{Aff}(N) \text{ and all } x \in N.$$

So an element of $\mathrm{Aff}(N)$ consists of a translational part $n \in N$ and a linear part $\alpha \in \mathrm{Aut}(N)$ (as a set $\mathrm{Aff}(N)$ is just $N \times \mathrm{Aut}(N)$) and $\mathrm{Aff}(N)$ acts on N by first applying the linear part and then multiplying on the left by the translational part). In this way, $\mathrm{Aff}(N)$ can also be seen as a subgroup of $\mathrm{Diff}(N)$.

Now, let C be a compact subgroup of $\mathrm{Aut}(N)$ and consider any torsion free discrete subgroup Γ of $N \rtimes C$, such that the orbit space $\Gamma \setminus N$ is compact. Note that Γ acts on N as being also a subgroup of $\mathrm{Aff}(N)$.

The action of Γ on N will be free and properly discontinuous, so $\Gamma \setminus N$ is a manifold, which is called an infra-nilmanifold. It follows from the (correct) Theorem 1 of L. Auslander in [1], that $\Gamma \cap N$ is a uniform lattice of N and that $\Gamma/(\Gamma \cap N)$ is a finite group. This shows that the fundamental group of an infra-nilmanifold $\Gamma \setminus N$ is virtually nilpotent (i.e. has a nilpotent normal subgroup of finite index). In fact, $\Gamma \cap N$ is a maximal nilpotent subgroup of Γ and it is the only normal subgroup of Γ with this property. (This also follows from [1]).

If we denote by $p \colon N \rtimes C \to C$ the natural projection on the second factor, then $p(\Gamma) \cong \Gamma/(\Gamma \cap N)$. Let F denote this finite group $p(\Gamma)$, then we will refer to F as being the holonomy group of Γ (or of the infra-nilmanifold $\Gamma \setminus N$). It follows that $\Gamma \subseteq N \rtimes F$. In case $F = 1$, so $\Gamma \subseteq N$, the manifold $N \setminus G$ is a nilmanifold. Hence, any infra-nilmanifold $\Gamma \setminus N$ is finitely covered by a nilmanifold $(\Gamma \cap N) \setminus N$. This also explains the prefix "infra".

When the Lie group N is abelian, so N is the additive group $\mathbb{R}^n$ for some n, it is enough to consider the case $C = O(n)$, the orthogonal group, because $O(n)$ is a maximal compact subgroup of $\mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})$ and so any other compact subgroup is conjugate to a subgroup of $O(n)$. It follows that in this situation $N \rtimes C = \mathbb{R}^n \rtimes O(n)$ is the group of isometries of Euclidean space $\mathbb{R}^n$. In this setting, the infra-nilmanifolds are compact flat Riemannian manifolds and the nilmanifolds are just tori.

Remark 2.1. Many authors (e.g. see [7], [12]) start from discrete subgroups of $N \rtimes F$ for various finite groups F to define the notion of an infra-nilmanifold. The discussion above shows that this is not a restriction.

In [9] and [11], an infra-nilmanifold is defined as a quotient $\Gamma \setminus N$, where Γ is a subgroup of the whole affine group $\mathrm{Aff}(N)$ acting freely and properly discontinuously on N. This is not a correct definition, for in this case, the linear parts do not have to form a finite group and hence Γ need not be a virtually nilpotent group. As an example, let $\varphi\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}^2)$ be any morphism and regard $\varphi(z)$ as being a 2×2-matrix. Then,

$$\Gamma = \left\{ \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \varphi(z) & 0 \\ 0 & 1 \end{pmatrix} \right) \middle| \; x, y, z \in \mathbb{Z} \right\}$$

is a subgroup of $\mathrm{Aff}(\mathbb{R}^3)$ acting freely and properly discontinuously on $\mathbb{R}^3$. The group Γ is isomorphic to the semi-direct product group $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is given via φ. Such a group is often not virtually nilpotent. E.g. there is a unique morphism $\varphi\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}^2)$, with $\varphi(1) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. The corresponding group $\mathbb{Z}^2 \rtimes \mathbb{Z}$ is not virtually nilpotent. Actually, the manifolds which are obtained in this way are called complete affinely flat manifolds (see [20]).

Let us now discuss why Theorem 2 of [1] is not correct. In fact, L. Auslander proves this theorem as a generalization of the second Bieberbach theorem. Unfortunately, even L. Auslander's formulation of this second Bieberbach theorem is not correct. This was first observed, without further explanation, by K.B. Lee and F. Raymond in [15]. As this theorem plays an important role in the work of J. Franks, I will explain in full detail what goes wrong and what can be saved.

We recall the statement of Auslander's theorem using the notations we introduced above.

Formulation of Theorem 2 in [1]**.** Let Γ_1 and Γ_2 be discrete uniform subgroups of $N \rtimes C$. Let $\psi\colon \Gamma_1 \to \Gamma_2$ be an isomorphism. Then ψ can be uniquely extended to a continuous automorphism ψ^* of $N \rtimes C$ onto itself.

It is very easy to produce a counterexample to this statement. In fact, the statement is almost never correct. Let $N = \mathbb{R}^2$ the additive group and $C = O(2)$. Let $\Gamma_1 = \Gamma_2 = \mathbb{Z}^2$ and let $\psi \in \mathrm{Aut}(\mathbb{Z}^2)$ be the automorphism represented by the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ (almost any matrix will do). Now assume that ψ extends to a continuous automorphism ψ^* of $\mathbb{R}^2 \rtimes O(2)$. The group $\mathbb{R}^2$ (seen as a subgroup of $\mathbb{R}^2 \rtimes O(2)$) is normal and maximal abelian and is the unique subgroup of $\mathbb{R}^2 \rtimes O(2)$ satisfying this condition, so we must have that $\psi^*(\mathbb{R}^2) = \mathbb{R}^2$. It follows that the restriction of ψ^* to $\mathbb{R}^2$ is the linear map, given by the matrix A. So $\psi^*(r, 1) = (Ar, 1)$ for all $r \in \mathbb{R}^2$. (Here 1 denotes the trivial automorphism of $\mathbb{R}^2$ or the 2×2 identity matrix).

Now let $B \in O(2)$, so $(0, B) \in \mathbb{R}^2 \rtimes O(2)$, and assume that $\psi^*(0, B) = (b, B')$ for some $b \in \mathbb{R}^2$ and some $B' \in O(2)$. Let us perform a small computation, where $r \in \mathbb{R}^2$ is arbitrary:

$$\psi^*((0,B)(r,1)(0,B)^{-1}) = \psi^*(0,B)\psi^*(r,1)\psi^*(0,B)^{-1}$$
$$\Downarrow$$
$$\psi^*(Br,1) = (b,B')(Ar,1)(-B'^{-1}b, B'^{-1})$$
$$\Downarrow$$
$$(ABr,1) = (B'Ar,1).$$

As this holds for any r we must have that $AB = B'A$, or $B' = ABA^{-1}$. It is now trivial to see that such a B' does not have to belong to $O(2)$. E.g. when $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We have that $B' = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} \notin O(2)$. We can conclude that ψ does not extend to a continuous morphism of $\mathbb{R}^2 \rtimes O(2)$, contradicting the statement made by L. Auslander. At this point I want to remark that Auslander's argumentation is very short, so it is difficult to point out where exactly the error is situated.

A correct formulation of a generalization of the second Bieberbach theorem is given in [15].

Theorem 2.2 ([15], see also [4, p. 16]). *Let N be a connected and simply connected nilpotent Lie group and C a compact subgroup of* $\mathrm{Aut}(N)$. *Let Γ_1 and Γ_2 be two discrete and uniform subgroups of $N \rtimes C$ and assume that $\psi\colon \Gamma_1 \to \Gamma_2$ is an isomorphism, then there exists an $\alpha \in \mathrm{Aff}(N)$ such that*

$$\psi(\gamma) = \alpha\gamma\alpha^{-1} \quad \textit{for all } \gamma \in \Gamma_1.$$

So, any isomorphism between the groups Γ_1 and Γ_2 is induced by a conjugation inside $\mathrm{Aff}(N)$.

At this point, I would like to mention a corollary, which can be seen as a fix to the false statement of L. Auslander.

Corollary 2.3. *Let N be a connected and simply connected nilpotent Lie group and C a compact subgroup of* $\mathrm{Aut}(N)$ *and let Γ be a discrete and uniform subgroup of $N \rtimes C$. Let $p\colon N \rtimes C \to C$ denote the natural projection. If $\psi\colon \Gamma \to \Gamma$ is a monomorphism, then $p(\Gamma) = p(\psi(\Gamma))$. Moreover, in this case ψ extends to an automorphism ψ^* of $N \rtimes p(\Gamma)$, such that $\psi^*(N) = N$.*

Proof. ψ is an isomorphism from Γ onto $\psi(\Gamma)$, so by Theorem 2.2, ψ can be realized as a conjugation, say by $\alpha \in \mathrm{Aff}(N)$, inside $\mathrm{Aff}(N)$. As N is a normal subgroup of $\mathrm{Aff}(N)$, we have that $\alpha N \alpha^{-1} = N$. On the other hand, we also have that $\alpha\Gamma\alpha^{-1} \subseteq \Gamma$. Therefore, $\alpha(N \rtimes p(\Gamma))\alpha^{-1} = \alpha N\Gamma\alpha^{-1} \subseteq N\Gamma = N \rtimes p(\Gamma)$. In fact, we can see that $\alpha(N \rtimes p(\Gamma))\alpha^{-1} = N \rtimes p(\Gamma)$ (and not a proper subset of it). To prove this, we must show that for any $\mu \in p(\Gamma)$, there is a $n \in N$, with

$(n,\mu) \in \psi(\Gamma) = \alpha\Gamma\alpha^{-1}$. This is however easy, because any morphism ψ of Γ induces a morphism

$$\overline{\psi}: p(\Gamma) = \Gamma/(\Gamma \cap N) \to p(\Gamma) = \Gamma/(\Gamma \cap N).$$

Now, as ψ is conjugation with an element $\alpha \in \mathrm{Aff}(N)$, it is easy to see that $\overline{\psi}$ is conjugation with the linear part of α in $\mathrm{Aut}(N)$. Therefore, $\overline{\psi}$ is bijective, showing that $p(\psi(\Gamma)) = p(\Gamma)$ and $\alpha(N \rtimes p(\Gamma))\alpha^{-1} = N \rtimes p(\Gamma)$. The proof now finishes by taking ψ^* to be conjugation with α inside $\mathrm{Aff}(N)$. □

3. Infra-nilmanifold endomorphisms

In this section, we will discuss the notion of an infra-nilmanifold endomorphism as introduced by M.W. Hirsch [12] and by J. Franks [7].

To do this, we fix an infra-nilmanifold $\Gamma \backslash N$, so N is a connected and simply connected nilpotent Lie group and Γ is a torsion free, uniform discrete subgroup of $N \rtimes F$, where F is a finite subgroup of $\mathrm{Aut}(N)$. We will assume that F is the holonomy group of Γ (so for any $\mu \in F$, there exists an $n \in N$ such that $(n,\mu) \in \Gamma$).

In what follows, we will identify N with the subgroup $N \times \{1\}$ of $N \rtimes \mathrm{Aut}(N) = \mathrm{Aff}(N)$, F with the subgroup $\{1\} \times F$ and $\mathrm{Aut}(N)$ with the subgroup $\{1\} \times \mathrm{Aut}(N)$. Hence, we can say that an element of Γ is of the form $n\mu$ for some $n \in N$ and some $\mu \in F$. Also, any element of $\mathrm{Aff}(N)$ can uniquely be written as a product $n\psi$, where $n \in N$ and $\psi \in \mathrm{Aut}(N)$. The product in $\mathrm{Aff}(N)$ is then given as

$$n_1\psi_1 n_2\psi_2 = n_1\psi_1(n_2)\psi_1\psi_2 \quad \text{for all } n_1, n_2 \in N \text{ and all } \psi_1, \psi_2 \in \mathrm{Aut}(N).$$

We will first look at the way M.W. Hirsch introduced the notion of an infra-nilmanifold endomorphism. Actually, Hirsch defines endomorphisms on a larger class of spaces, called infra homogeneous spaces, but we immediately specialise to the case of infra-nilmanifolds.

M.W. Hirsch starts with a given automorphism φ of the Lie group $N \rtimes F$, with $\varphi(F) = F$. Note that we also have that $\varphi(N) = N$, because N is the connected component of the identity element in $N \rtimes F$. Before we continue, let us give a description of these automorphisms.

Lemma 3.1. *Let N be a connected, simply connected nilpotent Lie group and F be a finite subgroup of* $\mathrm{Aut}(N)$. *Let φ be an automorphism of $N \rtimes F$ with $\varphi(F) = F$ and denote by $\psi \in \mathrm{Aut}(N)$ the restriction of φ to N, then*

$$\varphi(x) = \psi x \psi^{-1} \quad \textit{for all } x \in N \rtimes F$$

where $\psi x \psi^{-1}$ is a conjugation in the group $\mathrm{Aff}(N)$.

PROOF. For any $n \in N$ and any $\xi \in \mathrm{Aut}(N)$, the equality $\xi(n) = \xi n \xi^{-1}$ is valid, where $\xi n \xi^{-1}$ is a conjugation in $\mathrm{Aff}(N)$. So, we also have that

$$\varphi(n) = \psi(n) = \psi n \psi^{-1}. \tag{3.1}$$

Let us now consider an element $\mu \in F$. For any $n \in N$, we have the following equation in the group $N \rtimes F$:

$$\mu(n) = \mu n \mu^{-1}.$$

By applying φ to both sides of this equation, we find that

$$\varphi(\mu(n)) = \varphi(\mu)\psi(n)\varphi(\mu)^{-1} \;\Rightarrow\; \psi(\mu(n)) = \varphi(\mu)(\psi(n))$$

Since this holds for any $n \in N$, we have that $\psi \circ \mu = \varphi(\mu) \circ \psi$ showing that

$$\varphi(\mu) = \psi \mu \psi^{-1}. \tag{3.2}$$

Now, combining (3.1) and (3.2) we find that for $x = n\mu$, with $n \in N$ and $\mu \in F$:

$$\varphi(x) = \varphi(n)\varphi(\mu) = \psi n \psi^{-1} \psi \mu \psi^{-1} = \psi x \psi^{-1},$$

which finishes the proof. □

Now, let φ still be an automorphism of $N \rtimes F$ with $\varphi(F) = F$ and assume that $\varphi(\Gamma) \subseteq \Gamma$, where Γ is a torsion free, discrete and uniform subgroup of $N \rtimes F$. Now, let $\gamma = m\mu$ be any element of Γ, where $m \in N$ and $\mu \in F$. We denote the action of Γ on $n \in N$ by $\gamma \cdot n$, so $\gamma \cdot n = m\mu(n)$. Now we compute that

$$\begin{aligned} \varphi(\gamma \cdot n) &= \varphi(m\mu(n)) = \varphi(m)\varphi(\mu(n)) \\ &= \varphi(m)\varphi(\mu)(\varphi(n)) = \varphi(m\mu) \cdot \varphi(n) = \varphi(\gamma) \cdot \varphi(n). \end{aligned}$$

We are now ready to introduce the notion of an infra-nilmanifold endomorphism.

DEFINITION 3.2 (Infra-nilmanifold endomorphism following Hirsch). Let N be a connected and simply connected nilpotent Lie group and $F \subseteq \mathrm{Aut}(N)$ a finite group. Assume that Γ is a torsion free, discrete and uniform subgroup of $N \rtimes F$. Let $\varphi\colon N \rtimes F \to N \rtimes F$ be an automorphism, such that $\varphi(F) = F$ and $\varphi(\Gamma) \subseteq \Gamma$, then, the map

$$\overline{\varphi}\colon \Gamma \setminus N \to \Gamma \setminus N : \quad \Gamma \cdot n \mapsto \Gamma \cdot \varphi(n),$$

is the infra-nilmanifold endomorphism induced by φ. In case $\varphi(\Gamma) = \Gamma$, we call $\overline{\varphi}$ an infra-nilmanifold automorphism.

In the definition above, $\Gamma \cdot n$ denotes the orbit of n under the action of Γ. The computation above shows that $\overline{\varphi}$ is well defined. Note that infra-nilmanifold automorphisms are diffeomorphisms, while in general an infra-nilmanifold endomorphism is a self-covering map.

Remark 3.3. It is easy to check that for an infra-nilmanifold endomorphism $\overline{\varphi}$, the induced morphism $\overline{\varphi}_\sharp$ on the fundamental group $\Pi_1(\Gamma\setminus N, x_0)\cong\Gamma$ is exactly the restriction of φ to Γ (see also Proposition 3.7 below and note that one can always choose as basepoint $x_0=\Gamma\cdot e$, the orbit of the identity element of N). By Lemma 3.1 we know that $\overline{\varphi}_\sharp$ is induced by a conjugation with an automorphism inside $\mathrm{Aff}(N)$. On the other hand, Theorem 2.2 shows that in general an injective endomorphism of Γ is induced by a conjugation with a general element of $\mathrm{Aff}(N)$ and not necessarily by an automorphism. This already indicates that there might exist (interesting) diffeomorphisms and self–covering maps of an infra-nilmanifold which are not even homotopic to an infra-nilmanifold endomorphism. Further on, we will explicitly construct such examples and obtain an Anosov diffeomorphism (resp. an expanding map) of an infra-nilmanifold which is not homotopic to an infra-nilmanifold automorphism (resp. infra-nilmanifold endomorphism) of that infra-nilmanifold.

As already indicated above, we will also consider the definition of an infra-nilmanifold endomorphism as introduced by J. Franks in [7, p. 63], the definition which is in fact most often referred to. Using our notation introduced above, J. Franks writes that when $\varphi\colon N\rtimes F\to N\rtimes F$ is an automorphism for which $\varphi(\Gamma)\subseteq\Gamma$ and $\varphi(N)=N$, it induces a map

$$\overline{\varphi}\colon\Gamma\setminus N\to\Gamma\setminus N.$$

(In fact, J. Franks requires that $\varphi(\Gamma)=\Gamma$ and not that it is only a subgroup, but I believe this is a typo).

It is this kind of maps that he calls infra-nilmanifold endomorphisms. As J. Franks does not impose the condition that $\varphi(F)=F$, this seems to be a generalization of the notion introduced by M.W. Hirsch. Exactly the same definition was given by M. Shub in [24, p. 274] (without the typo).

Unfortunately, there seems to be a problem with this definition. It is *not true* that the map $\overline{\varphi}\colon\Gamma\setminus N\to\Gamma\setminus N\colon\Gamma\cdot n\mapsto\Gamma\cdot\varphi(n)$ is in general well defined. As many authors refer to the work of J. Franks when talking about infra-nilmanifold endomorphisms, we give a detailed example to show where it goes wrong.

Let $N=\mathbb{R}^3$, the additive group. We let $F\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\subseteq\mathrm{GL}_3(\mathbb{R})$ be the group with elements

$$1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad \alpha=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},$$

$$\beta=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\qquad \alpha\beta=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

Moreover, we pick

$$a = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Let $A = (a, \alpha) \in \text{Aff}(\mathbb{R}^3)$ and $B = (b, \beta) \in \text{Aff}(\mathbb{R}^3)$ and consider the group $\Gamma \subseteq \mathbb{R}^3 \rtimes F$ to be the group generated by $\mathbb{Z}^3 \cup \{A, B\}$. Then Γ is a torsion free, uniform discrete subgroup of $\mathbb{R}^3 \rtimes F$. In fact $\Gamma \setminus \mathbb{R}^3$ is the well known Hantzsche-Wendt manifold with fundamental group Γ.

Now, let

$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} \frac{1}{4} \\ 0 \\ 0 \end{pmatrix}$$

and take $D = (d, \delta) \in \text{Aff}(\mathbb{R}^3)$. Let $\varphi\colon \text{Aff}(\mathbb{R}^3) \to \text{Aff}(\mathbb{R}^3)\colon X \mapsto DXD^{-1}$ be the inner automorphism determined by D. A calculation shows that $\varphi(\mathbb{R}^3 \rtimes F) = \mathbb{R}^3 \rtimes F$, so φ restricts to an automorphism $\mathbb{R}^3 \rtimes F$ for which of course $\varphi(\mathbb{R}^3) = \mathbb{R}^3$ (but $\varphi(F) \neq F$!). Moreover,

$$\varphi(\mathbb{Z}^3) = \mathbb{Z}^3 \subseteq \Gamma, \quad \varphi(A) = A \in \Gamma \quad \text{and} \quad \varphi(B) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} AB \in \mathbb{Z}^3\Gamma = \Gamma.$$

So $\varphi(\Gamma) \subseteq \Gamma$ (in fact equality holds). I claim that in this case the map $\overline{\varphi}$ is not well defined. To prove this claim, we need to provide a $n \in \mathbb{R}^3$ and a $\gamma \in \Gamma$, such that $\Gamma \cdot \varphi(n) \neq \Gamma \cdot \varphi(\gamma \cdot n)$. Let

$$n = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad \text{and} \quad \gamma = B, \quad \text{then} \quad \gamma \cdot n = \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{6} \\ \frac{1}{6} \end{pmatrix}.$$

It follows that

$$\varphi(n) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad \text{and} \quad \varphi(\gamma \cdot n) = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}.$$

To check that $\Gamma \cdot \varphi(\gamma \cdot n) \neq \Gamma \cdot \varphi(n)$, it suffices to check that $\varphi(\gamma \cdot n) \notin \Gamma \cdot \varphi(n)$, or that $\varphi(\gamma \cdot n) \neq \gamma' \cdot \varphi(n)$ for any $\gamma' \in \Gamma$. Any $\gamma' \in \Gamma$ can uniquely be written in one of the following ways:

$$\gamma'_1 = z, \quad \gamma'_2 = zA, \quad \gamma'_3 = zB \quad \text{or} \quad \gamma'_3 = zAB, \quad \text{with } z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{Z}^3.$$

Computing $\gamma' \cdot \varphi(n)$ in each of these four case, we obtain:

$$\gamma'_1 \cdot \varphi(n) = \begin{pmatrix} z_1 + \frac{1}{3} \\ z_2 + \frac{1}{3} \\ z_3 + \frac{1}{3} \end{pmatrix}, \qquad \gamma'_2 \cdot \varphi(n) = \begin{pmatrix} z_1 + \frac{5}{6} \\ z_2 - \frac{1}{3} \\ z_3 - \frac{1}{3} \end{pmatrix},$$

$$\gamma_3' \cdot \varphi(n) = \begin{pmatrix} z_1 - \frac{1}{3} \\ z_2 + \frac{5}{6} \\ z_3 + \frac{1}{6} \end{pmatrix}, \qquad \gamma_4' \cdot \varphi(n) = \begin{pmatrix} z_1 + \frac{1}{6} \\ z_2 - \frac{5}{6} \\ z_3 - \frac{1}{6} \end{pmatrix}.$$

It is obvious that none of these expressions equals $\varphi(\gamma \cdot n)$, proving the claim.□

At the end of this section, we want to explain that in a certain sense, the definition of an infra-nilmanifold endomorphism as given by M.W. Hirsch is the best possible. In fact, we will show that the only maps of an infra-nilmanifold, that lift to an automorphism of the corresponding nilpotent Lie group are exactly the infra-nilmanifold endomorphisms defined in Definition 3.2. When reading the work of J. Franks, it is clear that he also only considers those maps on an infra-nilmanifold $\Gamma \setminus N$ which lift to an automorphism of the Lie group N (e.g. see the first few lines of the proof of Theorem 2.2 of [7]). In fact, when talking about infra-nilmanifold endomorphisms most authors, including J. Franks, M. Schub and M. Hirsch (but e.g. also in [2], [10], [11], [26] and in many other papers) are talking about maps which lift to an automorphism of the Lie group N.

Theorem 3.4. *Let N be a connected and simply connected nilpotent Lie group, $F \subseteq \mathrm{Aut}(N)$ a finite group and Γ a torsion free discrete and uniform subgroup of $N \rtimes F$ and assume that the holonomy group of Γ is F. If $\varphi: N \to N$ is an automorphism for which the map*

$$\overline{\varphi}: \Gamma \setminus N \to \Gamma \setminus N : \quad \Gamma \cdot n \mapsto \Gamma \cdot \varphi(n)$$

is well defined (meaning that $\Gamma \cdot \varphi(n) = \Gamma \cdot \varphi(\gamma \cdot n)$ for all $\gamma \in \Gamma$), then

$$\Phi: N \rtimes F \to N \rtimes F : \quad x \mapsto \varphi x \varphi^{-1} \quad (\textit{conjugation in } \mathrm{Aff}(N))$$

is an automorphism of $N \rtimes F$, with $\Phi(F) = F$ and $\Phi(\Gamma) \subseteq \Gamma$. Hence, $\overline{\varphi}$ is a infra-nilmanifold endomorphism (as in Definition 3.2).

Proof. The fact that $\overline{\varphi}$ is well defined, means that φ is a lift of $\overline{\varphi}$ to the universal cover N of $\Gamma \setminus N$. Now, for all $\gamma \in \Gamma$, also the composition $\varphi\gamma$ is a lift of $\overline{\varphi}$, since Γ is the group of covering transformations of the covering $N \to \Gamma \setminus N$. It follows that there exists a γ' such that $\varphi\gamma = \gamma'\varphi$. Now, since φ is an automorphism of N, we can write this as $\varphi\gamma\varphi^{-1} = \gamma'$ for some $\gamma' \in \Gamma$ so

$$\varphi\Gamma\varphi^{-1} \subseteq \Gamma.$$

Now, consider the inner automorphism Ψ of $\mathrm{Aff}(N)$ induced by φ:

$$\Psi: \mathrm{Aff}(N) \to \mathrm{Aff}(N) : \quad x \mapsto \varphi x \varphi^{-1}.$$

For all $n \in N$, we have that $\Psi(n) = \varphi(n)$, so $\Psi(N) = \varphi(N) = N$. We showed above that that $\Psi(\Gamma) \subseteq \Gamma$. It follows that $\Psi(N \rtimes F) = \Psi(N\Gamma) \subseteq N\Gamma$. Hence, Ψ induces an injective endomorphism of $N \rtimes F$. As F is mapped into itself by Ψ (because $\mathrm{Aut}(N)$ is mapped into itself by Ψ) and F is finite, we must have that $\Psi(F) = F$. Together with the fact that $\Psi(N) = N$, this implies that $\Psi(N \rtimes F) = N \rtimes F$ and hence Ψ restricts to an automorphism Φ of $N \rtimes F$, satisfying the conditions mentioned in the statement of the theorem. □

Remark 3.5. When checking literature, it seems that most authors that are talking about infra-nilmanifold endomorphisms, seem to assume that such a map lifts to an automorphism of the covering Lie group N. Hence, this implies that they are actually using the definition of M.W. Hirsch (which is probably also the definition that J. Franks meant to give). So from now onwards, when we use the term infra-nilmanifold endomorphism, we are referring to the only correct Definition 3.2.

We are now ready to define the generalization of the concept of an infra-nilmanifold endomorphism we announced in the introduction.

Definition 3.6. Let N be a connected and simply connected nilpotent Lie group, $F \subseteq \mathrm{Aut}(N)$ a finite group, Γ a torsion free discrete and uniform subgroup of $N \rtimes F$. Let $\alpha \in \mathrm{Aff}(N)$ be an element such that $\alpha\Gamma\alpha^{-1} \subseteq \Gamma$, then α induces a map

$$\overline{\alpha}\colon \Gamma \setminus N \to \Gamma \setminus N : \quad \Gamma \cdot n \mapsto \Gamma \cdot \alpha(n).$$

We call $\overline{\alpha}$ an affine endomorphism of the infra-nilmanifold $\Gamma \setminus N$ induced by α. When $\alpha\Gamma\alpha^{-1} = \Gamma$, the map $\overline{\alpha}$ is a diffeomorphism, and we call $\overline{\alpha}$ an affine automorphism.

As it is so crucial for what follows, we briefly recall from the theory of covering transformations how the group Γ can be seen as the fundamental group of $\Gamma \setminus N$ and what the effect of an affine endomorphism is on the fundamental group. Details of what follows can be found in any text book dealing with this topic, e.g. [25, Chapter 2] and [19, Chapter 5].

Choose any basepoint $n_0 \in \Gamma \setminus N$ and choose a point $\widetilde{n}_0 \in N$ whose orbit corresponds to the point n_0. Now, any loop $f\colon I \to \Gamma \setminus N$ at n_0 (I is the unit interval $[0,1]$) has a unique lift to a path $\widetilde{f}\colon I \to N$ starting at $\widetilde{n}_0$ (i.e $\widetilde{f}(0) = \widetilde{n}_0$). The endpoint $\widetilde{n}_1 = \widetilde{f}(1)$ of $\widetilde{f}$ lies in the same orbit as $\widetilde{n}_0$ (because they both project onto n_0) and hence, there exists a $\gamma_f \in \Gamma$ with $\gamma_f \cdot \widetilde{n}_0 = \widetilde{n_1}$. In this way, we associate to any loop f at n_0 an element $\gamma_f \in \Gamma$. It is a general fact that this correspondence does not depend on the path homotopy class of f and defines an isomorphism $\Phi\colon \Pi_1(\Gamma \setminus N, n_0) \to \Gamma$. Note that this isomorphism depends on the choice of the point $\widetilde{n}_0$ and that a different choice, say $\widetilde{n}_1$, changes the isomorphism by an inner automorphism of Γ.

Now, let $\overline{\alpha}$ be an affine endomorphism induced by an affine map $\alpha \in \mathrm{Aff}(N)$ (with $\alpha\Gamma\alpha^{-1} \subseteq \Gamma$). Choose a basepoint $n_0 \in \Gamma \setminus N$ and a point $\widetilde{n}_0 \in N$ projecting onto n_0. Then $\widetilde{n}_1 = \alpha(\widetilde{n}_0) \in N$ is a point projecting onto $n_1 = \alpha(n_0)$. Now, let us use $\widetilde{n}_0$ resp. $\widetilde{n}_1$ to identify $\Pi_1(\Gamma\setminus N, n_0)$, resp. $\Pi_1(\Gamma\setminus N, n_1)$ with Γ. Let $\overline{\alpha}_\sharp\colon \Pi_1(\Gamma\setminus N, n_0)\to \Pi_1(\Gamma\setminus N, n_1)$ denote the morphism induced by $\overline{\alpha}$. We claim that $\overline{\alpha}$ is exactly conjugation with α. Indeed, consider again a loop f based at n_0 and let $\widetilde{f}$ be the lift of f to N starting at $\widetilde{n}_0$. Let $\gamma\in\Gamma$ be the element such that $\gamma\cdot\widetilde{n}_0$ is the endpoint of $\widetilde{f}$ (so the path class $[f]\in\Pi_1(\Gamma\setminus N, n_0)$ corresponds to $\gamma\in\Gamma$). It is obvious that $\alpha\circ\widetilde{f}$ is the unique lift, beginning in $\widetilde{n}_1$, of the loop $\overline{\alpha}\circ f$. The endpoint of $\alpha\circ\widetilde{f}$ is

$$\alpha(f(1)) = \alpha(\gamma\cdot\widetilde{n}_0) = \alpha(\gamma(\alpha^{-1}(\alpha(\widetilde{n}_0)))) = (\alpha\gamma\alpha^{-1})\cdot\widetilde{n}_1.$$

This shows that the element of Γ corresponding to $\overline{\alpha}_\sharp[f] = [\overline{\alpha}\circ f]$ is exactly $\alpha\gamma\alpha^{-1}$.

Note that in the discussion above, we have chosen $\widetilde{n}_1$ based on our knowledge of α. In practice, this is often not possible or even not desirable. E.g. in this paper we often choose a fixed point n_0 of a selfmap $\overline{\alpha}$ on an infra-nilmanifold as a base point. To study then the induced morphism $\overline{\alpha}_\sharp\colon\Pi_1(\Gamma\setminus N, n_0)\to\Pi_1(\Gamma\setminus N, n_0)$ we will of course use two times the same $\widetilde{n}_0$ when identifying $\Pi_1(\Gamma\setminus N, n_0)$ with Γ. This implies that $\overline{\alpha}_\sharp$ will only be the same as conjugation with α in $\mathrm{Aff}(N)$ up to an inner conjugation by an element of Γ.

It follows that we have the following:

Proposition 3.7. *Let $\overline{\alpha}$ be an affine endomorphism of an infra-nilmanifold $\Gamma\setminus N$ and let $\psi\colon\Gamma\to\Gamma : \gamma\mapsto\alpha\gamma\alpha^{-1}$ be the corresponding monomorphism of Γ. Then, the map $\overline{\alpha}_\sharp\colon\Gamma=\Pi_1(\Gamma\setminus N, x)\to\Gamma=\Pi_1(\Gamma\setminus N,\overline{\alpha}(x))$ is, up to composition with an inner automorphism of Γ, precisely ψ.*

Remark 3.8. At this point, it is worthwhile to indicate that [7, Proposition 3.5], which is crucially used at other places in the work of Franks (e.g. in the basis theorem [7, Theorem 8.2] on which Gromov's result is based), is not correct. This proposition claims that for any covering $f\colon K\to K$, where $\Pi_1(K)$ is a finitely generated, torsion free and virtually nilpotent group, there exists an infra-nilmanifold M and an infra-nilmanifold endomorphism $g\colon M\to M$ which is Π_1–conjugate to f. This is not true (see the example below and the examples in the following sections) and one really also needs to consider affine endomorphisms of the infra-nilmanifolds as well. On the other hand, when $\Pi_1(K)$ is nilpotent (or abelian) the proposition is correct.

The problem in the alleged proof is situated at the very end of it on page 78. First of all, the wrong result of Auslander is used (but this can be solved by using Corollary 2.3). However, as indicated by the example above, the automorphism $\overline{g}$ (where I now use the notations of [7, p. 78]) does not necessarily induce a map on the infra-nilmanifold M (and even if it does, the induced map on the fundamental group is not necessarily the map g_*).

We finish this section by giving a counter-example to Franks' "Existence of a Model" – Proposition 3.5 in [7]. Consider the Klein Bottle K and choose a base point $x_0 \in K$. Then, the fundamental group $\Pi_1(K, x_0) \cong \Gamma = \langle a, b \mid ba = a^{-1}b \rangle$. It is easy to find a homeomorphism $f\colon K \to K$, with $f(x_0) = x_0$ and such that $f_\sharp\colon \Pi_1(K, x_0) \to \Pi_1(K, x_0)$ satisfies $f_\sharp(a) = a$ and $f_\sharp(b) = ab$. Now, consider any embedding of Γ into $\mathrm{Isom}(\mathbb{R}^2)$ as a discrete subgroup, then the translation subgroup of Γ will be $\Gamma \cap \mathbb{R}^2 = \langle a, b^2 \rangle \cong \mathbb{Z}^2$. Now, assume that $\overline{\varphi}\colon \Gamma\backslash\mathbb{R}^2 \to \Gamma\backslash\mathbb{R}^2$ is an infra-nilmanifold endomorphism (induced by the automorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$), which is Π_1-conjugate to f. This means that there is a commutative diagram

$$\begin{array}{ccc}
\Pi_1(K, x_0) & \xrightarrow{\Phi} & \Pi_1(\Gamma \setminus \mathbb{R}^2, \Gamma \cdot 0) \cong \Gamma \\
{\scriptstyle f_\sharp}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}_\sharp} \\
\Pi_1(K, x_0) & \xrightarrow[\Phi]{} & \Pi_1(\Gamma \setminus \mathbb{R}^2, \Gamma \cdot 0) \cong \Gamma
\end{array}$$

for some isomorphism Φ. As $f_\sharp(a) = a$ and $f_\sharp(b^2) = b^2$, it follows that $\overline{\varphi}_\sharp$ has to be the identity on the translation subgroup $\langle a, b^2 \rangle$ of Γ. But as the restriction of $\overline{\varphi}_\sharp$ to the translation subgroup is exactly the same as the restriction of φ to this translation subgroup, it follows that φ is the identity on this translation subgroup and hence φ is just the identity automorphism of $\mathbb{R}^2$. But this means that $\overline{\varphi}$ is the identity map also, hence $\overline{\varphi}_\sharp$ is the identity automorphism, which contradicts the commutativity of the diagram above.

4. An expanding map not topologically conjugate to an infra-nilmanifold endomorphism

Already on the smallest example of an infra-nilmanifold which is not a nilmanifold (or a torus) we can construct an expanding map which is not topologically conjugate to an infra-nilmanifold endomorphism of that infra-nilmanifold. Our example will be an affine endomorphism of the Klein Bottle. This example shows that there are problems with the proof of the geometric corollary on page 55 of [9], which we will explain below. Of course, this does not cast any doubt on the (very nice) main result of [9] stating that finitely generated groups of polynomial growth are virtually nilpotent!

For completeness, let us recall the definition of an expanding map.

Definition 4.1. Let M be a closed smooth manifold. A C^1-map $f\colon M \to M$ is an expanding map if there exist constants $C > 0$ and $\mu > 1$ such that

$$\|Df^n\| \geq C\mu^n \|v\|, \quad \text{for all } v \in TM,$$

for some (and hence any) Riemannian metric $\| \cdot \|$ on M.

To obtain the example mentioned above, we consider the Klein Bottle which is constructed by taking the group $\Gamma \subseteq \mathbb{R}^2 \rtimes \mathbb{Z}_2$, where

$$\mathbb{Z}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_2(\mathbb{R}). \tag{4.1}$$

The torsion free discrete and uniform subgroup Γ of $\mathbb{R}^2 \rtimes \mathbb{Z}_2$ we use to construct the Klein Bottle is generated by the following 2 elements:

$$a = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \quad \text{and} \quad b = \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

Note that a and b^2 generate the group of translations $\mathbb{Z}^2$. Let α be the affine map

$$\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2 : \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3x + \frac{1}{2} \\ 3y \end{pmatrix}.$$

So

$$\alpha = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) \in \mathrm{Aff}(\mathbb{R}^2).$$

One easily checks that

$$\alpha a \alpha^{-1} = a^3 \quad \text{and} \quad \alpha b \alpha^{-1} = a b^3$$

showing that $\alpha \Gamma \alpha^{-1} \subseteq \Gamma$. Hence α induces an affine endomorphism $\overline{\alpha}\colon \Gamma \backslash \mathbb{R}^2 \to \Gamma \backslash \mathbb{R}^2$ of the Klein bottle $K = \Gamma \backslash \mathbb{R}^2$. Moreover, as the linear part of α has only eigenvalues of modulus > 1, the map $\overline{\alpha}$ is an expanding map of the Klein bottle.

I claim that this map is not topologically conjugate to an expanding infra-nilmanifold endomorphism of this Klein Bottle.

To see this, suppose that $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear isomorphism inducing an endomorphism $\overline{\varphi}\colon \Gamma \backslash \mathbb{R}^2 \to \Gamma \backslash \mathbb{R}^2$ of the Klein bottle. By Theorem 3.4, we know that $\varphi \Gamma \varphi^{-1} \subseteq \Gamma$. From this, it also follows that $\varphi \mathbb{Z}_2 \varphi^{-1} = \mathbb{Z}_2$, where $\mathbb{Z}_2 \subseteq \mathrm{GL}(2, \mathbb{R})$ is as in (4.1). Hence,

$$\varphi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \varphi,$$

from which it follows that

$$\varphi = \begin{pmatrix} k & 0 \\ 0 & l \end{pmatrix}$$

for some $k, l \in \mathbb{R}$. Now, requiring that $\varphi a \varphi^{-1} \in \Gamma$ and $\varphi b \varphi^{-1} \in \Gamma$ leads to the condition that $k \in \mathbb{Z}$ and $l = 2m + 1$, for $m \in \mathbb{Z}$ (so l is odd).

As recalled in some detail in the discussion before Proposition 3.7, there is an isomorphism $\Pi_1(\Gamma \backslash \mathbb{R}^2, \overline{0}) \cong \Gamma$. Here, we use $\overline{0}$ to denote the image of the zero vector in the Klein Bottle $\Gamma \backslash \mathbb{R}^2$ and we use the zero vector as the point $\widetilde{n}_0$ (see discussion before Proposition 3.7) to establish the isomorphism between $\Pi_1(\Gamma \backslash \mathbb{R}^2, \overline{0})$ and Γ. From Proposition 3.7, we know that the map induced by $\overline{\varphi}$ is the same as conjugation with φ inside $\mathrm{Aff}(\mathbb{R}^2)$.

Now, suppose that $\overline{\alpha}$ is topologically conjugate to $\overline{\varphi}$, then there must exist a homeomorphism $h\colon \Gamma\backslash\mathbb{R}^2 \to \Gamma\backslash\mathbb{R}^2$, such that $h\circ\overline{\alpha} = \varphi\circ h$. Now, choose $h^{-1}(\overline{0})$ as another basepoint of $\Gamma\backslash\mathbb{R}^2$. It is obvious that $h^{-1}(\overline{0})$ is a fixed point of $\overline{\alpha}$. We know that we can also fix an isomorphism of $\Pi_1(\Gamma\backslash\mathbb{R}^2, h^{-1}(\overline{0}))$ with Γ and that under this identification the map $\overline{\alpha}_\sharp\colon \Gamma \to \Gamma$ is , up to an inner automorphism, exactly the same as conjugation with $\alpha \in \mathrm{Aff}(\mathbb{R}^2)$.

Using the above, we find a commutative diagram of groups and morphisms

$$\begin{array}{ccc} \Gamma & \xrightarrow{h_\sharp} & \Gamma \\ {\scriptstyle \overline{\alpha}_\sharp}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}_\sharp} \\ \Gamma & \xrightarrow[h_\sharp]{} & \Gamma \end{array}$$

This diagram leads to an induced diagram of morphisms on the abelianization of Γ:

$$\begin{array}{ccc} \Gamma/[\Gamma,\Gamma] & \xrightarrow{h_*} & \Gamma/[\Gamma,\Gamma] \\ {\scriptstyle \overline{\alpha}_*}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}_*} \\ \Gamma/[\Gamma,\Gamma] & \xrightarrow[h_*]{} & \Gamma/[\Gamma,\Gamma] \end{array}$$

We have that $\Gamma/[\Gamma,\Gamma] = \mathbb{Z}_2 \oplus \mathbb{Z}$, where $\mathbb{Z}_2$ (resp. $\mathbb{Z}$) is generated by the natural projection $\overline{a}$ of a (resp. $\overline{b}$ of b).

As $\overline{\alpha}_\sharp$ was, up to an inner automorphism of Γ, the same as conjugation with α inside $\mathrm{Aff}(\mathbb{R}^2)$, we know exactly what $\overline{\alpha}_*$ is, and we also already obtained some information on φ_*:

$$\overline{\alpha}_*(\overline{a}) = \overline{a}^3 = \overline{a}, \quad \overline{\alpha}_*(\overline{b}) = \overline{a}\overline{b}^3, \quad \overline{\varphi}_*(\overline{a}) = \overline{a}^l, \quad \overline{\varphi}_*(\overline{b}) = \overline{b}^{2m+1} \quad \text{with } l, m \in \mathbb{Z}.$$

As h is a homeomorphism of the Klein bottle, we know that h_* is an isomorphism of $\Gamma/[\Gamma,\Gamma]$. It follows that $h_*(\overline{a}) = \overline{a}$ while for $h_*(\overline{b})$ we have one of the following four possibilities:

$$h_*(\overline{b}) = \overline{b}, \quad h_*(\overline{b}) = \overline{b}^{-1}, \quad h_*(\overline{b}) = \overline{a}\overline{b} \quad \text{or} \quad h_*(\overline{b}) = \overline{a}\overline{b}^{-1}.$$

It is now easy to see that for none of these four possibilities, we can have that

$$h_* \circ \overline{\alpha}_* = \overline{\varphi}_* \circ h_*,$$

contradicting the fact that $h \circ \overline{\alpha} = \varphi \circ h$ and hence showing that $\overline{\varphi}$ is not topologically conjugate to $\overline{\alpha}$. □

This example indicates a real problem in the proof of the geometric corollary on page 55 of [9]. In fact, this geometric corollary follows from Gromov's main result by applying [7, Theorem 8.3] and [24, Theorem 5] (or the equivalent [7, Theorem 8.2]). Now looking at the proof of [24, Theorem 5] (or [7, Theorem 8.2]) ones sees that actually the incorrect "Existence of a Model" – proposition of

Franks is used (see Remark 3.8). In fact, both Shub and Franks are claiming that an expanding map on an infra-nilmanifold is topologically conjugate to an expanding infra-nilmanifold endomorphism of the same infra-nilmanfiold, which is actually wrong by the example above.

However, in the sequel of this section, we will show that any expanding map of a given infra-nilmanifold is topologically conjugate to an expanding affine endomorphism of the same infra-nilmanifold, from which it will follow that any expanding map of a compact manifold M will be topologically conjugate to an expanding affine infra-nilmanifold endomorphism of any infra-nilmanifold with the same fundamental group as M.

In order to prove this result, we need some more results concerning affine maps of infra-nilmanifolds. Let N be a connected and simply connected nilpotent Lie group, $\delta \in \mathrm{Aut}(N)$ and $d \in N$. Then $D = (d, \delta)$ is an affine map of N. As $\delta \in \mathrm{Aut}(N)$, we know that its differential $\delta_* \in \mathrm{Aut}(\mathfrak{n})$, where $\mathfrak{n}$ is the Lie algebra of N. When we talk about the eigenvalues of D (or the eigenvalues of δ) we will mean the eigenvalues of δ_*.

Lemma 4.2. *Let N be a connected and simply connected nilpotent Lie group and $D \in \mathrm{Aff}(N)$. If 1 is not an eigenvalue of D, then there is a unique fixed point $n_0 \in N$ for the affine map D.*

Proof. This is a special case of Lemma 2 in [3]. □

Now, consider a finitely generated and torsion free nilpotent group Λ and an injective endomorphism $\varphi \in \mathrm{Aut}(\Lambda)$. Up to isomorphism there is a unique connected and simply connected nilpotent Lie group N, containing Λ as a uniform discrete subgroup. This N is called the Mal'cev completion of Λ. The endomorphism φ extends uniquely to a continuous automorphism $\widetilde{\varphi} \in \mathrm{Aut}(N)$ and we can talk about the eigenvalues of φ, by which we will mean the eigenvalues of $\widetilde{\varphi}$ (which in their turn are the eigenvalues of the differential $\widetilde{\varphi}_* \in \mathrm{Aut}(\mathfrak{n})$ of $\widetilde{\varphi}$).

More generally, we can consider as before a torsion free uniform discrete subgroup $\Gamma \subseteq N \rtimes F$, where N is a connected and simply connected nilpotent Lie group and F is a finite subgroup of $\mathrm{Aut}(N)$. We assume that F is the holonomy group of Γ. We know that $\Lambda = \Gamma \cap N$ is a uniform discrete subgroup of N and so N is the Mal'cev completion of Λ. Let $\varphi\colon \Gamma \to \Gamma$ be an injective endomorphism of Γ. It follows from Corollary 2.3 that φ extends uniquely to an automorphism of $N \rtimes F$ and restricts to an injective endomorphism of Λ. We define the eigenvalues of φ to be the eigenvalues of the restriction of φ to Λ.

On the other hand, we know that φ can also be realized as conjugation by some element $D = (d, \delta)$ in $\mathrm{Aff}(N)$. It turns out that the eigenvalues of φ are exactly the same as the eigenvalues of D.

Lemma 4.3. *Let N be a connected and simply connected nilpotent Lie group and let F be a finite subgroup of* $\mathrm{Aut}(N)$. *Assume that Γ is a uniform discrete subgroup of $N \rtimes F$ with holonomy group F, φ is an injective endomorphism of Γ and that $D = (d, \delta) \in \mathrm{Aff}(N)$ realizes this endomorphism via conjugation in* $\mathrm{Aff}(N)$*:*

$$\varphi(\gamma) = (d, \delta)\gamma(d, \delta)^{-1} \quad \text{for all } \gamma \in \Gamma.$$

Then, the set of eigenvalues of φ is exactly the same as the set of eigenvalues of D.

Proof. To compute the eigenvalues of φ, we have to find the eigenvalues of the induced automorphism $\widetilde{\varphi}$ of N (obtained by first extending φ to $N \rtimes F$ and then taking the restriction to N). But this automorphism is also obtained by conjugation with D:

$$\widetilde{\varphi}(n) = DnD^{-1} = (d, \delta)n(d, \delta)^{-1} = d\delta(n)d^{-1} = (\mu(d) \circ \delta)(n) \quad \text{for all } n \in N.$$

where $\mu(d)$ denotes conjugation with $d \in N$. It follows that the eigenvalues of φ are precisely the same as the eigenvalues of $\mu(d) \circ \delta$. It is a standard argument to show that an inner automorphism of a nilpotent Lie group has no influence on the eigenvalues: indeed, to find the eigenvalues of a given automorphism $\psi \in \mathrm{Aut}(N)$, we can consider the filtration of N by the terms of its lower central series (which goes to 1 as N is nilpotent)

$$N = \gamma_1(N) \supseteq \gamma_2(N) \supseteq \ldots \supseteq \gamma_i(N) \supseteq \gamma_{i+1}(N) = [\gamma_i(N), N] \supseteq \ldots \supseteq \gamma_c(N) = 1.$$

Each term in this filtration is invariant under ψ and analogously the corresponding terms of the lower central series of the Lie algebra $\mathfrak{n}$ of N:

$$\mathfrak{n} = \gamma_1(\mathfrak{n}) \supseteq \gamma_2(\mathfrak{n}) \supseteq \ldots \supseteq \gamma_i(\mathfrak{n}) \supseteq \gamma_{i+1}(\mathfrak{n}) = [\gamma_i(\mathfrak{n}), \mathfrak{n}] \supseteq \ldots \supseteq \gamma_c(\mathfrak{n}) = 1$$

are then invariant under the differential ψ_* of ψ. It follows that to find the eigenvalues of ψ, we have to find the eigenvalues of the induced automorphism on each quotient $\gamma_i(\mathfrak{n})/\gamma_{i+1}(\mathfrak{n})$. However, an inner automorphism of N induces the identity on each quotient $\gamma_i(N)/\gamma_{i+1}(N)$ and so its differential induces the identity on $\gamma_i(\mathfrak{n})/\gamma_{i+1}(\mathfrak{n})$. It follows that δ and $\mu(d) \circ \delta$ induce the same linear map on each quotient $\gamma_i(\mathfrak{n})/\gamma_{i+1}(\mathfrak{n})$ and hence, they have the same eigenvalues.□

In what follows it will be crucial to know when an affine map does not have 1 as an eigenvalue (so that we will be able to apply Lemma 4.2). The following lemma can serve as a criterion for this.

Lemma 4.4. *Let N be a connected and simply connected nilpotent Lie group and Let F be a finite subgroup of* $\mathrm{Aut}(N)$. *Assume that Γ is a uniform discrete subgroup of $N \rtimes F$ with holonomy group F and φ is an injective endomorphism* *of* Γ*. If* φ *has* 1 *as an eigenvalue, then there exists a non-trivial subgroup* Δ *of* Γ *such that for all* $\gamma \in \Delta$ *such that* $\varphi(\gamma) = \gamma$*.*

Proof. Let $\Lambda = \Gamma \cap N$. As already argued above, φ restricts to an injective endomorphism of Λ and this restriction extends uniquely to an automorphism of N. We will use the same symbol φ to denote all these endomorphisms. Recall that for nilpotent Lie groups, the exponential map $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism ($\mathfrak{n}$ is the Lie algebra of N) and we denote its inverse by log. Consider now $\mathfrak{n}_{\mathbb{Q}} = \mathbb{Q}\log(\Lambda)$ (the rational span of $\log(\Lambda)$) and $N_{\mathbb{Q}} = \exp(\mathfrak{n}_{\mathbb{Q}})$. The vector space $\mathfrak{n}_{\mathbb{Q}}$ is a rational Lie algebra and the differential φ_* of φ restricts to an automorphism of $\mathfrak{n}_{\mathbb{Q}}$. For more details about this and following facts on these rational Lie algebras, we refer to [22, Chapter 6]. As ψ_* has 1 as an eigenvalue, there exists a nonzero vector $X \in \mathfrak{n}_{\mathbb{Q}}$ with $\varphi_*(X) = X$. This implies that $1 \neq x = \exp(X) \in N_{\mathbb{Q}}$ is an element with $\varphi(x) = x$. Now, $N_{\mathbb{Q}}$ is the radicable hull (see [22, p.107]) of Λ, and so there exists a positive integer $k > 0$ such that $1 \neq x^k \in \Lambda$. It follows that x^k is a nontrivial element of Λ with $\varphi(x^k) = x^k$. The proof now finishes by taking Δ to be the group generated by x^k. □

We are now ready to prove the main result of this paper in which we will adopt J. Franks' original approach for infra-nilmanifold endomorphisms [7, Section 8] to the more general case of affine endomorphisms.

Theorem 4.5. *Let* $f\colon M \to M$ *be an expanding map of a compact manifold* M*. Then, there exists an infra-nilmanifold* $\Gamma \setminus N$ *whose fundamental group* Γ *is isomorphic to* $\Pi_1(M)$*. And for any such* $\Gamma \setminus N$*, there exists an expanding affine endomorphism of that infra-nilmanifold which is topologically conjugate to* f*.*

Proof. By [23, Theorem 1] we can choose a fixed point $m_0 \in M$ of f. From [7, Theorem 8.3] we know that $\Pi_1(M, m_0)$ has polynomial growth and so by the main result of [9] it follows that $\Pi_1(M, m_0)$ has a nilpotent subgroup of finite index. Moreover, by [23, Proposition 3], we know that $\Pi_1(M, m_0)$ is torsion free, M is a $K(\Pi_1(M, m_0), 1)$-space and the induced map $f_\sharp\colon \Pi_1(M, m_0) \to \Pi_1(M, m_0)$ is an injective endomorphism.

Every finitely generated torsion free virtually nilpotent group can be realized as a uniform and discrete subgroup of a semi-direct product $N \rtimes F$, where N is a connected and simply connected nilpotent Lie group and F is a finite subgroup of $\mathrm{Aut}(N)$ (e.g. [4, Theorem 3.1.3]). Fix such an embedding $i\colon \Pi_1(M, m_0) \to N \rtimes F$ realizing $\Pi_1(M, m_0)$ as such a uniform discrete subgroup and denote $\Gamma = i(\Pi_1(M, m_0))$. Without loss of generality we assume that F is the holonomy group of Γ. So there is an isomorphism $A\colon \Gamma \to \Pi_1(M, m_0)$ (where A is in fact the inverse of i), already showing the existence of the infra-nilmanifold $\Gamma \setminus N$.

We continue our proof with a fixed choice of such an infra-nilmanifold. Let $B = A^{-1} \circ f_\sharp \circ A$, then B is an injective endomorphism of Γ and so there exists an affine map $\alpha = (d, \delta) \in \mathrm{Aff}(N)$ with $B(\gamma) = \alpha\gamma\alpha^{-1}$, for all $\gamma \in \Gamma$ (see Theorem 2.2 and Corollary 2.3). By [23, Corollary 1] we know that the identity element is the unique fixed element of $f_\sharp$ and so the identity element is also the only fixed point for B. By Lemma 4.4 it follows that α does not have 1 as one of its eigenvalues and so, by Lemma 4.2 there exists a unique fixed point $\widetilde{n}_0 \in N$ for α. Let n_0 be the corresponding point in the infra-nilmanifold $\Gamma\backslash N$ and use $\widetilde{n}_0$ to identify the fundamental group $\Pi_1(\Gamma\backslash N, n_0)$ with Γ. By the discussion before Proposition 3.7, we know that α induces an affine endomorphism $\overline{\alpha}$ of $\Gamma\backslash N$, with n_0 as a fixed point, and that the induced endomorphism $\overline{\alpha}_\sharp$ of $\Pi_1(\Gamma \backslash N, n_0) = \Gamma$ is exactly B. We therefore have a commutative diagram

$$\begin{array}{ccc} \Pi_1(\Gamma \backslash N, n_0) = \Gamma & \xrightarrow{A} & \Pi_1(M, m_0) \\ \Big\downarrow{\scriptstyle \overline{\alpha}_\sharp} & & \Big\downarrow{\scriptstyle f_\sharp} \\ \Pi_1(\Gamma \backslash N, n_0) = \Gamma & \xrightarrow[A]{} & \Pi_1(M, m_0) \end{array}$$

By [23, Theorem 4] there exists a unique continuous map $h\colon (\Gamma\backslash N, n_0) \to (M, m_0)$ with $f \circ h = h \circ \overline{\alpha}$ and for which $h_\sharp\colon \Pi_1(\Gamma \backslash N, n_0) \to \Pi_1(M, m_0)$ is exactly A. (As usual, by a map $g\colon (X, x) \to (Y, y)$ we mean a map from the space X to the space Y, with $g(x) = y$ where x and y are given points of X and Y respectively). As A is an isomorphism, h is a homotopy equivalence, since we are working with $K(\Pi, 1)$-spaces.

Let $\widetilde{M}$ denote the universal covering space of M, with covering projection $p_M\colon \widetilde{M} \to M$ and let $\widetilde{m}_0 \in \widetilde{M}$ be a point with $p_M(\widetilde{m}_0) = m_0$. Now, consider the unique lift $\widetilde{h}\colon (N, \widetilde{n}_0) \to (\widetilde{M}, \widetilde{m}_0)$ of h and the unique lift $\widetilde{f}\colon (\widetilde{M}, \widetilde{m}_0) \to (\widetilde{M}, \widetilde{m}_0)$ of f, then

$$\widetilde{f} \circ \widetilde{h} = \widetilde{h} \circ \alpha.$$

Let $L_{\widetilde{n}_0}\colon N \to N$, $x \mapsto \widetilde{n}_0\, x$ denote left translation by $\widetilde{n}_0$ in N. As $\widetilde{n}_0$ is a fixed point of $\alpha = (d, \delta)$, we have that $L_{\widetilde{n}_0} \circ \delta = \alpha \circ L_{\widetilde{n}_0}$. Summarizing the above, we obtain the following commutative diagram of maps and spaces in which exp and $L_{\widetilde{n}_0}$ are diffeomorphisms.

$$\begin{array}{ccccccc} \mathfrak{n} & \xrightarrow{\exp} & N & \xrightarrow{L_{\widetilde{n}_0}} & N & \xrightarrow{\widetilde{h}} & \widetilde{M} \\ \Big\downarrow{\scriptstyle \delta_*} & & \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \widetilde{f}} \\ \mathfrak{n} & \xrightarrow[\exp]{} & N & \xrightarrow[L_{\widetilde{n}_0}]{} & N & \xrightarrow[\widetilde{h}]{} & \widetilde{M} \end{array}$$

Let $k = \widetilde{h} \circ L_{\widetilde{n}_0} \circ \exp$. By [7, Lemma 3.4], the map $\widetilde{h}$ and hence also k is a proper map. We can now continue as in Franks' paper to show that δ_* only has eigenvalues of modulus > 1.

From $\widetilde{f} \circ k = k \circ \delta_*$ it immediately follows that $\widetilde{f}^n \circ k = k \circ \delta_*^n$. Now, assume that δ_* has an eigenvalue of modulus ≤ 1. It then follows that there exists a non-zero element $x \in \mathfrak{n}$ with $\|\delta_*^n(x)\| \leq \|x\|$ (where $\|\cdot\|$ denotes a chosen norm on $\mathfrak{n}$). (Note that the argument given in [7] is not completely correct, because he considers an eigenvector of the corresponding eigenvalue of modulus ≤ 1. However this eigenvalue can be complex and a corresponding eigenvector does not have to exist in the real Lie algebra $\mathfrak{n}$. It is however not difficult to see that also in this case, we can find an x as claimed). As $\widetilde{f}$ is expanding (see [23, Lemma 6]), we have that $\widetilde{f}^n(m)$ tends to infinity as n goes to infinity for all $m \in M$ which are not equal to the (unique) fixed point $\widetilde{m}_0$ of $\widetilde{f}$. As $\widetilde{f}^n(k(x)) = k(\delta_*^n(x))$, this implies that $k(x) = \widetilde{m}_0$. Moreover, the same argument applies to any point of the form $rx \in \mathfrak{n}$. Hence, the whole line $\mathbb{R}x$ is mapped onto the point $\widetilde{m}_0$ by k, which contradicts the fact that k is a proper map. So, the assumption that there exists an eigenvalue of modulus ≤ 1 is wrong. This shows that δ_* is an expanding linear map and hence $\overline{\alpha}$ is an expanding affine endomorphism of the infra-nilmanifold $\Gamma \setminus N$.

Now, since we have the information that $\overline{\alpha}$ is expanding, we can apply [23, Theorem 5] to conclude that h is actually a homeomorphism and hence $\overline{\alpha}$ and f are topologically conjugate. □

Note that in the above theorem it did not matter in which way we realised the fundamental group Γ as a uniform discrete subgroup of $N \rtimes F$. It turns out that if we choose the embedding in a good way (depending on the expanding map f!) we can recover completely Gromov's result.

Theorem 4.6. *Let $f\colon M \to M$ be an expanding map of a compact manifold M, then f is topologically conjugate to an expanding infra-nilmanifold endomorphism.*

Proof. We already know that f is topologically conjugate to an expanding affine endomorphism $\overline{\alpha}$ of an infra-nilmanifold $\Gamma \setminus N$, by Theorem 4.5. So it is enough to show that this affine infra-nilmanifold endomorphism is topologically conjugate to an expanding infra-nilmanifold endomorphism of a possibly other infra-nilmanifold.

Let $\alpha = (d, \delta) \in \mathrm{Aff}(N)$ be a lift of $\overline{\alpha}$, hence $\alpha\Gamma\alpha^{-1} \subseteq \Gamma$. As $\overline{\alpha}$ is expanding, the map $\alpha\colon N \to N$ has a fixed point, say x_0. Now let $h\colon N \to N\colon n \mapsto x_0^{-1}n$ and $\Gamma' = x_0^{-1}\Gamma x_0 \subseteq \mathrm{Aff}(N)$. Then $\Gamma' \setminus N$ is also an infra-nilmanifold (with $\Gamma' \cong \Gamma$) and h determines a homeomorphism $\overline{h}\colon \Gamma \setminus N \to \Gamma' \setminus N\colon \Gamma \cdot n \mapsto \Gamma' \cdot x_0^{-1}n$. One also easily checks that $\delta\Gamma'\delta^{-1} \subseteq \Gamma'$ so that δ induces an expanding infra-nilmanifold endomorphism $\overline{\delta}$ of $\Gamma' \setminus N$, for which the following diagram commutes:

$$\begin{array}{ccc} \Gamma \setminus N & \xrightarrow{\overline{h}} & \Gamma' \setminus N \\ \overline{\alpha}\downarrow & & \downarrow\overline{\delta} \\ \Gamma \setminus N & \xrightarrow[\overline{h}]{} & \Gamma' \setminus N \end{array}.$$

This shows that $\overline{\alpha}$, and hence also f, is topologically conjugate to the expanding infra-nilmanifold endomorphism $\overline{\delta}$. □

Remark 4.7. We want to stress the fact here that the infra-nilmanifold which is obtained in the theorem does not only depend on M, but also on the expanding map f itself.

5. An Anosov diffeomorphism not topologically conjugate to an infra-nilmanifold automorphism

Analogously as in the previous section, we will show that there exists an infra-nilmanifold $M = \Gamma \setminus N$ and an Anosov diffeomorphism $f\colon M \to M$ which is not topologically conjugate to an infra-nilmanifold automorphism of M.

For this example, we will use a 4-dimensional flat manifold. Again the holonomy group of the corresponding Bieberbach group will be $\mathbb{Z}_2$, where we embed $\mathbb{Z}_2$ as the subgroup $\{\mathbb{I}_4, L_f\} \subseteq \mathrm{GL}_2(\mathbb{R})$, where $\mathbb{I}_4$ is the 4×4 identity matrix and

$$L_f = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, let Γ be the torsion free, discrete and uniform subgroup of $\mathbb{R}^4 \rtimes \mathbb{Z}_2$ generated by

$$a = (e_1, \mathbb{I}_4), \quad b = (e_2, \mathbb{I}_4), \quad c = (e_3, \mathbb{I}_4), \quad d = (e_4, \mathbb{I}_4), \quad f = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, L_f \right),$$

where e_i is the standard basis vector with a 1 on the i-th spot and 0 elsewhere. It follows that Γ is a Bieberbach group, with translation subgroup $\mathbb{Z}^4$ generated by a, b, c and d.

We consider the affine map

$$\alpha\colon \mathbb{R}^4 \to \mathbb{R}^4 : \quad \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \begin{pmatrix} 13x + 8y + \frac{1}{2} \\ 8x + 5y + \frac{1}{2} \\ 13z + 8t \\ 8z + 5t \end{pmatrix}.$$

So

$$\alpha = \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 13 & 8 & 0 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 0 & 13 & 8 \\ 0 & 0 & 8 & 5 \end{pmatrix} \right) \in \mathrm{Aff}(\mathbb{R}^4).$$

One can compute that

$$\begin{aligned} \alpha a\, \alpha^{-1} &= a^{13}b^8, & \alpha b \alpha^{-1} &= a^8 b^5, & \alpha c \alpha^{-1} &= c^{13}d^8, \\ \alpha d \alpha^{-1} &= c^8 d^5, & \alpha f \alpha^{-1} &= abc^{10}d^6 f. \end{aligned}$$

From this, one can see that $\alpha\Gamma\alpha^{-1} = \Gamma$ and hence, α induces a diffeomorphism $\overline{\alpha}$ on $\Gamma \setminus \mathbb{R}^4$. Moreover, as the linear part of α only has eigenvalues of modulus different than 1, $\overline{\alpha}$ is an Anosov diffeomorphism. We will show that this Anosov diffeomorphism is not topologically conjugate to an infra-nilmanifold automorphism of $\Gamma \setminus \mathbb{R}^4$. Suppose on the contrary that $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear automorphism inducing a map $\overline{\varphi}$ on $\Gamma \setminus \mathbb{R}^4$ which is topologically conjugate to $\overline{\alpha}$. We have seen that in this case $\varphi\Gamma\varphi^{-1} = \Gamma$ and $\varphi\mathbb{Z}_2\varphi^{-1} = \mathbb{Z}_2$, which now implies that the matrix representation of φ is of the form:

$$\varphi = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{with } A, B \in \mathrm{GL}_2(\mathbb{Z}),$$

where we also used that $\varphi\mathbb{Z}^4\varphi^{-1} = \mathbb{Z}^4$. The matrix form of φ implies that

$$\varphi f \varphi^{-1} = c^k d^l f \quad \text{for some } k, l \in \mathbb{Z}.$$

The fact that we suppose that $\overline{\varphi}$ is topologically conjugate to $\overline{\alpha}$ implies the existence of a homeomorphism $h\colon \Gamma \setminus \mathbb{R}^4 \to \Gamma \setminus \mathbb{R}^4$ with $\overline{\alpha} = h^{-1} \circ \overline{\varphi} \circ h$. Let $\overline{\alpha}_\sharp, \overline{\varphi}_\sharp$ and $h_\sharp$ denote the induced maps on the fundamental group Γ of $\Gamma \setminus \mathbb{R}^4$. Then, we know that, up to an inner conjugation of Γ, $\overline{\alpha}_\sharp$ resp. $\overline{\varphi}_\sharp$ is the same as conjugation with α, resp. φ in $\mathrm{Aff}(\mathbb{R}^4)$ and $h_\sharp(\mathbb{Z}^4) = \mathbb{Z}^4$. We already remark here that we will be dividing out by the derived subgroup of Γ in a moment, so that without any problems we can forget about the possible inner conjugations.

From $\overline{\alpha} = h^{-1} \circ \overline{\varphi} \circ h$, it follows that $\overline{\alpha}_\sharp = h_\sharp^{-1} \circ \overline{\varphi}_\sharp \circ h_\sharp$. We claim that this condition leads to a contradiction. To easily see this, note that the derived subgroup of Γ is $[\Gamma, \Gamma] = \mathrm{grp}\{a^2, b^2\}$ and the centre of Γ is $Z(\Gamma) = \mathrm{grp}\{c, d\}$. So $Z(\Gamma)[\Gamma, \Gamma]$ is a normal subgroup of Γ and

$$\Gamma / Z(\Gamma)[\Gamma, \Gamma] = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

where we view the first $\mathbb{Z}_2$ factor as being generated by $\overline{a}$, the second factor by $\overline{b}$ and the last one by $\overline{f}$, where $\overline{a}$, $\overline{b}$, $\overline{f}$ denote the images of a, b, f under the natural projection $\Gamma \to \Gamma / Z(\Gamma)[\Gamma, \Gamma]$. Any automorphism of Γ induces an automorphism of $\Gamma/Z(\Gamma)[\Gamma,\Gamma]$, which can also be seen as a linear map of the 3-dimensional vector space $\mathbb{Z}_2^3$ over the field $\mathbb{Z}_2$. So, we can represent the induced automorphism on $\Gamma/Z(\Gamma)[\Gamma,\Gamma]$ by means of a matrix in $\mathrm{GL}_3(\mathbb{Z}_2)$.

From the conjugation relations given above, we see that $\overline{\alpha}_\sharp(\overline{a}) = \overline{a}^{13}\overline{b}^8 = \overline{a}$, $\overline{\alpha}_\sharp(\overline{b}) = \overline{b}$ and $\overline{\alpha}_\sharp(\overline{f}) = \overline{a}\,\overline{bf}$, so the corresponding matrix in $\mathrm{GL}_3(\mathbb{Z}_2)$ is

$$M_\alpha = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Analogously, one can see that the matrix representations of the linear automorphisms induced by $\overline{\varphi}_\sharp$ and $h_\sharp$ are of the form

$$M_\varphi = \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_h = \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_4 & h_6 \\ 0 & 0 & 1 \end{pmatrix}$$

where the a_i are obtained by reducing the entries of A modulo 2. Now, the relation $\overline{\alpha}_\sharp = h_\sharp^{-1} \circ \overline{\varphi}_\sharp \circ h_\sharp$ implies that $M_\alpha = M_h^{-1} M_\varphi M_h$. By focussing on the upper left 2×2 corner, one immediately gets that $M_\varphi = \mathbb{I}_3$. But this then implies that also $M_\alpha = \mathbb{I}_3$ which is clearly a contradiction.

This example casts a lot of doubts on the main result of [18] (Theorem C). Note that [18] does not really contain a proof for Theorem C, but refers to the proof of Franks' Theorem for Anosov diffeormorphisms on tori [6, Theorem 1]. There is, up to my knowledge, indeed nothing wrong with [6, Theorem 1] or its proof, but to be able to generalize this to the class of infra-nilmanifolds, it is assumed in [18] (see the sentence immediately after the statement of Theorem A on page 423) that each homotopy class of maps from an infra-nilmanifold to itself inducing a hyperbolic automorphism of the fundamental group, contains a hyperbolic infra-nilmanifold automorphism. In [18], the author refers to the wrong result of Auslander for this, but even with the use of Corollary 2.3 of the current paper, the claim does not follow.

In fact, the example above shows that this is not correct and one really needs also to consider hyperbolic affine automorphisms! Of course, an affine automorphism $\overline{\alpha}$ is hyperbolic if α (or the linear part of α) does not have any eigenvalue of modulus 1.

Unfortunately, I have not been able to give an alternative proof for the analogous version of [18, Theorem C] for the case of affine automorphisms. So, we are left with the following open question:

QUESTION. Let $f\colon M \to M$ be an Anosov diffeomorphism of an infra-nilmanifold. Is it true that f is topologically conjugate to a hyperbolic affine automorphism of the infra-nilmanifold M?

It is very tempting to believe that the answer to this question is indeed positive. In fact, for nilmanifolds, the arguments of A. Manning in [18] are correct (every map on a nilmanifold is homotopic to a nilmanifold endomorphism) and so a correct partial version of [18, Theorem C] is

Theorem 5.1. *Any Anosov diffeomorphism of a nilmanifold M is topologically conjugate to a hyperbolic nilmanifold automorphism.*

So for nilmanifolds there is no need to consider affine maps (this is also true for expanding maps).

More generally one can even ask whether or not it is true that an Anosov diffeomorphism on any given compact manifold M is conjugate to a hyperbolic affine automorphism of an infra-nilmanifold. For this, it would be very useful to have a generalization of [7, Theorem 2.1] to the case of hyperbolic affine automorphisms. However, the proof of [7, Theorem 2.1] is very dependent on the fact that the lift of an infra-nilmanifold automorphism is really an automorphism of the covering Lie group and it seems rather impossible to generalize this approach to the case of affine automorphisms.

Recently, there has been a lot of interest in the existence question of Anosov diffeomorphisms on infra-nilmanifolds (e.g. [5], [8], [13], [14], [16], [17], [21]). Often, one refers to [18, Theorem C] to reduce the question to a pure algebraic question. Luckily, in case one is only dealing with nilmanifolds, there is by the above theorem no problem at all. On the other hand, for infra-nilmanifolds one has to be a bit more careful. However, for the existence question, there is not really a problem.

Theorem 5.2. *Let M be an infra-nilmanifold. Then the following are equivalent:*

(a) *M admits an Anosov diffeomorphism,*
(b) *M admits a hyperbolic affine automorphism,*
(c) *M admits a hyperbolic infra-nilmanifold automorphism,*

Proof. The implications (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) are obviously true, so we only have to show (a) $\Rightarrow$ (c).

Let $M = \Gamma\backslash N$ where N is a connected simply connected nilpotent Lie group and Γ is a uniform discrete subgroup of $N \rtimes F$ where F is a finite subgroup of $\mathrm{Aut}(N)$. We also assume that F is the holonomy group of Γ. Moreover, as is explained in [5, Section 3], we can assume that any element of F restricts to an automorphism of $N_{\mathbb{Q}}$ (see [5] or the proof of Lemma 4.4 for the meaning of $N_{\mathbb{Q}}$) and that Γ is actually a subgroup of $N_{\mathbb{Q}} \rtimes F$ (which we called a rational realization in [5]).

Assume that $f\colon M \to M$ is an Anosov diffeomorphism. By [18, Theorem A] f induces a hyperbolic automorphism

$$f_\sharp\colon \Pi_1(M, m_0) \cong \Gamma \to \Pi_1(M, f(m_0)) \cong \Gamma.$$

We recall here that for different choices of isomorphisms of $\Pi_1(M, x)$ with Γ the induced map $f_\sharp\colon \Gamma \to \Gamma$ will change by an inner automorphism of Γ. Anyhow, the existence of an Anosov diffeomorphism of M implies the existence of a hyperbolic automorphism $\varphi = f_\sharp$ of Γ. In the second part of the proof of Theorem A in [5, p. 564], we show that for some positive power φ^k there exists a $\psi \in \mathrm{Aut}(N)$ such that φ^k is just conjugation by $\psi \in \mathrm{Aff}(N)$:

$$\varphi^k(\gamma) = (1, \psi)\gamma(1, \psi)^{-1} \quad \text{for all } \gamma \in \Gamma.$$

As φ is a hyperbolic, the same holds for φ^k and hence also for ψ (Lemma 4.3). It follows that $\Psi\colon N \rtimes F \to N \rtimes F$, $x \mapsto (1, \psi)x(1, \psi)^{-1}$ is an automorphism of $N \rtimes F$ with $\Psi(F) = F$ and $\Psi(\Gamma) = \Gamma$. Hence, Ψ determines a hyperbolic infra-nilmanifold automorphism of $\Gamma \setminus N$. □

Actually, the proof given above also shows the following

Theorem 5.3. *An infra-nilmanifold M admits an Anosov diffeomorphism if and only if $\Pi_1(M)$ admits a hyperbolic automorphism.*

Moreover, we also showed that for any Anosov diffeomorphism f on a given infra-nilmanifold M, there is some positive power f^n of f such that f^n is homotopic to an infra-nilmanifold endomorphism of M. Actually, this is true for any homeomorphism of an infra-nilmanifold. We note here that this does not hold for expanding maps.

As a conclusion of this paper we can state that in the study of selfmaps of a given infra-nilmanifold, which play a crucial role in the theory of expanding maps and Anosov diffeomorphisms, the class of infra-nilmanifold endomorphisms is just not rich enough to contain at least one map from each homotopy class and one really should consider the more general class of affine endomorphisms on that infra-nilmanifold.

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