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PLANAR NONAUTONOMOUS POLYNOMIAL EQUATIONS III. ZEROS OF THE VECTOR FIELD

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Abstract. We give a few sufficient conditions for the existence of periodic solutions of the equation $\dot{z} = \sum_{j=0}^{n} a_j(t)z^j$. We prove the existence of one up to n periodic solutions and heteroclinic ones.

1. Introduction

The presented paper is a continuation of [27]–[29]. We study a planar nonautonomous differential equations of the form

$$\dot{z} = v(t, z) = v_t(z) = \sum_{j=0}^{n} a_j(t)z^j, \tag{1.1}$$

where $n \geq 2$ and $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ are T-periodic. Our main tool is the topological method of isolating segments.

An extensive study of the set of periodic solutions of the equation (1.1) was initiated in [21] and continued in many papers e.g. [1]–[3], [8]–[14], [16], [19], [20], [22]. In those papers the coefficients a_j are real. One of the most important problem is to examin the structure of the set of periodic solutions. The second one is the investigation of a centre which is motivated by the Poincaré centre–focus problem. The third one is connected with the XVIth Hilbert problem for

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degree two equations in the plane which can be reduced to the problem of finding the maximal number of closed solutions of the equation (1.1) with $n = 3$ and special coefficients a_j. This leads to investigations of the maximal number of periodic solutions of (1.1). It is proved in [15] that in the general case there is no upper bound for this number provided that $n \geq 3$.

The complex coefficients are considered in [4], [5] and a few sufficient conditions for the existence of periodic solutions are presented. The upper bound of the number of periodic solutions and structure of the centre variety is considered in [7]. The problem of nonexistence of periodic solutions is investigated in [31] where it is proved that there exist coefficients a_j such that the equation (1.1) has no periodic solutions.

In the presented paper we develop the ideas from [27]–[29] and give a few sufficient conditions for the existence of one up to n periodic solutions. We seek periodic solutions close to the branches of the vector field v. We try to construct tubes containing the branches. Then we examine behaviour of the vector field $(1, v)$ on the boundary of the tubes. If in every point of the sides the vector field $(1, v)$ points outward (inward) the tube, then there exists a periodic solution inside it. These tubes are simple examples of isolating segments (see [24]–[26] for the notion of isolating segments). By the special properties of holomorphic functions we can use the Denjoy–Wolff fixed point theorem (cf. [6], [23]) instead of the Brouwer one. It allows us to obtain asymptotic unstability or asymptotic stability of detected periodic solutions. Moreover, they are attracting or repelling in the whole tube, which leads to the heteroclinic solutions connecting periodic ones.

Dealing with polynomial equations leads (as in [17]) to analytic case i.e. presented method is valid also in holomorphic case. That is why we formulate and prove main theorems of the paper (Theorems 3.1, 4.2, 4.5) in the case of holomorphic vector field.

The paper is organised as follows. In Section 2 we give definitions and introduce notion. The next section is devoted to (1.1) with branches of zeros of v of multiplicity one. We deal also with the special equation of polynomial type. In Section 4 we focus on the branches of multiplicity greater then one. We also treat polynomial equation of a special type more carefully.

2. Definitions

2.1. Processes. Let X be a topological space and $\Omega \subset \mathbb{R} \times X \times \mathbb{R}$ be an open set.

By a *local process* on X we mean a continuous map $\varphi\colon \Omega \to X$, such that three conditions are satisfied:

(a) $I_{(\sigma,x)} = \{t \in \mathbb{R} : (\sigma, x, t) \in \Omega\}$ is an open interval containing 0, for every $\sigma \in \mathbb{R}$ and $x \in X$,
(b) $\varphi(\sigma, \cdot\,, 0) = \mathrm{id}_X$, for every $\sigma \in \mathbb{R}$,
(c) $\varphi(\sigma, x, s+t) = \varphi(\sigma+s, \varphi(\sigma, x, s), t)$, for every $x \in X$, $\sigma \in \mathbb{R}$ and $s, t \in \mathbb{R}$ such that $s \in I_{(\sigma,x)}$ and $t \in I_{(\sigma+s,\varphi(\sigma,x,s))}$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma, x, t)$.

Let M be a smooth manifold and let $v: \mathbb{R} \times M \to TM$ be a time-dependent vector field. We assume that v is so regular that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t, x), \tag{2.1}$$

$$x(t_0) = x_0 \tag{2.2}$$

has unique solution. Then the equation (2.1) generates a local process φ on X by $\varphi_{(t_0,t)}(x_0) = x(t_0, x_0, t + t_0)$, where $x(t_0, x_0, \cdot\,)$ is the solution of the Cauchy problem (2.1), (2.2).

Let T be a positive number. In the sequel T denotes the period. We assume that v is T-periodic in t. It follows that the local process φ is T-periodic, i.e.

$$\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)} \quad \text{for all } \sigma,\ t \in \mathbb{R},$$

hence there is a one-to-one correspondence between T-periodic solutions of (2.1) and fixed points of the Poincaré map $\varphi_{(0,T)}$.

2.2. Periodic isolating segments. Let X be a topological space. We assume that φ is a T-periodic local process on X.

For any set $Z \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$ we put $Z_t = \{x \in X : (t, x) \in Z\}$.

Let $\pi_1: \mathbb{R} \times X \to \mathbb{R}$ be the projection on the time variable.

We call a compact set $W \subset [a, b] \times X$ an *isolating segment over* $[a, b]$ *for* φ if the exit and entrance sets W^-, W^+ of W are also compact and there exist compact subsets $W^{--}, W^{++} \subset W$ (called, respectively, the *proper exit set* and the *proper entrance set*) such that

(1) $\partial W = W^- \cup W^+$,
(2) $W^- = W^{--} \cup (\{b\} \times W_b)$,
(3) $W^+ = W^{++} \cup (\{a\} \times W_a)$, $W_a^{++} = \mathrm{cl}(\partial(W_a) \setminus W_a^{--})$,
(4) there exists homeomorphism $h: [a, b] \times W_a \to W$ such that $\pi_1 \circ h = \pi_1$ and $h([a, b] \times W_a^{--}) = W^{--}$, $h([a, b] \times W_a^{++}) = W^{++}$.

An isolating segment W over $[a, b]$ is said to be $(b - a)$-*periodic* (or simply *periodic*) if $W_a = W_b$, $W_a^{--} = W_b^{--}$ and $W_a^{++} = W_b^{++}$.

The definition of periodic isolated segment from [25] does not contain the second equality from the point (3). Presented definition is more general then the ones from [24], [26]. All segments which appear in the paper satisfy also definitions introduced in [24]–[26].

The simplest isolating segments are of the form $W = [0,T] \times B$, where $W^{--} = [0,T] \times \partial B$, $W^{++} = \emptyset$ or $W^{--} = \emptyset$, $W^{++} = [0,T] \times \partial B$ and B is some arbitrary compact subset of X. All segments in the sequel are of one of this form with B such that $\operatorname{int} B$ is holomorphic equivalent to the unit disc.

Let a local process φ be given by the equation (2.1). To prove that a set $W \subset \mathbb{R} \times M$ is an isolating segment for φ it is enough to check the behaviour of the vector field $(1, v)$ on the boundary of W. Then, by an appropriate fixed point theorem, there exists a periodic solution inside the segment. The Lefschetz fixed point theorem is used in the general case but, by the simplicity of our segments, we need only the Brouwer one. In fact, we use the Denjoy–Wolff fixed point theorem because the Poincaré map $\varphi_{(0,T)}$ is holomorphic.

2.3. Basic notions. Let $g\colon M \to M$ and $n \in \mathbb{N}$. We denote by g^n the n-th iterate of f, and by g^{-n} the n-th iterate of g^{-1} (if exists).

We say that the point z_0 is *attracting* (*repelling*) *for g in the set $W \subset M$* if the equality $\lim_{n\to\infty} g^n(w) = z_0$ ($\lim_{n\to\infty} g^{-n}(w) = z_0$) holds for every $w \in W$.

We call a T-periodic solution of (2.1) *attracting* (*repelling*) *in the set $W \subset M$* if the corresponding fixed point of the Poincaré map $\varphi_{(0,T)}$ is attracting (repelling) in the set W.

Let $-\infty \leq \alpha < \omega \leq \infty$ and $s\colon (\alpha, \omega) \to \mathbb{C}$ be a full solution of (1.1). We call s *forward blowing up* (shortly *f.b.*) or *backward blowing up* (*b.b.*) if $\omega < \infty$ or $\alpha > -\infty$, respectively.

We call the set $\{z \in \mathbb{C} : \mathfrak{Im}(z) = 0,\ \mathfrak{Re}(z) \leq 0\}$ the *critical line*. We say that the function $f\colon \mathbb{R} \to \mathbb{C}$ fulfils the *critical line condition* if and only if the formula

$$f(\mathbb{R}) \cap \{z \in \mathbb{C} : \mathfrak{Im}(z) = 0, \mathfrak{Re}(z) \leq 0\} = \emptyset \tag{2.3}$$

holds. This condition was introduced in the context of discriminant of the right hand side of Riccati equation (cf. [28]).

We define the sector $\mathcal{S}(\alpha, \beta) = \{z \in \mathbb{C} : \alpha < \operatorname{Arg}(z) < \beta\}$, where $-\pi \leq \alpha < \beta \leq \pi$. Moreover, for $0 < \alpha \leq \pi$ we define $\mathcal{S}(\alpha) = \mathcal{S}(-\alpha, \alpha)$ and $\widehat{\mathcal{S}}(\alpha)$ to be a set symmetric with respect to the origin to sector $\mathcal{S}(\alpha)$. Obviously, $0 \notin \mathcal{S}(\alpha, \beta)$.

Let $I \subset \mathbb{R}$. We denote the *angular width* of function $f\colon I \to \mathbb{C}$ by

$$\sphericalangle(f) = \inf\{\beta - \alpha : \text{ there exists } \theta \in \mathbb{R} \text{ such that } e^{i\theta} f(I) \subset \mathcal{S}(\alpha, \beta) \cup \{0\}\}.$$

It is easy to see that for $f, g, h\colon \mathbb{R} \to \mathbb{C}$ with $f(t) = e^{i \sin(t)}$ we have $\sphericalangle(f) = 2$, for $g(t) = 1 + e^{it}$ it is $\sphericalangle(g) = \pi$ and for $h(t) = e^{it}$ the angular width $\sphericalangle(h)$ is not defined.

Let us recall that the inner product of two vectors $a, b \in \mathbb{C}$ is given by the formula $\langle a, b\rangle = \mathfrak{Re}(a\overline{b}) = \mathfrak{Re}(\overline{a}b)$.

We write $\mathbb{R}^+ = [0, +\infty)$. By $\#K$ we denote the cardinality of the set K. By $B(a, r)$ we denote the closed ball centered at a with radius r.

Let $f \in \mathcal{C}(\mathbb{R}, \mathbb{C} \setminus \{0\})$. We let $\Gamma_f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ denote the function such that

$$\Gamma_f(t) \in \arg(f(t)) \text{ and } \Gamma_f(0) = \operatorname{Arg}(f(0)). \tag{2.4}$$

We also write

$$\beta_T(f) = \frac{\Gamma_f(T) - \Gamma_f(0)}{2\pi}. \tag{2.5}$$

Let $f \colon \mathbb{R} \to \mathbb{C}$ satisfy the local Lipschitz condition and U be an open subset of $\mathbb{R}$. We denote the Lipschitz constant of f in U by $L(U, f)$ (if it does not exist, we set $L(U, f) = \infty$). Write

$$L_f(t) = \inf\{L(U, f) : U \text{ is a neighbourhood of } t\}.$$

It is easy to see that $L_f : \mathbb{R} \to [0, \infty)$ is upper semi-continuous and for $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ we get $L_f \equiv |f'|$.

The family $\{A_\iota\}$ is called a *decomposition* of the set X when

(1) $\emptyset \neq A_\iota \subset X$,
(2) $\bigcup\{A_\iota\} = X$,
(3) $A_\iota \cap A_\kappa = \emptyset$

holds for all ι, κ, $\iota \neq \kappa$.

Let $v \in \mathcal{C}(\mathbb{R} \times \Omega, \mathbb{C})$. We say that v *is T-periodic with respect to the first variable* if and only if $v(t, z) = v(t + T, z)$ for every $t \in \mathbb{R}$. We say that v *is holomorphic with respect to the second variable* if and only if $v(t, \cdot) : \Omega \to \mathbb{C}$ is holomorphic for every $t \in \mathbb{R}$.

Let $\xi \in \mathcal{C}(\mathbb{R}, \mathbb{C})$, and $k \in \mathbb{N}$, $k > 0$. We say that ξ is *a branch of zeros of v of multiplicity k* if and only if for every $t \in \mathbb{R}$ and $0 < j < k$ the formulas $\frac{d^j v}{dz^j}(t, f(t)) = 0$ and $\frac{d^k v}{dz^k}(t, f(t)) \neq 0$ hold. We say that ξ is a branch of *simples zeros of v* if and only if it is a branch of zeros of multiplicity 1. We say that ξ is a branch of *multiply zeros of v* if and only if ξ is a branch of zeros of multiplicity greater than 1.

3. Simple zeros of the vector field

In the present section we investigate the existence of periodic solutions which are close to the branch of simple zeros ξ of the vector field v. If for an arbitrary time t the point $\xi(t)$ is a repelling (or attracting) fixed point of v_t then we try to construct an isolating segment W containing the graph of ξ (cf. Figure 1).

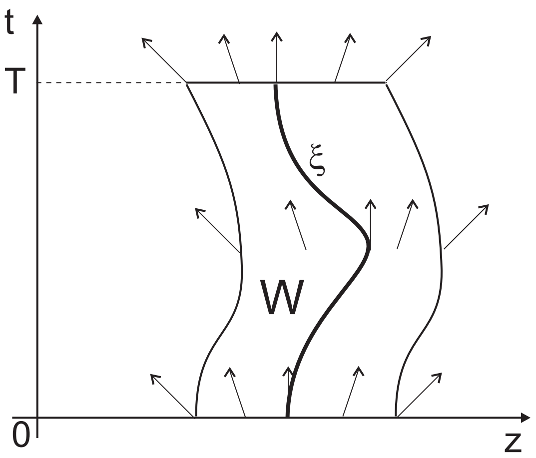

Figure 1. Schematic picture of the vector field $(1, v)$ in a neighborhood of the graph of ξ and an isolating segment W.

Thus we seek ξ, χ branches of simple zeros of the vector field v satisfying the following inequalities

$$\mathfrak{Re}\left[\left(\frac{d}{dz}v(t,z)\right)\bigg|_{z=\xi(t)}\right] > 0, \tag{3.1}$$

$$\mathfrak{Re}\left[\left(\frac{d}{dz}v(t,z)\right)\bigg|_{z=\chi(t)}\right] < 0. \tag{3.2}$$

To simplify notation we write

$$\mathfrak{Re}\left[\left(\frac{d}{dz}v(t,\xi(t))\right)\right] > 0 \quad \text{and} \quad \mathfrak{Re}\left[\left(\frac{d}{dz}v(t,\chi(t))\right)\right] < 0$$

instead of (3.1) and (3.2) respectively.

It is possible to construct an isolating segment W if at every point of the sides of W the dominating term of an inner product in $\mathbb{R}^3$ of the vector field $(1, v)$ and an outward normal vector is the space term (it comes from v).

The following theorems allows us to find periodic solutions close to these branches even if some perturbation is introduced.

Theorem 3.1. *Let $T > 0$, Ω be an open subset of a complex plain, $v, f \in \mathcal{C}(\mathbb{R} \times \Omega, \mathbb{C})$ be T-periodic with respect to the first variable and holomorphic with respect to the second one. Let the sets*

$$\begin{aligned} J^+ = \{\xi \in \mathcal{C}(\mathbb{R}, \Omega) : \xi &\text{ is } T\text{-periodic branch of simple zeros of } v \\ &\text{which for every } t \in \mathbb{R} \text{ satisfies (3.1)}\}, \\ J^- = \{\xi \in \mathcal{C}(\mathbb{R}, \Omega) : \xi &\text{ is } T\text{-periodic branch of simple zeros of } v \\ &\text{which for every } t \in \mathbb{R} \text{ satisfies (3.2)}\} \end{aligned}$$

be finite. Then the equation

$$\dot z = Rv(t,z) + f(t,z), \tag{3.3}$$

has at least $\#J^+$ T-periodic asymptotically unstable solutions and at least $\#J^-$ T-periodic asymptotically stable ones, provided $R \in \mathbb{R}$ is big enough.

Proof. Write $\#J^+ = l$, $\#J^- = m$, $l + m = n$. Let $\eta \in J^+ \cup J^-$. We write the equation (3.3) in the form

$$\dot z = u(t,z) = R(z - \eta(t))\frac{dv}{dz}(t,\eta(t)) + R\widehat{v}(t,z) + f(t,z),$$

where $\widehat{v}$ is T-periodic with respect to the first variable, holomorphic with respect to the second one and for every $t \in \mathbb{R}$ satisfies the equality

$$\lim_{z\to\eta(t)} \frac{\widehat{v}(t,z)}{z-\eta(t)} = 0. \tag{3.4}$$

(1) Let now $\eta \in J^+$. Our goal is to prove the existence of one T-periodic asymptotically unstable solution which graph is close to the graph of η.

(1a) We assume that $\eta \in \mathcal{C}^1(\mathbb{R},\mathbb{C})$ and construct isolating segment W containing the graph of $\eta|_{[0,T]}$ and prove that there exists exactly one T-periodic solution inside W. Moreover, it is asymptotically unstable.

Let $M > 0$. We define the segment W by

$$W = \{(t,z) \in [0,T] \times \mathbb{C} : |z - \eta(t)| \le M\}.$$

Our goal is to show that $W^{--} = K$, where

$$K = \{(t,z) \in [0,T] \times \mathbb{C} : |z - \eta(t)| = M\}.$$

We parameterize K by

$$s\colon [0,T] \times (0,2\pi] \ni (t,o) \mapsto (t, \eta(t) + Me^{io}) \in [0,T] \times \mathbb{C}.$$

It easy to see that an outward orthogonal vector to W at every point of K has the form $n(t,o)$ where

$$n\colon [0,T] \times (0,2\pi] \ni (t,o) \mapsto [-\mathfrak{Re}[\eta'(t)e^{-io}], e^{io}]^T \in \mathbb{R} \times \mathbb{C}.$$

Thus the inner product of an outward normal vector n and the vector field $(1,u)$ is equal to

$$\begin{aligned} \langle n,(1,u(s))\rangle &= -\mathfrak{Re}[\eta_k' e^{-io}] + \mathfrak{Re}\left[e^{-io}RMe^{io}\frac{dv}{dz}(t,\eta(t))\right] \\ &\quad + \mathfrak{Re}[e^{-io}R\widehat{v}(t,\eta(t)+Me^{io})] + \mathfrak{Re}[e^{-io}f(t,\eta_k(t)+Me^{io})] \\ &\ge -|\eta_k'| + RM\mathfrak{Re}\left[\frac{dv}{dz}(t,\eta(t))\right] - R|\widehat{v}(t,\eta(t)+Me^{io})| - N(M) = (\star), \end{aligned} \tag{3.5}$$

where $N(M) = \sup\{|f(s(t,o))| : (t,o) \in [0,T] \times (0,2\pi]\}$.

By (3.1), T-periodicity of η and (3.4), there exists $\varepsilon > 0$ such that

$$M\mathfrak{Re}\left[\frac{dv}{dz}(t,\eta(t))\right] - |\widehat{v}(t,\eta(t)+Me^{io})| > \varepsilon M$$

holds for every $t \in \mathbb{R}$ provided $M > 0$ is small enough. Thus taking M small enough and R big enough one can see that $(\star) > 0$ so $W^{--} = K$ and W is an isolating segment. T-periodicity of W follows by T-periodicity of η. By the Denjoy–Wolff fixed point theorem there exists exactly one T-periodic solution inside W. It is asymptotically unstable.

(1b) Let now η be continuous. We define $\widetilde{\eta} \in \mathcal{C}^1(\mathbb{R},\mathbb{C})$ which is T-periodic and close enough in the supremum norm to the η_k. We construct the set W for $\widetilde{\eta}$ as at the point (1a). By the continuity of multiplication and the inner product the set W is an isolating segment for the vector field u. The existence of T-periodic solution inside W follows as in (1a).

(2) Let us fix $\eta \in J^-$. Our goal is to prove the existence of one T-periodic asymptotically stable solution which graph is close to the graph of η. The only difference from the point (1) is that at every point of the sides of W the vector field $(1,u)$ points inward W. Thus the existence of asymptotically stable periodic solution follows.

By taking M small enough all isolating segments can be pairwise disjoint. Thus detected periodic solutions are pairwise different. $\square$

Example 3.2. The equation

$$\dot{z} = R(e^{-2it}z^3 - 6e^{-it}z^2 + 11z - 6e^{it}) + z^4 = Re^{-2it}(z-e^{it})(z-2e^{it})(z-3e^{it}) + z^4$$

has at least three 2π-periodic solutions provided that R is big enough. Two of them which are close to e^{it} and $3e^{it}$, respectively, are asymptotically unstable and the third one which is close to $2e^{it}$ is asymptotically stable.

The following corollaries show that the presented method works good for branches of zeros $R\eta$ for big R and long periods of the equation.

Corollary 3.3. *Let $T > 0$, $n \geq 3$ and $a_j \in \mathcal{C}(\mathbb{R},\mathbb{C})$ for $j \in \{1,\dots,n\}$ be T-periodic. Moreover, let $g \in \mathcal{C}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ be T-periodic in the first variable i.e. $g(t,z) = g(t+T,z)$ holds for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$ and the following condition*

$$\lim_{|z|\to\infty} \frac{g(t,z)}{|z|^{n-1}} = 0 \tag{3.6}$$

is satisfied uniformly with respect to t. We also assume that $l,m \in \mathbb{N}$, $l+m \leq n$ and $\xi_p,\chi_q\colon \mathbb{R} \to \mathbb{C}$ for $p \in \{1,\dots,l\}$, $q \in \{1,\dots,m\}$ are T-periodic and continuous branches of simple zeros of the vector field v given by (1.1)*. If for* *every* $p \in \{1,\dots,l\}$ $(q \in \{1,\dots,m\})$ *the inequality* (3.1), ((3.2)) *holds for every* $t \in \mathbb{R}$*, then the equation*

$$\dot{z} = \sum_{j=0}^{n} R^{n-j} a_j(t) z^j + g(t,z) \tag{3.7}$$

has at least l (m) T*-periodic asymptotically unstable* (*asymptotically stable*) *solutions, provided* $R \in \mathbb{R}$ *is big enough.*

Proof. We write the equation (3.7) in the form

$$\dot{z} = u(t,v) = a_n(t) \prod_{r=1}^{n} (z - R\eta_r(t)) + g(t,z),$$

where $\eta_r\colon \mathbb{R} \to \mathbb{C}$ is the branch of zeros of the vector field v.

After the substitution $z = Ru$ we get the equation

$$\dot{u} = a(t) \prod_{r=1}^{n} (u - \eta_r(t)) + g(t, Ru)$$

which is of the form (3.3). Now it is enough to apply Theorem 3.1. $\square$

Example 3.4. The equation

$$\begin{aligned} \dot{z} &= z^3 - 6\cos(t)Rz^2 + (9 + 2e^{2it})R^2 z - 6R^3 e^{it} + z + 1 \\ &= (z - Re^{it})(z - 2Re^{it})(z - 3Re^{-it}) + z + 1 \end{aligned}$$

has at least two 2π-periodic solutions provided that R is big enough. The one which is close to Re^{it} is asymptotically unstable and the other close to $2Re^{it}$ is asymptotically stable.

Corollary 3.5. *Let* $T > 0$*,* Ω *be an open subset of a complex plain,* $v \in \mathcal{C}(\mathbb{R} \times \Omega, \mathbb{C})$ *be* 1*-periodic with respect to the first variable and holomorphic with respect to the second one. Let the sets*

$$\begin{aligned} J^+ = \{\xi \in \mathcal{C}(\mathbb{R},\Omega) : &\ \xi \textit{ is } 1-\textit{periodic branch of simple zeros of } v \\ &\textit{which for every } t \in \mathbb{R} \textit{ satisfies } (3.1)\}, \\ J^- = \{\xi \in \mathcal{C}(\mathbb{R},\Omega) : &\ \xi \textit{ is } 1-\textit{periodic branch of simple zeros of } v \\ &\textit{which for every } t \in \mathbb{R} \textit{ satisfies } (3.2)\} \end{aligned}$$

be finite. Then the equation

$$\dot{z} = v\left(\frac{t}{T}, z\right), \tag{3.8}$$

has at least $\#J^+$ T-periodic asymptotically unstable solutions and at least $\#J^-$ T-periodic asymptotically stable ones, provided $T \in \mathbb{R}$ is big enough.

Proof. We make a change of variables given by $w(t) = z(tT)$. Then we get equation $\dot{w}(t) = T\dot{z}(tT) = Tv(t, z(tT)) = Tv(t, w(t))$ which, by Theorem 3.1, has at last $\#J^+$ 1-periodic asymptotically unstable solutions and at least $\#J^-$ 1-periodic asymptotically stable ones, provided $T \in \mathbb{R}$ is big enough. But these 1 periodic solutions in w-coordinate are T-periodic in z-coordinate. $\square$

Example 3.6. The equation

$$\dot{z} = z^4 - e^{iTt}z^3 - z(4 + e^{iTt})^3 + e^{iTt}(4 + e^{iTt})^3 = [z^3 - (4 + e^{iTt})^3][z - e^{iTt}]$$

has at least four $2\pi/T$-periodic solutions provided that $T > 0$ is small enough. Three of them which are close to $e^{(2k\pi/3)i}(4 + e^{iTt})$ are asymptotically unstable and the one close to e^{iTt} is asymptotically stable.

Remark 3.7. Some versions of the above theorems in the case of $n = 2$ can be found in [28].

Remark 3.8. The above theorems also allow to find p-harmonic solutions for $p > 1$ (cf. Example 3.9 and 3.10).

Example 3.9. By Theorem 3.1, the 2π-periodic equation

$$\dot{z} = R(2e^{-it}z^3 + z^2 - 2z - e^{it}) + z^{10} = R(z^2 - e^{it})(2e^{-it}z + 1) + z^{10}$$

has at least two 4π-periodic asymptotically unstable solutions provided that R is big enough. The solutions are close to the branches $\xi_1 = e^{it/2}$ or $\xi_2 = -\xi_1$, respectively.

Example 3.10. By Corollary 3.3, the equation

$$\dot{z} = e^{-it}z^4 - Re^{-2it}z^3 - 8R^3z + 8R^4e^{-it} + z^2 = e^{-it}(z^3 - 8R^3e^{it})(z - Re^{-it}) + z^2$$

has at least three asymptotically unstable 6π-periodic solutions provided that R is big enough.

Remark 3.11. It is possible to state Theorem 3.1 and Corollary 3.5 in more general settings. It is enough to assume that there exists D an open subset of $\mathbb{R} \times \mathbb{C}$ such that $v \in \mathcal{C}(D, \mathbb{C})$. In this case, every branch of zeros η must satisfy

$$(t, \eta(t)) \in D \quad \text{for every } t \in \mathbb{R}. \tag{3.9}$$

Example 3.12. By Theorem 3.1 and Remark 3.11, the equation

$$\dot{z} = v(t, z) = Re^{-2it}z^3\frac{z - ie^{it}}{z - e^{it}}$$

has at least one 2π-periodic asymptotically stable solution which is close to ξ provided that R is big enough. Here $\xi(t) = ie^{it}$ and $\frac{d}{dz}v(t, \xi(t)) = R\frac{i-1}{2}$.

Write $\Omega = \{z \in \mathbb{C} : |z| \neq 1\}$ and consider $v\colon \mathbb{R} \times \Omega \to \mathbb{C}$. But the branch of simple zeros $\xi(t) = ie^{it}$ does not fulfill the crucial condition $\xi(\mathbb{R}) \subset \Omega$, so Theorem 3.1 is useless in this situation.

Now write $D = \{(t,z) \in \mathbb{R} \times \mathbb{C} : z \neq e^{it}\}$, so (3.9) holds. Thus Remark 3.1 can be applied.

Remark 3.13. The proof of Theorem 3.1 is not valid in the case of infinite sets J^+ or J^- as shown in the Example 3.14.

Example 3.14. Let us consider the equation

$$\dot{z} = v(t,z) = Re^{-3it}\frac{e^{e^{it}z} - 1}{z^2},$$

where $R > 0$. It has infinitely many branches of simple zeros $\eta_k(t) = e^{-it}2ki\pi$ for $k \in \mathbb{Z} \setminus \{0\}$. But $\frac{dv}{dz}(t, \eta_k(t)) = \frac{-R}{4k^2\pi^2}$, $J^- = \{\eta_k : k \in \mathbb{Z} \setminus \{0\}\}$,

$$\lim_{k\to\pm\infty} \frac{-R}{4k^2\pi^2} = 0, \qquad \lim_{k\to\pm\infty} |\eta_k'(t)| = \infty,$$

so for a particular value of R only in case of finite number of η_k's the crucial inequality (3.5) may be satisfied. So the bigger R, the more periodic solutions, but for every R only finite number may be produced by Theorem 3.1.

It is possible to use the method presented in the current subsection to detect the existence of asymptotically stable (asymptotically unstable) periodic solution when neither (3.1) nor (3.2) is satisfied for all $t \in \mathbb{R}$. This is possible if for every time t such that (3.1) (or (3.2)) holds the dominating term of the inner product in $\mathbb{R}^3$ of the vector field $(1, v_t)$ and an outward normal vector of the segment W is the space term. But when (3.1) (or (3.2)) is close to zero the dominating term is the time one (it comes from 1). It is possible when the isolating segment broadens or narrows rapidly. The presented procedure can be used only when the branch of simple zeros is close to the one fulfilling (3.1) (or (3.2)) for all $t \in \mathbb{R}$.

3.1. Special case. In the present subsection we deal with the special case of the equation (1.1) which is given by

$$\dot{z} = v(t,z) = a(t)z^n + b(t), \tag{3.10}$$

where $a, b \in \mathcal{C}(\mathbb{R}, \mathbb{C} \setminus \{0\})$ are T-periodic and $n \geq 2$. By the simplicity of the vector field, it is possible to present estimates for a and b which guarantee the existence of one up to n periodic solutions.

Let us start with a few more pieces of notation: we write

$$I_n = \begin{cases} \{0, 1, \ldots, 2n-3\} & \text{when } n \text{ is odd,} \\ \{0, 1, \ldots, n-2\} & \text{when } n \text{ is even,} \end{cases}$$

$$\gamma_n = \begin{cases} \dfrac{\pi}{2(n-1)} & \text{when } n \text{ is odd,} \\ \dfrac{\pi}{n-1} & \text{when } n \text{ is even.} \end{cases}$$

For a set $J \subset I_n$ we write

$$m_J = \begin{cases} n - \#J & \text{when } n \text{ is odd,} \\ n - 2\#J & \text{when } n \text{ is even.} \end{cases}$$

We set $\tau_a = \Gamma_a$, $\tau_b = \Gamma_{-b/a}$, where Γ is given by (2.4) and define $\alpha_J \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ by

$$\alpha_J(t) = \begin{cases} \min\limits_{k \in I_n \setminus J} \left\{ \left| \tau_b(t) - (2k+1)\gamma_n + \tau_a(t)\dfrac{n}{n-1} \right| \mod 2\pi \right\} & \text{if } 4 \nmid n, \\ \min\limits_{k \in I_n \setminus J} \left\{ \left| \tau_b(t) - 2k\gamma_n + \tau_a(t)\dfrac{n}{n-1} \right| \mod 2\pi \right\} & \text{if } 4 \mid n. \end{cases}$$

Moreover, we write

$$g_t(z) = \frac{dv}{dz}(t, z) = na(t)z^{n-1}, \qquad h(z) = z^n$$

and

$$C(t) = g_t^{-1}(i\mathbb{R}) = \bigcup_{k=0}^{2n-3} e^{i\left(\frac{\pi}{2(n-1)} - \frac{\tau_a(t)}{n-1} + \frac{k\pi}{n-1}\right)}\mathbb{R}^+,$$

$$D_J(t) = \begin{cases} \bigcup\limits_{k \in I_n \setminus J} e^{i\left[(2k+1)\gamma_n - \frac{n}{n-1}\tau_a(t)\right]}\mathbb{R}^+ & \text{if } 4 \nmid n, \\ \bigcup\limits_{k \in I_n \setminus J} e^{i\left[2k\gamma_n - \frac{n}{n-1}\tau_a(t)\right]}\mathbb{R}^+ & \text{if } 4 \mid n. \end{cases}$$

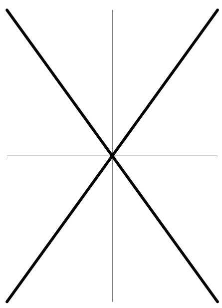

FIGURE 2. The set $D_\emptyset(t) = C(t)$ for the equation (3.10) with $n = 3$ and $\tau_a(t) = 0$.

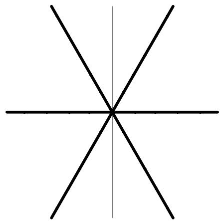

Figure 3. The set $C(t)$ for the equation (3.10) with $n = 4$ and $\tau_a(t) = 0$.

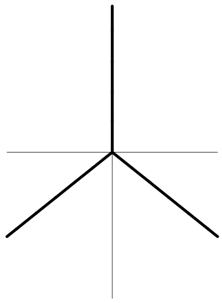

Figure 4. The set $D_\emptyset(t)$ for the equation (3.1) with $n = 4$ and $\tau_a(t) = 0$.

It is easy to see that $D_\emptyset(t) = h(C(t))$ holds. Every set of the form $D_J(\mathbb{R})$ is an analogue of the critical line for the Riccati equation (cf. [28]).

In the case $a \equiv 1$ the above sets are less complicate, namely $\tau_a \equiv 0$ and so $C(t) = C(0)$, $D_J(t) = D_J(0)$ for every $t \in \mathbb{R}$ and $J \subset I_n$.

We now state the main theorem of the present subsection (the definition of β_T is given by (2.5)).

Theorem 3.15. *Let $T > 0$, $n \geq 2$ and T-periodic functions $a, b\colon \mathbb{R} \to \mathbb{C}\setminus\{0\}$ satisfy the local Lipschitz condition. If there exist $J \subset I_n$, $l \in \mathbb{R}$ and $E > 0$ such that the conditions:*

$$m_J > 0, \tag{3.11}$$

$$\beta_T(a) = -(n-1)\beta_T(b), \tag{3.12}$$

$$\begin{aligned} |a(t)| \sin\left(\frac{(n-1)\alpha_J(t)}{n}\right) \\ > \frac{1}{n^2} L_{b/a}(t)\left[|l|\left|\frac{b(t)}{a(t)}\right|^{(1-2n)/n} + \frac{1}{E}\left|\frac{b(t)}{a(t)}\right|^{(2-l-2n)/n}\right] \\ + E\frac{n-1}{2}|a(t)|\left|\frac{b(t)}{a(t)}\right|^{(l-1)/n}\left(1 + E\left|\frac{b(t)}{a(t)}\right|^{(l-1)/n}\right)^{n-2} \end{aligned} \tag{3.13}$$

are satisfied for every $t \in \mathbb{R}$, then the equation (3.10) *has at least m_J T-periodic solutions which are asymptotically stable or asymptotically unstable.*

Proof. Our goal is to construct m_J T-periodic isolating segments such that the vector field $(1, v)$ points inwards or outwards in the whole side. Then we apply the Denjoy–Wolff fixed point theorem.

Let $\psi \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be a branch of simple zeros of v i.e. $v(t, \psi(t)) = 0$ holds for every $t \in \mathbb{R}$. Then $\psi^n(t) = -b(t)/a(t)$ is satisfied. We write $\psi_m = e^{i2\pi m/n}\psi$ for $m \in \{0, \ldots, n-1\}$. We seek ψ_m such that the equality

$$\operatorname{sgn}(\mathfrak{Re}[g_t(\psi_m)]) = \text{const.} \tag{3.14}$$

is satisfied.

The number $\alpha_J(t)$ is the angle between $-b(t)/a(t)$ and the nearest half-line which is the part of $D_J(t)$. By (3.11), we get $0 \le \alpha_J \le \pi n/(2(n-1))$, and by (3.13), it must be $\alpha_J > 0$.

Let $J = \emptyset$ and $\tau_a \equiv 0$ hold. Then $(-b/a)(\mathbb{R}) \cap D_J(\mathbb{R}) = \emptyset$ and (3.14) is satisfied for every ψ_m. Let us notice that $\sphericalangle(g_t(\psi)) = \frac{n-1}{n}\sphericalangle\big(-\frac{b}{a}\big)$ holds.

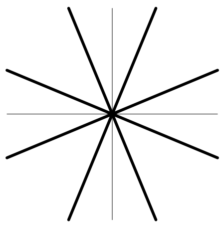

Figure 5. The set $D_\emptyset(t) = C(t)$ for the equation (3.10) with $n = 5$ and $\tau_a(t) = 0$.

We now allow $(-b/a)(t_0) \in D_J(t_0)$ to hold for exactly one $t_0 \in (0, T]$. If n is odd then there exists exactly one ψ_m such that the inclusion $g_{t_0}(\psi_m(t_0)) \in i\mathbb{R}$ is satisfied. The uniqueness of ψ_m comes from the fact that at most one number of the form $e^{i(2\pi/n)j} g_{t_0}(\psi(t_0))$ where $j \in \{0, \ldots, n-1\}$ can be a purely imaginary one. It follows that the number of functions ψ_m such that the equality (3.14) holds is equal to $n-1$. By the same argument, it can be proved that if the set $(-b/a)(\mathbb{R})$ has nonempty intersections with exactly k half-lines which are parts of $D_\emptyset(\mathbb{R})$ then exactly k functions of the form ψ_m does not longer satisfy the condition (3.14).

If n is odd then either both ψ_m and $-\psi_m$ satisfy the condition $g_{t_0}(\psi_k(t_0)) \in i\mathbb{R}$ or both do not satisfy it. It is for this reason that the nonempty intersection of $(-b/a)(\mathbb{R})$ with one of the half-lines which are parts of $D_\emptyset(\mathbb{R})$ makes the number of ψ_m satisfying (3.14) smaller by two.

Finally, for an arbitrary $J \subset I_n$ the number of the function of the type ψ_m which satisfy (3.14) equals at least $\max\{m_J, 0\}$. Moreover, for every such function ψ_m, the angle between $g_t(\psi_m(t))$ and the imaginary axis is at least

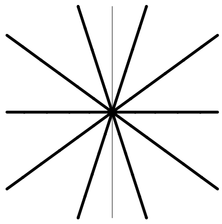

FIGURE 6. The set $C(t)$ for the equation (3.10) with $n = 6$ and $\tau_a(t) = 0$.

$(n-1)\alpha_J(t)/n$, so

$$|\mathfrak{Re}[g_t(\psi_m(t))]| \geq n|a(t)||\psi_m(t)|^{n-1} \sin\left(\frac{(n-1)\alpha_J(t)}{n}\right) \tag{3.15}$$

holds.

The above considerations are valid also in the case $\tau_a \not\equiv 0$. It is worth noting that (3.12) implies $\beta_T(a) \in (n-1)\mathbb{Z}$ and $D_J(0) = D_J(T)$.

The function $|\psi|$ is T-periodic. Indeed, by (3.12) we get

$$\beta_T(\psi^n) = \beta_T(b) - \beta_T(a) = n\beta_T(b) \in n\mathbb{Z}, \tag{3.16}$$

so ψ must be T-periodic.

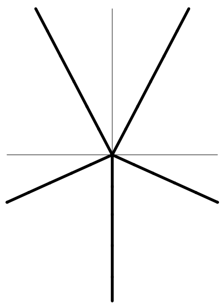

FIGURE 7. The set $D_\emptyset(t)$ for the equation (3.10) with $n = 6$ and $\tau_a(t) = 0$.

Let us assume that $b/a \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$. We define T-periodic segment $W^m \subset [0,T] \times \mathbb{C}$ by

$$W^m_t = \{z \in \mathbb{C} : |z - \psi_m(t)| \leq M(t)\},$$

where the mapping $M \in \mathcal{C}^1(\mathbb{R}, (0,\infty))$ is T-periodic. We parameterize its side by function $s_m\colon [0,T] \times (0, 2\pi] \to \mathbb{R} \times \mathbb{C}$ given by

$$s_m(t, o) = (t, \psi_m(t) + e^{io}M(t)).$$

An outward orthogonal vector has the form $n_m = [-M' - \mathfrak{Re}[\psi_m' e^{-io}], e^{io}]^T$. We do a substitution

$$\begin{aligned} v(s_m) &= a(\psi_m + Me^{io})^n + b = a\sum_{k=0}^{n}\binom{n}{k}M^k e^{iok}\psi_m^{n-k} + b \\ &= anMe^{io}\psi_m^{n-1} + a\sum_{k=2}^{n}\binom{n}{k}M^k e^{iok}\psi_m^{n-k}. \end{aligned}$$

By (3.15), we estimate the modulus of inner product of the vector field $(1, v)$ and an outward orthogonal vector at every point of the side of W^m by

$$\begin{aligned} &|\langle n_m, (1, v(s_m))\rangle| \\ &\quad \geq \bigg| -M' - \mathfrak{Re}[\psi_m' e^{-io}] + \mathfrak{Re}\bigg[naM\psi_m^{n-1} + a\sum_{k=2}^{n}\binom{n}{k}M^k e^{io(k-1)}\psi_m^{n-k}\bigg]\bigg| \\ &\quad \geq n|a|M|\psi_m|^{n-1}\sin\bigg(\frac{(n-1)\alpha_J}{n}\bigg) - |M'| - |\psi_m'| \\ &\qquad - |a|M^2\sum_{k=2}^{n}\binom{n-2}{k-2}M^{k-2}|\psi_m|^{n-2-(k-2)}\frac{\binom{n}{k}}{\binom{n-2}{k-2}} \\ &\quad \geq n|a|M|\psi_m|^{n-1}\sin\bigg(\frac{(n-1)\alpha_J}{n}\bigg) \\ &\qquad - |M'| - |\psi_m'| - |a|M^2\frac{n(n-1)}{2}(M + |\psi_m|)^{n-2}, \end{aligned}$$

where the last inequality follows by

$$\max\bigg\{\frac{\binom{n}{k}}{\binom{n-2}{k-2}} : k \in \{2, \dots, n\}\bigg\} = \frac{n(n-1)}{2}.$$

Finally, to get $|\langle n_m, (1, v(s_m))\rangle| > 0$ it is enough the inequality

$$\begin{aligned} n|a|M|\psi_m|^{n-1}&\sin\bigg(\frac{(n-1)\alpha_J}{n}\bigg) \\ &> \frac{n(n-1)}{2}|a|M^2(M + |\psi_m|)^{n-2} + |M'| + |\psi_m'| \end{aligned} \tag{3.17}$$

holds.

Write $M(t) = E|\psi_m(t)|^l$. Then $M' = El|\psi_m|^{l-2}\mathfrak{Re}[\psi_m'\overline{\psi_m}]$. Moreover, $\psi_m^n = -b/a$ holds, so $\psi_m' = -(b/a)'\psi_m^{1-n}/n$ and $|\psi_m| = |b/a|^{1/n}$ are satisfied. Thus the formulas

$$|\psi_m'| = \frac{1}{n}\bigg|\bigg(\frac{b}{a}\bigg)'\bigg|\bigg|\frac{b}{a}\bigg|^{(1-n)/n} \quad \text{and} \quad |M'| \leq \frac{E|l|}{n}\bigg|\bigg(\frac{b}{a}\bigg)'\bigg|\bigg|\frac{b}{a}\bigg|^{(l-n)/n}$$

are satisfied. The inequality (3.17) is fulfilled provided that

$$\begin{aligned} nE|a|\left|\frac{b}{a}\right|^{(n+l-1)/n} & \sin\left(\frac{(n-1)\alpha_J}{n}\right) \\ & > \frac{n(n-1)}{2}|a|E^2|\psi_m|^{n-2+2l}(1+E|\psi_m|^{l-1})^{n-2} \\ & \qquad + \frac{E|l|}{n}\left|\left(\frac{b}{a}\right)'\right|\left|\frac{b}{a}\right|^{(l-n)/n} + \frac{1}{n}\left|\left(\frac{b}{a}\right)'\right|\left|\frac{b}{a}\right|^{(1-n)/n} \end{aligned}$$

holds. But the last formula is equivalent to (3.13).

Finally, every set of the form W^m is an isolating segment and the Poincaré mapping satisfies $P_T(W_0^m) \subset \operatorname{int} W_0^m$ or $P_{-T}(W_0^m) \subset \operatorname{int} W_0^m$ when the vector field on the side of W^m points respectively inward or outward the set W^m. By the Denjoy–Wolff fixed point theorem, there exists exactly one T-periodic solution inside every W^m. It is asymptotically stable or asymptotically unstable.

What is left is to show that all obtained periodic solutions are distinct. It suffices to prove that the segments are pairwise disjoint. Let us fix $t \in [0,T)$. We show that the distance between centres A_k, A_{k+1} of two consecutive balls W_t^k and W_t^{k+1} is greater than $2M(t) = 2E|\psi(t)|^l$. The triangle with vertices 0, A_k, A_{k+1} is isosceles and the 0 vertex angle is equal to $2\pi/n$. Since $|A_k| = |A_{k+1}| = |\psi(t)|$, it is enough to show that $\sin(\pi/n)|\psi| > E|\psi|^l$ holds. But the last inequality is equivalent to

$$\left|\frac{b}{a}\right|^{(1-l)/n} > \frac{E}{\sin(\pi/n)}. \tag{3.18}$$

By (3.13), after dropping some terms, we get

$$\sin\left(\frac{(n-1)\alpha_J}{n}\right) > E\frac{n-1}{2}\left|\frac{b}{a}\right|^{(l-1)/n}\left(1+(n-2)\left|\frac{b}{a}\right|^{(l-1)/n}\right).$$

We multiply it by $|b/a|^{2(1-l)/n}$ and get the inequality

$$\sin\left(\frac{(n-1)\alpha_J}{n}\right)\left|\frac{b}{a}\right|^{2(1-l)/n} - E\frac{n-1}{2}\left|\frac{b}{a}\right|^{(1-l)/n} - \frac{(n-1)(n-2)}{2}E > 0, \tag{3.19}$$

which solution fulfills

$$\begin{aligned} \left|\frac{b}{a}\right|^{(1-l)/n} & > \frac{\frac{n-1}{2}E + \sqrt{\left(\frac{n-1}{2}E\right)^2 + 2E(n-1)(n-2)\sin\left(\frac{(n-1)\alpha_J}{n}\right)}}{2\sin\left(\frac{(n-1)\alpha_J}{n}\right)} \\ & > \frac{n-1}{2\sin\left(\frac{(n-1)\alpha_J}{n}\right)}E \geq \frac{n-1}{2}E. \end{aligned}$$

Thus (3.18) is satisfied for $n \geq 4$, by $\sin(\pi/n) \geq 2\sqrt{2}/n$. Let now $n = 3$. If $m_J = 1$, then there is nothing to prove. If $m_J \geq 2$, then $\#J \leq 1$, so there is at most one half-line removed from the set $D_\emptyset(t)$. Thus the maximal angle between an arbitrary vector from the complex plane and the closest half-line from the set $D_J(t)$ equals $\pi/2$, so $\alpha_J(t) \leq \pi/2$ holds and the solution of (3.19) fulfills (3.18). If $n = 2$ then one periodic solution is asymptotically stable and the other is asymptotically unstable (cf. [28]) so they are distinct.

If $b/a \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ is not longer satisfied, then we modify the above arguments as at the point (1b) of the proof of Theorem 3.1. □

Remark 3.16. If the number $\mathfrak{Re}[g_t(\psi_m)]$ is positive or negative then the associated periodic solution is asymptotically unstable or asymptotically stable respectively.

The following examples are the straightforward application of Threorem 3.15 and Remark 3.16.

Example 3.17. The equation

$$\dot{z} = z^4 - Re^{i\sin(t)\pi/2} \tag{3.20}$$

has at least two 2π-periodic solutions provided that $R > R_0$, where numeric calculation gives $R_0 < 8.8$. One of the them is asymptotically stable and the other is asymptotically unstable. Here $J = \{0\}$, $\alpha_J \geq \pi/6$ and $l = -0.2$, $E = 0.2$. It is worth noting that, by $\sphericalangle(-Re^{i\sin(t)\pi/2}) = \pi/2$, [29, Theorem 4] cannot be here applied. Moreover, Corollary 3.3 can be applied to the equation (3.20) but it gives no estimation of R.

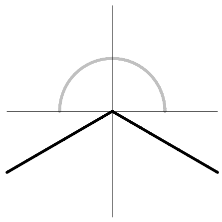

Figure 8. Sets $D_J(0)$ and $-\left(\frac{b}{a}\right)(\mathbb{R})$ for the equation (3.20).

Example 3.18. The equation

$$\dot{z} = e^{2it}z^3 - Re^{-it} + 1 \tag{3.21}$$

has at least three 2π-periodic solutions provided that $R > R_0$, where numeric calculation gives $R_0 < 22.5$. Two of them are asymptotically stable and the other is asymptotically unstable. Here $J = \emptyset$, $\alpha_\emptyset \geq \pi/4 - \arcsin(1/R)$ and $l = 0$, $E = 0.5$.

Moreover, if $R > R_1$, where numeric calculation gives $R_1 < 3.4$, then the equation has at one 2π-periodic asymptotically unstable solution. Here $J = \{0, 3\}$, $\alpha_{\{0,3\}} \geq (3/4)\pi - \arcsin(1/R)$ and $l = 0$, $E = 0.5$.

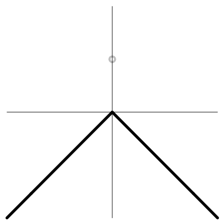

FIGURE 9. The sets $D_{\{0,3\}}(0)$ and $-\left(\frac{b}{a}\right)(0)$ for the equation (3.21) and $D_{\{0,3\}}(0)$, $D_{\{0,3\}}(\pi)$ and $-\left(\frac{b}{a}\right)(0)$, $-\left(\frac{b}{a}\right)(\pi)$ for the equation (3.22).

Setting neither $J = \{0\}$ nor $J = \{3\}$ gives anything new because $\alpha_J \geq \pi/4 - \arcsin(1/R)$ which is similar to $\alpha_\emptyset$.

EXAMPLE 3.19. The equation

$$\dot{z} = e^{i\sin(t)\pi/2} z^3 - Re^{-i\sin(t)\pi/2} \tag{3.22}$$

has at least one 2π-periodic asymptotically unstable solution provided that $R > R_0$, where numeric calculation gives $R_0 < 4$. Here $J = \{0, 3\}$, $\alpha_{\{0,3\}} \geq \pi/2$ and $l = 0$, $E = 0.45$.

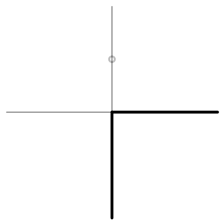

FIGURE 10. The sets $D_{\{0,3\}}\left(\frac{\pi}{2}\right)$ and $-\left(\frac{b}{a}\right)\left(\frac{\pi}{2}\right)$ for the equation (3.22).

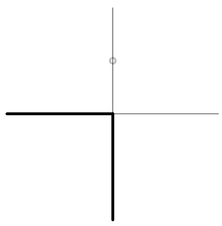

FIGURE 11. The sets $D_{\{0,3\}}\left(\frac{3}{2}\pi\right)$ and $-\left(\frac{b}{a}\right)\left(\frac{3}{2}\pi\right)$ for the equation (3.22).

REMARK 3.20. In the case of equation (3.10) the condition

$$\left(-\frac{b}{a}\right)(t) \notin D_J(t) \quad \text{for every } t \in \mathbb{R}, \tag{3.23}$$

plays an analogous role to the critical line condition for the Riccati equation (cf. [28]).

Corollary 3.21. *Let $T > 0$, $n \geq 2$ and T-periodic functions $a, b\colon \mathbb{R} \to \mathbb{C} \setminus \{0\}$ satisfy the local Lipschitz condition. If there exists $J \subset I_n$ such that* (3.11), (3.12) *and* (3.23) *are satisfied, then every equation*

$$\dot{z} = Ra(t)z^n + b(t), \tag{3.24}$$

$$\dot{z} = Ra(t)z^n + Rb(t), \tag{3.25}$$

$$\dot{z} = a(t)z^n + Rb(t) \tag{3.26}$$

has at least m_J T-periodic solutions provided that R is big enough. Every such solution is either asymptotically stable or asymptotically unstable.

Proof. It suffices to fix such an l and E that the inequality (3.13) holds. By (3.24), the left-hand side of the inequality (3.13) is positive. In the case of (3.24) and (3.23) it is linear with respect to R. Write $l = 1$. We fix $E > 0$ so small that the inequality

$$E\,\frac{n-1}{2}(1+E)^{n-2} < \frac{1}{2}\min\left\{\sin\left(\frac{(n-1)\alpha_J(t/T)}{n}\right) : t \in \mathbb{R}\right\} \tag{3.27}$$

holds.

In the case of (3.24) the formula

$$\frac{1}{n^2}L_{b/a}\left[|l|\left|\frac{b}{a}\right|^{(1-2n)/n} + \frac{1}{E}\left|\frac{b}{a}\right|^{(2-l-2n)/n}\right]$$

is linear with respect to $R^{(n-1)/n}$, and for (3.25) it does not depend on R.

For (3.26) the left-hand side of (3.13) does not depend on R. Let us fix $l = 1/2$ and $E = 1$. Then every term of the right-hand side is linear with respect to some R^p where $p < 0$.

Finally, the inequality (3.13) holds provided R is big enough. □

Remark 3.22. (a) The conclusion of the above corollary for the equations (3.25) and (3.26) follows also by Theorem 3.1 and Corollary 3.3, respectively.

(b) By any of Corollary 3.5 and Theorem 3.15, it is possible to state corollary for the equation (3.10) in the case of long periods.

Presented method fails in the case of small coefficients and short periods (cf. [28, Subsection 3.2]) as can be seen in the following examples.

Example 3.23. The equation

$$\dot{z} = z^3 + Re^{i\sin^2(t)\pi/6} \tag{3.28}$$

has at least three π-periodic solutions provided that $R > R_0$, where numeric calculation gives $R_0 < 25$. Two of them are asymptotically stable and the other is asymptotically unstable. Here $J = \emptyset$, $\alpha_\emptyset \geq \pi/12$ and $l = 0$, $E = 0.22$.

Moreover, if $R > R_1$, where numeric calculation gives $R_1 < 1.2$, the equation has at least two π-periodic solutions. One of them is asymptotically stable and

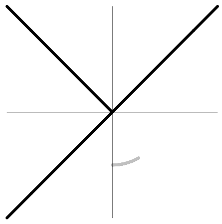

FIGURE 12. The sets $D_{\{2\}}(0)$ and $-\left(\frac{b}{a}\right)(\mathbb{R})$ for the equation(3.28).

the other is asymptotically unstable. Here $J = \{2\}$, $\alpha_{\{2\}} \geq \pi/4$ and $l = 0$, $E = 0.2$.

What is more, by setting $J = \{1, 2\}$, we prove the existence of at least one π-periodic asymptotically unstable solution provided that $R > R_2$, where $R_2 > 0$. But, by [29, Theorem 4], such a solution exists for every $R > 0$.

EXAMPLE 3.24. Let us consider the equation

$$\dot{z} = z^n + Rb(t) \tag{3.29}$$

where $n \geq 3$ and $b \in \mathcal{C}(\mathbb{R}, \mathbb{C} \setminus \{0\})$ is T-periodic and satisfies the local Lipschitz condition. By [29, Theorem 4], it has at least one T-periodic asymptotically unstable solution p rovided that $(-b)(\mathbb{R}) \subset \mathcal{S}(\pi/(n-1))$ and $R > 0$. But Theorem 3.15 gives the existence of the solution provided that $(-b)(\mathbb{R}) \subset \mathcal{S}((n/(2(n-1))\pi)$ and R is big enough. To see this it is enough to set

$$J = \begin{cases} \left\{0, \dots, \dfrac{n-3}{2}\right\} \cup \left\{\dfrac{3n-3}{2}, \dots, 2n-4, 2n-3\right\} & \text{for } 2 \nmid n, \\ \left\{0, \dots, \dfrac{n-6}{4}\right\} \cup \left\{\dfrac{3n-2}{4}, \dots, n-3, n-2\right\} & \text{for } 2 \mid n \text{ and } 4 \nmid n, \\ \left\{0, \dots, \dfrac{n-4}{4}\right\} \cup \left\{\dfrac{3n}{4}, \dots, n-3, n-2\right\} & \text{for } 4 \mid n. \end{cases}$$

REMARK 3.25. By (3.16), Theorem 3.15 does not allow to detect p-harmonic solutions for $p > 1$.

REMARK 3.26. Theorem 3.15 can be applied to the Riccati equation ($n = 2$) instead of [28, Theorem 11]. The inequality [28, (14)] can be more restrictive than (3.13). It is due to the difference in definition of the radius of the segment. In the inequality [28, (14)] the radius depends on $\mathfrak{Re}[\psi_m]$ and in (3.13) it depends on $|\psi_m|$.

EXAMPLE 3.27. Let f be such as in Example 41 from [28] i.e.

$$f(t) = \begin{cases} e^{it} & \text{for } t \in \left[-\dfrac{3}{4}\pi, \dfrac{3}{4}\pi\right], \\ e^{i((3/2)\pi - t)} & \text{for } t \in \left[\dfrac{3}{4}\pi, \dfrac{9}{4}\pi\right]. \end{cases}$$

By [28, Theorem 11], the equation $\dot z = z^2 + Rf(t)$ has two 3π-periodic solutions provided that $R > (3+2\sqrt{2})(7+4\sqrt{3})/2 \approx 40.59$. But, by Theorem 3.15, it has the solutions provided that $R > (1/4)\sin^{-4}(\pi/8) \approx 11.66$. Here $J = \emptyset$, $D_J(0) = -\mathbb{R}^+$, $\alpha_J \geq \pi/4$, $l = 0$ and $E = (1/2)\sin(\pi/8)$.

4. Multiple zeros of the vector field

In the present section we seek periodic solutions which are close to multiple zeros of the vector field v. If for an arbitrary time t the point $f(t)$ is a multiple zero of v_t, then we try to construct isolating segment W which sides contain the graph of f. If the multiplicity of zeros is an even number, then we try to construct another isolating segment Z (cf. Figure 13).

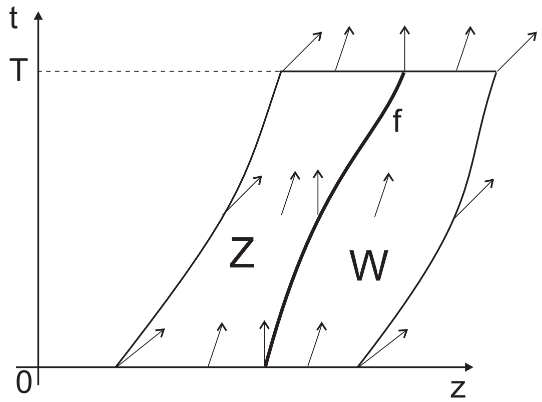

Figure 13. Schematic picture of the vector field $(1, v)$ in a neighborhood of the graph of f if f is a branch of zeros of even multiplicity. The segments W and Z are not periodic in the picture because the dimension of the z-space is only one.

To do that we need to estimate the inner product in $\mathbb{R}^3$ of the vector field $(1, v)$ and an outward normal vector at every point of the sides of W (and Z).

Let us observe that there exist holomorphic vector fields $v(t, z)$ with branches of multiply zeros such that the equation $\dot z = v(t, z)$ has no periodic solutions.

Example 4.1. The equation

$$\dot z = \frac{1}{2z}(z^2 - ire^{it})^2$$

has no periodic solutions for some values of r. To see this, we make the change of variables $w = z^2 - ire^{it}$ and get $\dot w = w^2 - re^{it}$ which has no periodic solutions for infinitely many values of r (cf. [18], [30]).

We try to obtain the existence of periodic solutions by making some assumptions on the derivatives along the branch of zeros.

We start with the following equation

$$\dot{z} = v(t, z) = Ru(t, z). \tag{4.1}$$

Let $f\colon \mathbb{R} \to \mathbb{C}$. We denote by $u_f\colon \mathbb{R} \to \mathbb{C}$ the function given by $u_f(t) = u(t, f(t))$ and write

$$\frac{\partial^k u_f}{\partial z^k}(t) = \frac{\partial^k u}{\partial z^k}(t, f(t)).$$

Theorem 4.2. *Let $k \geq 3$, $T > 0$, Ω be an open subset of a complex plain, $f \in \mathcal{C}^2(\mathbb{R}, \Omega)$ be T-periodic, $u \in \mathcal{C}^1(\mathbb{R} \times \Omega, \mathbb{C})$ be T-periodic with respect to the first variable and holomorphic with respect to the second one. Let for every $t \in \mathbb{R}$ the point $f(t)$ be a zero of $u(t, \cdot\,)$ of multiplicity k.*

If the function $[f']^{k-1} \frac{\partial^k u_f}{\partial z^k}$ fulfills the critical line condition (2.3)*, then the equation* (4.1) *has at least one T-periodic asymptotically unstable solution ξ provided that R is big enough.*

Moreover, if k is even, then the equation (4.1) *has at least one T-periodic asymptotically stable solution χ and infinitely many solutions which are heteroclinic from ξ to χ.*

If k is odd, R is big enough, and the function $-[f']^{k-1} \frac{\partial^k u_f}{\partial z^k}$ fulfills the critical line condition (2.3)*, then the equation* (4.1) *has at least one T-periodic asymptotically stable solution χ. If, additionally, $[f']^{k-1} \frac{\partial^k u_f}{\partial z^k}$ fulfills the critical line condition* (2.3)*, then there are infinitely many solutions which are heteroclinic from ξ to χ.*

Proof. Our goal is to construct T-periodic isolating segment W such that W^{--} consists of the sides of W. Then the existence of ξ is a consequence of the Denjoy–Wolff fixed point theorem.

Let us write the equation (4.1) in the following form

$$\dot{z} = v(t, z) = Rh(t, z)\,[z - f(t)]^k. \tag{4.2}$$

Obviously,

$$h(t, z) = \frac{1}{k!}\frac{\partial^k u}{\partial z^k}(t, z)$$

and $h \in \mathcal{C}^1(\mathbb{R} \times \Omega, \mathbb{C})$ is T-periodic with respect to the first variable and holomorphic with respect to the second one. We set $h_f(t) = h(t, f(t))$.

We make the change of variables given by

$$w = h^s(t, z)(z - f(t)), \tag{4.3}$$

where $s = 1/(k-1)$. Here, a neighbourhood of $f(t)$ in z-coordinates is transformed into a neighbourhood of zero in w-coordinates. By h^s we denote the branch of $(k-1)$-th root such that

$$(f' h_f^s)(\mathbb{R}) \subset \mathcal{S}\left(\frac{\pi}{k-1}\right) \tag{4.4}$$

holds. We denote by $z = \Xi(t, w)$ the inverse transformation to (4.3).

To show that there exists a branch of h^s such that (4.4) holds and (4.3) preserves the T-periodicity of solutions, we rewrite (4.3) in the form

$$w = h^s(t, [z - f(t)] + f(t))(z - f(t)).$$

Let us observe that the right hand side is T-periodic with respect to the variable t. It comes from the fact that, since $[f']^{k-1}\frac{\partial^k u_f}{\partial z^k}$ is T-periodic and fulfils the critical line condition (2.3), we have

$$\beta_T\bigg([f']^{k-1}\frac{\partial^k u_f}{\partial z^k}\bigg) = 0.$$

Therefore

$$\beta_T\bigg(\frac{\partial^k u_f}{\partial z^k}\bigg) = -(k-1)\beta_T(f')$$

and $\Big(\frac{\partial^k u_f}{\partial z^k}\Big)^s$ is T-periodic. Thus the change of coordinates (4.3) preserves T-periodicity of solutions. Moreover, since $\Big([f']^{k-1}\frac{\partial^k u_f}{\partial z^k}\Big)(\mathbb{R}) \subset \mathcal{S}(\pi)$, (4.4) holds for some branch of h^s.

Now we apply (4.3) to (4.2) and get

$$\begin{aligned} \text{(4.5)} \quad \dot{w}(t) &= Rw^k - h^s(t, \Xi(t, w))f'(t) + wsh^{-1}(t, \Xi(t, w))\frac{\partial h}{\partial t}(t, \Xi(t, w)) \\ &\quad + Rw^{k+1}sh^{-ks}(t, \Xi(t, w))\frac{\partial h}{\partial z}(t, \Xi(t, w)) = \Omega(t, w) \end{aligned}$$

where $\frac{\partial h}{\partial z}$ denotes the (complex) derivative of h with respect to the second variable and h^{-1} stands for $1/h$ (not for the inverse function).

Let us fix

$$0 < \alpha < \frac{\pi}{k-1} \le \frac{\pi}{2} \tag{4.6}$$

such that

$$([f']^{k-1}h_f)(\mathbb{R}) \subset \mathcal{S}((k-1)\alpha) \tag{4.7}$$

holds. It is possible because $(f'h_f^s)(\mathbb{R})$ is compact and (4.4) holds.

We define W to be equal to $[0, T] \times B(p)$, where $p > 0$ and $B(p) \subset \mathcal{S}(\pi/(k-1)) \cup \{0\}$ is a figure which is bouded by line segments connecting points 0, $pe^{i\alpha}$ and 0, $pe^{-i\alpha}$ and the arc between $pe^{i\alpha}$, $pe^{-i\alpha}$ parameterized by

$$r\colon \bigg[-\frac{k-1}{2}\alpha, \frac{k-1}{2}\alpha\bigg] \ni o \mapsto p\bigg[\cos\bigg(\frac{k-1}{2}\alpha\bigg)(1 + i\tan(o))\bigg]^{2/(1-k)} \in \mathbb{C}$$

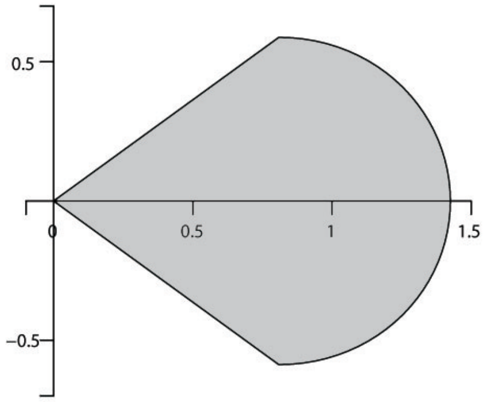

Figure 14. The set $B(1)$ for $k=4$, $\alpha=\pi/5$.

(cf. Figure 14), where r is such that $r(-(k-1)/2\alpha)=pe^{i\alpha}$ and $r((k-1)/2\alpha)=pe^{-i\alpha}$ hold.

We parameterize one side of W by function $s_1\colon[0,T]\times[0,1]\to\mathbb{R}\times\mathbb{C}$ given by

$$s_1(t,o)=(t,ope^{i\alpha}).$$

An outer normal vector is equal to $n_1\colon[0,T]\times[0,1]\ni(t,o)\mapsto(0,ie^{i\alpha})\in\mathbb{R}\times\mathbb{C}$.

We fix $p_0>0$ so small that $\Xi(s_1(t,o))$ is defined for every $(p,t,o)\in[0,p_0]\times[0,T]\times[0,1]$. We set $p\in(0,p_0]$.

The inner product of n_1 and the vector field $(1,\Omega(t,w))$ in every point of the side of W is equal to

$$\begin{aligned}
&\langle n_1(t,o),(1,\Omega(s_1(t,o)))\rangle\\
&\quad=\mathfrak{Re}\biggl[Ro^kp^ke^{ki\alpha}(-i)e^{-i\alpha}-h^s(t,\Xi(s_1(t,o)))f'(t)(-i)e^{-i\alpha}\\
&\qquad+ope^{i\alpha}s\left[h^{-1}\frac{\partial h}{\partial t}\right](t,\Xi(s_1(t,o)))(-i)e^{-i\alpha}\\
&\qquad-iR(op)^{k+1}e^{ki\alpha}s\left[h^{-ks}\frac{\partial h}{\partial z}\right](t,\Xi(s_1(t,o)))\biggr]\\
&\quad\geq R(op)^k\sin[(k-1)\alpha]+D-opE-R(op)^{k+1}F=(\star)
\end{aligned}$$

where, by (4.7)

$$D=\inf\{\mathfrak{Re}[ih^s(t,\Xi(s_1(t,o)))f'(t)e^{-i\alpha}]:(p,t,o)\in(0,p_1]\times[0,T]\times[0,1]\}>0$$

holds for $0<p_1\leq p_0$ small enough,

$$E=\max\left\{\left|s\left[h^{-1}\frac{\partial h}{\partial t}\right](t,\Xi(s_1(t,o)))\right|:(p,t,o)\in[0,p_0]\times[0,T]\times[0,1]\right\}$$

and

$$F = \max\left\{\left|s\left[h^{-ks}\frac{\partial h}{\partial z}\right](t,\Xi(s_1(t,o)))\right| : (p,t,o)\in[0,p_0]\times[0,T]\times[0,1]\right\}.$$

We fix $0 < p_1 \le p_0$ such that $D > 0$. It is easy to see, that $(\star) > 0$ for every $0 < p < \min\{p_1, D/E, \sin[(k-1)\alpha]/F\}$, so the vector field points outward the segment W.

Now we parameterize second side of W by function $s_2\colon [0,T]\times[0,1]\to\mathbb{R}\times\mathbb{C}$ given by

$$s_2(t,o) = (t, ope^{-i\alpha}).$$

An outer normal vector is equal to $n_2\colon [0,T]\times[0,1]\ni(t,o)\mapsto(0,-ie^{-i\alpha})\in\mathbb{R}\times\mathbb{C}$. The inner product of n_2 and the vector field $(1,\Omega(t,w))$ in every point of this side of W is positive for p small enough. The estimations are similar to the ones in the case of the side parameterized by s_1.

Finally, we parameterize the last side of W by

$$s_3\colon [0,T]\times\left[-\frac{k-1}{2}\alpha,\frac{k-1}{2}\alpha\right]\to\mathbb{R}\times\mathbb{C}$$

given by $s_3(t,o) = (t, r(o))$.

An outer orthogonal vector is equal to

$$n_3\colon [0,T]\times\left[-\frac{k-1}{2}\alpha,\frac{k-1}{2}\alpha\right]\ni(t,o)\mapsto(0,(1+i\tan(o))^{(k+1)/(1-k)})\in\mathbb{R}\times\mathbb{C}$$

where $n_3(t,0) = (0,1)$.

We fix $p_3 > 0$ so small that $\Xi(s_3(t,o))$ is defined for every $(p,t,o)\in[0,p_3]\times[0,T]\times[-(k-1)\alpha/2,(k-1)\alpha/2]$. We set $p\in(0,p_3]$.

The inner product of n_3 and the vector field $(1,\Omega(t,w))$ in every point of this side of W is equal to

$$\begin{aligned}
&\langle n_3(t,o),(1,\Omega(s_3(t,o)))\rangle\\
&\quad=\mathfrak{Re}\biggl\{Rp^k\left[\cos\left(\frac{k-1}{2}\alpha\right)(1+i\tan(o))\right]^{2k/(1-k)}(1-i\tan(o))^{(k+1)/(1-k)}\\
&\qquad - h^s(t,\Xi(s_3(t,o)))f'(t)(1-i\tan(o))^{(k+1)/(1-k)}\\
&\qquad + ps\left[\cos\left(\frac{k-1}{2}\alpha\right)(1+i\tan(o))\right]^{2/(1-k)}\\
&\qquad \cdot\left[h^{-1}\frac{\partial h}{\partial t}\right](t,\Xi(s_3(t,o)))(1-i\tan(o))^{(k+1)/(1-k)}\\
&\qquad + Rsp^{k+1}\left[\cos\left(\frac{k-1}{2}\alpha\right)(1+i\tan(o))\right]^{(2k+2)/(1-k)}\\
&\qquad \cdot\left[h^{-ks}\frac{\partial h}{\partial z}\right](t,\Xi(s_3(t,o)))(1-i\tan(o))^{(k+1)/(1-k)}\biggr\}\\
&\quad\ge Rp^kQ - G - pH - Rp^{k+1}J = (\star\star)
\end{aligned}$$

where

$$Q = \min\left\{\mathfrak{Re}\left(\left[\cos\left(\frac{k-1}{2}\alpha\right)(1+i\tan(o))\right]^{2k/(1-k)} \cdot (1-i\tan(o))^{(k+1)/(1-k)}\right) : o \in \left[-\frac{k-1}{2}\alpha, \frac{k-1}{2}\alpha\right]\right\} = \left[\cos\left(\frac{k-1}{2}\alpha\right)\right]^{2k/(k-1)} > 0,$$

$$G = \sup\left\{|h^s(t,\Xi(s_3(t,o)))f'(t)(1-i\tan(o))^{(k+1)/(1-k)}| : (p,t,o) \in (0,p_3] \times [0,T] \times \left[-\frac{k-1}{2}\alpha, \frac{k-1}{2}\alpha\right]\right\},$$

$$H = \sup\left\{\left|s\left[\cos\left(\frac{k-1}{2}\alpha\right)(1+i\tan(o))\right]^{2/(1-k)} \cdot \left[h^{-1}\frac{\partial h}{\partial t}\right](t,\Xi(s_3(t,o))) \cdot (1-i\tan(o))^{(k+1)/(1-k)}\right| : (p,t,o) \in (0,p_3] \times [0,T] \times \left[-\frac{k-1}{2}\alpha, \frac{k-1}{2}\alpha\right]\right\}$$

and

$$J = \sup\left\{\left|s\left[\cos\left(\frac{k-1}{2}\alpha\right)(1+i\tan(o))\right]^{(2k+2)/(1-k)} \cdot \left[h^{-ks}\frac{\partial h}{\partial z}\right](t,\Xi(s_3(t,o))) \cdot (1-i\tan(o))^{(k+1)/(1-k)}\right| : (p,t,o) \in (0,p_3] \times [0,T] \times \left[-\frac{k-1}{2}\alpha, \frac{k-1}{2}\alpha\right]\right\}.$$

It is easy to see, that $(\star\star) > 0$ for every fixed $0 < p < \min\{p_3, J/Q\}$, provided that R is big enough, so the vector field points outward the segment W.

Finally, in every point of the sides of W the vector field $(1,v)$ points outward W. Thus W is an isolating segment and, by the Denjoy–Wollf fixed point theorem, there exists exactly one T-periodic solution ξ inside W. It is asymptotically unstable.

If k is even, then it is possible to construct another isolating segment Z such that in every point of its sides the vector field $(1,\Omega)$ points inward Z. We define Z to be $[0,T]\times(-B(p))$. We omit calculations since they are quite similar to the above ones. Hence we get a T-periodic asymptotically stable solution χ inside Z.

Let η be a solution of (4.5) such that $\eta(t) = 0$ for some $t \in (0,T)$. Thus $\eta(\tau) \in B(p)$ for every $\tau \in [0,t]$ and $\eta(\tau) \in -B(p)$ for every $\tau \in [t,T]$ hold. Finally, η is heteroclinic from ξ to χ.

If k is odd and the function $-[f']^{k-1}\frac{\partial^k u_f}{\partial z^k}$ fulfils the critical line condition (2.3), then we make the change of variables (4.3), where $s = 1/(k-1)$. By h^s

we denote the branch of $(k-1)$-th root such that

$$(f'h^s_f)(\mathbb{R}) \subset \mathcal{S}\Big(-\frac{2\pi}{k-1}, 0\Big) \tag{4.8}$$

holds. Let us define Z to be the set $[0,T] \times e^{i(k-2)/(k-1)\pi}B(p)$ where $B(p)$ is as above. Now $(-f'h^s_f)(\mathbb{R}) \subset \mathcal{S}(((k-3)/(k-1))\pi, \pi/(k-1)) \supset e^{i(k-2)/(k-1)\pi}B(p)$. By calculations similar to the above ones (cf. [29, Section 4]), one can prove that in every point of the sides of Z the vector field $(1,\Omega)$ points inward Z. Thus we get a T-periodic asymptotically stable solution χ inside Z.

If, additionally, $[f']^{k-1}\frac{\partial^k u_f}{\partial z^k}$ fulfils the critical line condition (2.3), then we make the change of variables (4.3), where $s = 1/(k-1)$. Since $[f']^{k-1}\frac{\partial^k u_f}{\partial z^k}(\mathbb{R})$ is a subset of $\mathcal{S}(0,\pi)$ or $\mathcal{S}(-\pi,0)$, there exists h^s the branch of $(k-1)$-th root such that both (4.4) and (4.8) hold. Thus there exist two isolating segments W and Z, both of them containing exactly one periodic solution. Every solution η of (4.5) such that $\eta(t) = 0$ for some $t \in (0,T)$ is heteroclinic between the periodic ones. □

Remark 4.3. The above proof is not valid for $k = 2$. If $k = 2$, then the crucial estimation $\alpha < \pi/2$ in (4.6) does not hold.

Example 4.4. By Theorem 4.2, the equation

$$\dot z = Ru(t,z) = Rie^{-4it}(z - e^{it})^3 z^2$$

has at least two 2π-periodic solutions provided that R is big enough. One of them is asymptotically stable and the other one is asymptotically unstable. Here $f(t) = e^{it}$ and $[f']^2\frac{\partial^3 u_f}{\partial z^3} = -6i$ fulfils the critical line condition (2.3) and $-[f']^2\frac{\partial^3 u_f}{\partial z^3} = 6i$ also fulfils the critical line condition (2.3).

Now we state the version of Theorem 4.2 for $k = 2$.

Theorem 4.5. *Let $T > 0$, Ω be an open subset of complex plain, $f \in \mathcal{C}^2(\mathbb{R},\Omega)$ be T-periodic, $u \in \mathcal{C}^1(\mathbb{R}\times\Omega,\mathbb{C})$ be T-periodic with respect to the first variable and holomorphic with respect to the second one. Let for every $t \in \mathbb{R}$ the point $f(t)$ be a zero of $u(t,\cdot\,)$ of multiplicity 2.*

If there exists the decomposition $\{A, l, -A\}$ of the complex plane such that l is a line passing through the origin, A is a semi-plane and the following conditions

$$1 \in A, \tag{4.9}$$

$$\Big[f'\frac{\partial^2 u_f}{\partial z^2}\Big](\mathbb{R}) \subset A \tag{4.10}$$

hold, then the equation (4.1) *has at least one T-periodic asymptotically unstable solution ξ and at least one T-periodic asymptotically stable solution χ and infinitely many solutions which are heteroclinic from ξ to χ provided that R is big enough.*

PROOF. Our goal is to construct T-periodic isolating segments W and Z such that W^{--} consists of the sides of W and $Z^{++} = \emptyset$. Then the existence of ξ and χ is a consequence of the Denjoy–Wolff fixed point theorem.

Firstly we change coordinates. We do it in exactly the same way as in the proof of Theorem 4.2 i.e. by (4.3), keeping in mind that $k = 2$ (because the multiplicity of the f is now equal to 2) and so $s = 1$. We write

$$h(t, z) = \frac{1}{2}\frac{\partial^2 u}{\partial z^2}(t, z)$$

where $h \in \mathcal{C}^1(\mathbb{R} \times \Omega, \mathbb{C})$ is T-periodic with respect to the first variable and holomorphic with respect to the second one. We set $h_f(t) = h(t, f(t))$. By (4.3), a neighbourhood of $f(t)$ in z-coordinates is transformed into a neighbourhood of zero in w-coordinates. We denote by $z = \Xi(t, w)$ the inverse transformation to (4.3). Since the conditions (4.9) and (4.10) are more restrictive then the critical line condition, (4.3) preserves the T-periodicity of solutions. We fix $p_0 > 0$ so small that $\Xi(t, w)$ is defined for every $(t, w) \in [0, T] \times B(0, 3p_0)$. We set $p \in (0, p_0]$.

Now we apply (4.3) to (4.2) and get

$$\begin{aligned}(4.11)\quad \dot{w}(t) =& Rw^2 - h(t, \Xi(t, w))f'(t) + wh^{-1}(t, \Xi(t, w))\frac{\partial h}{\partial t}(t, \Xi(t, w)) \\ &+ Rw^3h^{-2}(t, \Xi(t, w))\frac{\partial h}{\partial z}(t, \Xi(t, w)) = \Omega(t, w)\end{aligned}$$

where $\frac{\partial h}{\partial z}$ denotes the (complex) derivative of h with respect to the second variable and h^{-1} stands for $1/h$ (not for the inverse function).

Let $\alpha \in (0, \pi)$ be such that $l = e^{i\alpha}\mathbb{R}$. Thus, by (4.10),

$$(4.12)\qquad \left\langle ie^{i\alpha}, \left[f'\frac{\partial^2 u_f}{\partial z^2}\right](t)\right\rangle < 0$$

holds for every $t \in \mathbb{R}$. Moreover, by periodicity of v, $\left[f'\frac{\partial^2 u_f}{\partial z^2}\right](\mathbb{R})$ is compact.

We define $W = [0, T] \times B(-pie^{i\alpha}, p)$. Parameterization of its side is given by

$$s_1 \colon [0, T] \times [0, 2\pi) \ni (t, o) \mapsto (t, pie^{i\alpha}(1 - e^{io})) \in \mathbb{R} \times \mathbb{C}.$$

An outward normal vector is given by

$$n_1 \colon [0, T] \times [0, 2\pi) \ni (t, o) \mapsto (0, ie^{i\alpha}e^{io}) \in \mathbb{R} \times \mathbb{C}.$$

The inner product of n_1 and the vector field $(1, \Omega(t,w))$ in every point of this side of W is equal to

$$
\begin{aligned}
&\langle n_1(t,o),(1,\Omega(s_1(t,o)))\rangle\\
&\quad= \mathfrak{Re}\biggl\{Rp^2 i^2 e^{2i\alpha}(1-e^{io})^2(-i)e^{-i\alpha}e^{-io} - h(t,\Xi(s_1(t,o)))f'(t)(-i)e^{-i\alpha}e^{-io}\\
&\qquad + pie^{i\alpha}(1-e^{io})\left[h^{-1}\frac{\partial h}{\partial t}\right](t,\Xi(s_1(t,o)))(-i)e^{-i\alpha}e^{-io}\\
&\qquad + Rp^3 i^3 e^{3i\alpha}(1-e^{io})^3\left[h^{-2}\frac{\partial h}{\partial z}\right](t,\Xi(s_1(t,o)))\cdot(-i)e^{-i\alpha}e^{-io}\biggr\}\\
&\quad= \mathfrak{Re}\biggl\{(-2)Rp^2 ie^{i\alpha}[1-\cos(o)] - h(t,\Xi(s_1(t,o)))f'(t)(-i)e^{-i\alpha}e^{-io}\\
&\qquad + p(e^{-io}-1)\left[h^{-1}\frac{\partial h}{\partial t}\right](t,\Xi(s_1(t,o)))\\
&\qquad - Rp^3 e^{2i\alpha}e^{-io}(1-e^{io})^3\left[h^{-2}\frac{\partial h}{\partial z}\right](t,\Xi(s_1(t,o)))\biggr\} = (\star)
\end{aligned}
$$

Let

$$l(t,p,o) = \langle f'(t)h(t,\Xi(s_1(t,o))), ie^{i\alpha}\rangle = \mathfrak{Re}[h(t,\Xi(s_1(t,o)))f'(t)(-i)e^{-i\alpha}].$$

Since, by (4.12), $l|_{[0,T]\times\{0\}\times[0,2\pi)} < 0$, we get $l(t,p,o) < 0$ for every $(t,p,o) \in [0,T]\times[0,p_1]\times[0,2\pi)$ where $0 < p_1 < p_0$ is small enough. Let us fix such a p_1. Since $l([0,T]\times[0,p_1]\times[0,2\pi))$ is compact, there exist $0<\delta<\pi$ and $B>0$ such that for every $o\in[0,\delta]\cup[2\pi-\delta,2\pi)$ and $(t,p)\in[0,T]\times[0,p_1]$ one get

$$\langle f'(t)h(t,\Xi(s_1(t,o)))e^{-io}, ie^{i\alpha}\rangle < -B < 0.$$

Let

$$
\begin{aligned}
K &= \max_{(t,p,o)\in[0,T]\times[0,p_1]\times[0,2\pi)}\{|-h(t,\Xi(s_1(t,o)))f'(t)(-i)e^{-i\alpha}e^{-io}|\},\\
G &= \max_{(t,p,o)\in[0,T]\times[0,p_1]\times[0,2\pi)}\left\{\left|(e^{-io}-1)\left[h^{-1}\frac{\partial h}{\partial t}\right](t,\Xi(s_1(t,o)))\right|\right\},\\
H &= \max_{(t,p,o)\in[0,T]\times[0,p_1]\times[0,2\pi)}\left\{\left|e^{2i\alpha}e^{-io}(1-e^{io})^3\left[h^{-2}\frac{\partial h}{\partial z}\right](t,\Xi(s_1(t,o)))\right|\right\},\\
J &= \max_{o\in[\delta,2\pi-\delta]}\{\mathfrak{Re}\left[(-2)ie^{i\alpha}[1-\cos(o)]\right]\} = 2\sin(\alpha)[1-\cos(\delta)].
\end{aligned}
$$

Now

$$(\star) \geq \begin{cases} B - pG - Rp^3H & \text{for } o\in[0,\delta]\cup[2\pi-\delta,2\pi),\\ Rp^2J - K - pG - Rp^3H & \text{for } o\in[\delta,2\pi-\delta]\end{cases}$$

and every $(t,p)\in[0,T]\times[0,p_1]$. Let

$$p_2 = \min\left\{\frac{1}{2}\,\frac{BJ}{GJ+H(K+B)}, p_1\right\}, \qquad R_0 = 2\,\frac{K+B}{Jp_2^2}.$$

So, to $(\star) > 0$ hold for every $R \geq R_0$, it is enough to find for every such R a $0 < p \leq p_2$ such that the inequalities

$$B - pG - Rp^3H > 0, \tag{4.13}$$

$$Rp^2J - K - pG - Rp^3H > 0. \tag{4.14}$$

hold. Given $R \geq R_0$ we fix $0 < p \leq p_2$ such that the formula

$$R = 2\,\frac{K+B}{Jp^2}$$

is satisfied. We claim that (4.13) and (4.14) are satisfied. Let us start with (4.13). It is easy to see that

$$\begin{aligned} B - pG - Rp^3H &= B - pG - 2\,\frac{K+B}{Jp^2}p^3H \\ &= B - p\,\frac{GJ + 2(K+B)H}{J} \\ &\geq B - \frac{1}{2}\,\frac{BJ}{GJ + H(K+B)}\,\frac{GJ + 2(K+B)H}{J} > 0 \end{aligned}$$

holds. In the case of (4.14) we get

$$\begin{aligned} Rp^2J - K - pG - Rp^3H &= 2\,\frac{K+B}{Jp^2}\,p^2J - K - pG - 2\,\frac{K+B}{Jp^2}\,p^3H \\ &= K + 2B - p\,\frac{GJ + 2(K+B)H}{J} \\ &\geq K + 2B - \frac{1}{2}\,\frac{BJ}{GJ+H(K+B)}\,\frac{GJ+2(K+B)H}{J} > 0. \end{aligned}$$

Finally, in every point of the sides of W the vector field $(1, v)$ points outward W. Thus W is an isolating segment and, by the Denjoy–Wollf fixed point theorem, there exists exactly one T-periodic solution ξ inside W. It is asymptotically unstable.

By the symmetry of the main part of equation (4.11), i.e. symmetry of $Rw^2 - h(t, \Xi(t, w))f'(t)$, the set $Z = [0, T] \times B(pie^{i\alpha}, p)$ is an isolating segment such that in every point of the sides of Z the vector field $(1, v)$ points inward Z (calculations are similar to the above). Thus, by the Denjoy–Wollf fixed point theorem, there exists exactly one T-periodic solution χ inside Z. It is asymptotically stable.

Every solution which passes through the origin is heteroclinic from ξ to χ.$\square$

Remark 4.6. As in the case of simple zeros (cf. Remark 3.11) it is possible to state Theorems 4.2 and 4.5 in more general settings. It is enough to assume that there exists D an open subset of $\mathbb{R} \times \mathbb{C}$ such that $v \in \mathcal{C}(D, \mathbb{C})$. In this case, every branch of zeros f must satisfy $(t, f(t)) \in D$ for every $t \in \mathbb{R}$.

Remark 4.7. If there are more than one branches of zeros satisfying assumptions of TheoremS 3.1, 4.2 or 4.5 we can select a finite number of them and then seek for an R big enough to guarantee for every selected branch the existence of one or two periodic solutions which are close to it. It may not be possible to use this theorem to detect infinite number of periodic solutions (cf. Remark 3.13 and Example 3.14).

Example 4.8. Let $T > 0$. By Theorem 4.2, using the change of variables from the proof of Corollary 3.5 the equation

$$\dot{z} = u\left(\frac{t}{T}, z\right) = e^{-5it/T}(z - e^{it/T})^4 z^2$$

has at least two $2\pi T$-periodic solutions provided that T is big enough. One of them is asymptotically stable and the other one is asymptotically unstable. Here $f(t) = e^{it}$ and $[f']^3 \frac{\partial^4 u_f}{\partial z^4} = -24i$ fulfils the critical line condition (2.3).

4.1. Special case. In the present subsection we deal with the equation

$$\dot{z} = u(t, z) = [a(t)\overline{f'}^{k-1}(t) + b(t)][z - f(t)]^k + c(t) \tag{4.15}$$

where $n \geq 3$ (the case $n = 2$ is investigated in [28, Theorem 20]). Here b and c are treated as perturbations. If they are zero, then the vector u field has a branch of zeros of multiplicity k.

We state the main theorem in the subsection.

Theorem 4.9. *Let $T > 0$, $k \in \mathbb{N}$, $k \geq 3$ and $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{C} \setminus \{0\})$, $b, c \in \mathcal{C}(\mathbb{R}, \mathbb{C})$, $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{C})$ be T-periodic. If there exist $p > 0$ and*

$$0 < \alpha < \frac{\pi}{k - 1} \tag{4.16}$$

such that the following conditions:

$$a(t) \in \mathcal{S}((k-1)\alpha), \tag{4.17}$$

$$|f'(t)|^2 \sin\left[\alpha - \frac{\operatorname{Arg}(a(t))}{k-1}\right] > p|a(t)|^{1/(1-k)}\left|\frac{1}{k-1}\frac{a'(t)}{a(t)} + \frac{\overline{f''}(t)}{\overline{f'}(t)}\right| + |f'(t)||c(t)|, \tag{4.18}$$

$$|b(t)| \leq \sin[(k-1)\alpha]|a(t)||f'(t)|^{k-1}, \tag{4.20}$$

$$\begin{aligned} p^k\left[1 - \left|\frac{b(t)}{\cos((k-1)\alpha/2)a(t)[\overline{f'}]^{k-1}}\right|\right] &> p\cos^{(2+2k)/(1-k)}\left(\frac{k-1}{2}\alpha\right)\left|\frac{1}{k-1}\frac{a'(t)}{a(t)} + \frac{\overline{f''}(t)}{\overline{f'}(t)}\right| \\ &\quad + \cos^{2k/(1-k)}\left(\frac{k-1}{2}\alpha\right)|f'(t)||a(t)|^{1/(k-1)}(|c(t)| + |f'(t)|) \end{aligned} \tag{4.21}$$

hold for every $t \in \mathbb{R}$, then the equation (4.15) *has at least one T-periodic asymptotically unstable solution ξ. Moreover, if k is even, then the equation* (4.15) *has additionally at least one T-periodic asymptotically stable solution χ and infinitely many solutions which are heteroclinic from ξ to χ.*

If k is odd and the conditions (4.20), (4.21) *and*

$$a(t) \in \widehat{\mathcal{S}}((k-1)\alpha), \tag{4.22}$$

$$\begin{aligned} |f'(t)|^2 \sin\left[\alpha - \frac{\operatorname{Arg}(-a(t))}{k-1}\right] \\ > p|a(t)|^{1/(1-k)}\left|\frac{1}{k-1}\frac{a'(t)}{a(t)} + \frac{\overline{f''}(t)}{\overline{f'}(t)}\right| + |f'(t)||c(t)| \end{aligned} \tag{4.23}$$

hold for every $t \in \mathbb{R}$, then the equation (4.15) *has at least one T-periodic asymptotically stable solution χ. If, additionally, the conditions* (4.17) *and* (4.18) *are satisfied, then there are infinitely many solutions which are heteroclinic from ξ to χ.*

Proof. The proof follows the line of the proof of Theorem 4.2. First of all we make the change of variables given by

$$w(t) = a^{1/(k-1)}(t)\overline{f'}(t)[z(t) - f(t)] \tag{4.24}$$

where $a^{1/(k-1)}\colon \mathbb{R} \to \mathbb{C} \setminus \{0\}$ is a continuous branch of $(k-1)$-th root of $a(t)$ such that $a^{1/(k-1)}(t) \in \mathcal{S}(\pi/(k-1))$ holds for every $t \in \mathbb{R}$.

The equation (4.15) in the new coordinates has the form

$$\begin{aligned} \dot{w} = u(t,w) = \left[1 + \frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\right]w^k - a^{1/(k-1)}(t)|f'(t)|^2 \\ + \left[\frac{1}{k-1}\frac{a'(t)}{a(t)} + \frac{\overline{f''}(t)}{\overline{f'}(t)}\right]w + a^{1/(k-1)}(t)\overline{f'}(t)c(t). \end{aligned} \tag{4.25}$$

The change of variables (4.24) preserves the T-periodicity of solutions (see the proof of Theorem 4.2).

By (4.17), one gets

$$a^{1/(k-1)}(t) \in \mathcal{S}(\alpha) \quad \text{for every } t \in \mathbb{R}. \tag{4.26}$$

Our goal is to construct an isolating segment W such that in every point of its sides the vector field $(1,u)$ points outwards. The theorem follows by the Denjoy–Wollff fixed point theorem.

We define a set W, parameterizations s_1, s_2, s_3 of its sides and outward normal vectors n_1, n_2, n_3 like in the proof of Theorem 4.2.

The inner product of n_1 and the vector field $(1, u(t,w))$ in every point of the side of W is equal to

$$
\begin{aligned}
&\langle n_1(t,o),(1,u(s_1(t,o)))\rangle \\
&= \mathfrak{Re}\bigg[-ie^{-i\alpha}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]p^k o^k e^{ik\alpha}\bigg] - \mathfrak{Re}[-ie^{-i\alpha}a^{1/(k-1)}(t)|f'(t)|^2] \\
&\quad + \mathfrak{Re}\bigg[-ie^{-i\alpha}\bigg[\frac{1}{k-1}\frac{a'(t)}{a(t)}+\frac{\overline{f''}(t)}{\overline{f'}(t)}\bigg]poe^{i\alpha}\bigg] + \mathfrak{Re}[-ie^{-i\alpha}a^{1/(k-1)}(t)\overline{f'}(t)c(t)] \\
&\geq p^k o^k \mathfrak{Re}\bigg[-ie^{(k-1)i\alpha}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]\bigg] \\
&\quad + |f'(t)|^2|a(t)|^{1/(k-1)}\sin\bigg[\alpha-\frac{\operatorname{Arg}(a(t))}{k-1}\bigg] \\
&\quad - p\bigg|\frac{1}{k-1}\frac{a'(t)}{a(t)}+\frac{\overline{f''}(t)}{\overline{f'}(t)}\bigg| - |f'(t)||a(t)|^{1/(k-1)}|c(t)| = (\star).
\end{aligned}
$$

To see that

$$\mathfrak{Re}\bigg[-ie^{(k-1)i\alpha}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]\bigg] \geq 0$$

holds it is enough to obtain

$$\operatorname{Arg}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg] \leq \min\{\pi-(k-1)\alpha,(k-1)\alpha\}.$$

But this is implied by the inequality

$$\arcsin\bigg(\bigg|\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg|\bigg) \leq \min\{\pi-(k-1)\alpha,(k-1)\alpha\}$$

which is equivalent to $|b(t)| \leq \sin((k-1)\alpha)|a(t)||f'(t)|^{k-1}$. The last inequality follows by (4.20). Thus $(\star) > 0$ provided that (4.18) holds.

Similar calculations show that the vector field $(1,u)$ points outwards W in every point of the side of W parameterized by s_2, so they are left to the reader.

Now we deal with the side parameterized by s_3. The inner product of n_3 and the vector field $(1,u(t,w))$ in every point of this side of W can be estimated by

$$
\begin{aligned}
&\langle n_3(t,o),(1,u(s_3(t,o)))\rangle \\
&\quad= \mathfrak{Re}\bigg[(1-i\tan(o))^{(k+1)/(1-k)}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]p^k \\
&\qquad \bigg[\cos\bigg(\frac{k-1}{2}\alpha\bigg)(1+i\tan(o))\bigg]^{2k/(1-k)}\bigg] \\
&\qquad - \mathfrak{Re}[(1-i\tan(o))^{(k+1)/(1-k)}a^{1/(k-1)}(t)|f'(t)|^2] \\
&\qquad + \mathfrak{Re}\bigg[(1-i\tan(o))^{(k+1)/(1-k)}\bigg[\frac{1}{k-1}\frac{a'(t)}{a(t)}+\frac{\overline{f''}(t)}{\overline{f'}(t)}\bigg]p
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left[\cos\left(\frac{k-1}{2}\alpha\right)(1+i\tan(o))\right]^{2/(1-k)}\Bigg]\\
&+\mathfrak{Re}[(1-i\tan(o))^{(k+1)/(1-k)}a^{1/(k-1)}(t)\overline{f'}(t)c(t)]\\
\geq{}& p^k\cos^{2k/(1-k)}\left(\frac{k-1}{2}\alpha\right)\cos^{4k/(k-1)}(o)\\
&\cdot\mathfrak{Re}\left[\left[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\right](1-i\tan(o))\right]\\
&-|f'(t)|a^{1/(k-1)}(t)[|f'(t)|+|c(t)|]\cos^{(k+1)/(k-1)}(o)\\
&+\mathfrak{Re}\Bigg[\cos^{(2k+2)/(k-1)}(o)(1+i\tan(o))p\\
&\cdot\cos^{2/(1-k)}\left(\frac{k-1}{2}\alpha\right)\left[\frac{1}{k-1}\frac{a'(t)}{a(t)}+\frac{\overline{f''}(t)}{\overline{f'}(t)}\right]\Bigg]\\
\geq{}& p^k\cos^{2k/(k-1)}\left(\frac{k-1}{2}\alpha\right)\left[1-\left|\frac{b(t)}{\cos((k-1)/2\alpha)a(t)[\overline{f'}]^{k-1}}\right|\right]\\
&-|f'(t)|a^{1/(k-1)}(t)[|f'(t)|+|c(t)|]\\
&-p\cos^{2/(1-k)}\left(\frac{k-1}{2}\alpha\right)\left[\frac{1}{k-1}\frac{a'(t)}{a(t)}+\frac{\overline{f''}(t)}{\overline{f'}(t)}\right]>0,
\end{aligned}
$$

provided that (4.21) holds.

Finally, W is an isolating segment and in every point of its sides the vector field $(1,u)$ points outwards. So, by the Denjoy–Wolff fixed point theorem there exists one T-periodic solution ξ inside W. It is asymptotically unstable.

If k is even, then we prove the existence of a T-periodic asymptotically stable solution χ and heteroclinic solutions just as in the proof of Theorem 4.2.

If k is odd, then we make the change of variables given by (4.24) and get (4.25). But this time, by (4.22), $a^{1/(k-1)}\colon\mathbb{R}\to\mathbb{C}\setminus\{0\}$ is a continuous branch of $(k-1)$-th root of $a(t)$ such that

$$a^{1/(k-1)}(t)\in\mathcal{S}\left(\frac{-2\pi}{k-1},0\right)$$

holds for every $t\in\mathbb{R}$.

Now we define the set $Z\subset\mathbb{R}\times\mathbb{C}$ like in the proof of Theorem 4.2 i.e. $Z=\{(t,z)\in\mathbb{R}\times\mathbb{C}:z\in e^{i((k-2)/(k-1))\pi}\pi_2(W)\}$ where $\pi_2\colon\mathbb{R}\times\mathbb{C}\to\mathbb{C}$ is given by $\pi_2(t,z)=z$. We parameterize sides of Z by s_4, s_5 and s_6, where

$$
\begin{aligned}
s_{j+3}(t,o)&=\big(t,e^{i((k-2)/(k-1))\pi}\pi_2(s_j(t,o))\big),\\
n_{j+3}(t,o)&=\big(0,e^{i((k-2)/(k-1))\pi}\pi_2(n_j(t,o))\big)
\end{aligned}
$$

holds for $j=1,2,3$.

Our goal is to show that the vector field $(1,u)$ points inward in every point of the sides of Z. We estimate

$$\begin{aligned}
&\langle n_4(t,o),(1,u(s_4(t,o)))\rangle\\
&=\mathfrak{Re}\bigg[-ie^{-i\alpha}e^{-i((k-2)/(k-1))\pi}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]p^k o^k e^{ik\alpha}e^{i((k-2)/(k-1))k\pi}\bigg]\\
&\quad-\mathfrak{Re}[-ie^{-i\alpha}e^{-i((k-2)/(k-1))\pi}a^{1/(k-1)}(t)|f'(t)|^2]\\
&\quad+\mathfrak{Re}\bigg[-ie^{-i\alpha}e^{-i((k-2)/(k-1))\pi}\bigg[\frac{1}{k-1}\frac{a'(t)}{a(t)}+\frac{\overline{f''}(t)}{\overline{f'}(t)}\bigg]poe^{i\alpha}e^{i((k-2)/(k-1))\pi}\bigg]\\
&\quad+\mathfrak{Re}[-ie^{-i\alpha}e^{-i((k-2)/(k-1))\pi}a^{1/(k-1)}(t)\overline{f'}(t)c(t)]\\
&\le p^k o^k\mathfrak{Re}\bigg[ie^{(k-1)i\alpha}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]\bigg]\\
&\quad-|f'(t)|^2|a(t)|^{1/(k-1)}\sin\bigg[\alpha-\frac{\operatorname{Arg}(-a(t))}{k-1}\bigg]\\
&\quad+p\bigg|\frac{1}{k-1}\frac{a'(t)}{a(t)}+\frac{\overline{f''}(t)}{\overline{f'}(t)}\bigg|+|f'(t)||a(t)|^{1/(k-1)}|c(t)|=(\star\star).
\end{aligned}$$

To see that

$$\mathfrak{Re}\bigg[ie^{(k-1)i\alpha}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]\bigg]\le 0$$

holds it is enough to obtain

$$\operatorname{Arg}\bigg[1+\frac{b(t)}{a(t)[\overline{f'}]^{k-1}(t)}\bigg]\le\min\{\pi-(k-1)\alpha,(k-1)\alpha\}.$$

But, as previously, this follows by (4.20). Thus $(\star\star)<0$ provided that (4.23) holds.

Estimations for s_5 are similar to the above, so we omit them.

Estimations for s_6 are quite similar to those for s_3, so also omit them.

Finally, we have shown that in every point of the sides of Z the vector field $(1,u)$ points inward Z. Thus, by the Denjoy–Wolff fixed point theorem, there exists exactly one T-periodic solution χ inside Z. It is asymptotically stable. Every solution which passes through the origin is heteroclinic from ξ to χ. □

REMARK 4.10. Combining (4.20) and (4.21) we get

$$|b(t)|\le\min\bigg\{\cos\bigg(\frac{k-1}{2}\alpha\bigg),\sin[(k-1)\alpha]\bigg\}|a(t)||f'(t)|^{k-1}$$

instead of (4.20).

EXAMPLE 4.11. By Theorem 4.9, the equation

$$\dot z=-Re^{2it}(z-Re^{-it})^3+1$$

has at least one 2π-periodic asymptotically unstable solution provided that R is big enough. Numerical estimations show that it is true for $R \geq 1.65$. Here $f = Re^{-it}$, $a \equiv 1/R$, $b \equiv 0$, $c \equiv 1$ and we fix $\alpha = 0.8$, $p = \sqrt{R}(R^3 \sin(\alpha) - 1.01)$.

Since $\operatorname{Arg}(a) \equiv 0$, the condition (4.22) does not hold, so Theorem 4.9 says nothing about asymptotically stable solutions of the equation.

Example 4.12. By Theorem 4.9, the equation

$$\dot{z} = (-2e^{-3it} + 1)(z - Re^{it})^4$$

has at least one 2π-periodic asymptotically unstable solution, one 2π-periodic asymptotically stable solution and infinitely many ones which are heteroclinic between them provided that R is big enough. Numerical estimations show that it is true for $R \geq 8.3$. Here $f = Re^{it}$, $a \equiv 2i/R^3$, $b \equiv 1$, $c \equiv 0$ and we fix $\alpha = 0.65$, $p = \sqrt[3]{R^5}(\sqrt[3]{2}\sin(\alpha - \pi/6) - 0.01)$.

Corollary 4.13. *Let $T > 0$, $k \in \mathbb{N}$, $k \geq 3$ and $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{C} \setminus \{0\})$, $b, c \in \mathcal{C}(\mathbb{R}, \mathbb{C})$, $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{C})$ be T-periodic. If a fulfills the critical line condition* (2.3)*, then every of the following equations:*

$$\dot{z} = u_1(t, z) = [Ra(t)\overline{f'}^{k-1}(t) + b(t)][z - f(t)]^k, \tag{4.27}$$

$$\dot{z} = u_2(t, z) = R^s a(t)\overline{f'}^{k-1}(t)[z - Rf(t)]^k + c(t), \tag{4.28}$$

$$\dot{z} = u_3(t, z) = [R^q a(t)\overline{f'}^{k-1}(t) + b(t)][z - Rf(t)]^k + c(t) \tag{4.29}$$

has at least one T-periodic asymptotically unstable solution ξ provided that R is big enough and

$$s > 1 - k, \qquad q > 0 \tag{4.30}$$

hold. Moreover, if k is even, then the equations (4.27)–(4.29) *has additionally at least one T-periodic asymptotically stable solution χ and infinitely many solutions which are heteroclinic from ξ to χ.*

If k is odd and $-a$ fulfills the critical line condition (2.3)*, then every of the equations* (4.27)–(4.29) *has at least one T-periodic asymptotically stable solution χ provided that R is big enough and* (4.30) *hold. If, additionally, a fulfills the critical line condition* (2.3)*, then there are infinitely many solutions which are heteroclinic from ξ to χ.*

Proof. By the periodicity of a, the set $a(\mathbb{R}) \subset \mathbb{C}$ is compact, so there exists α satisfying (4.16) such that (4.17) holds. So it is enough to prove that (4.18), (4.20) and (4.21) hold.

Let us start with the equation (4.27). We write it in the form

$$\dot{z} = \Big[\widetilde{a}(t)\overline{\widetilde{f}'}^{k-1}(t) + \widetilde{b}(t)\Big][z - \widetilde{f}(t)]^k + \widetilde{c}(t), \tag{4.31}$$

where $\widetilde{a} = Ra$, $\widetilde{b} = b$, $\widetilde{c} \equiv 0$ and $\widetilde{f} = f$. We fix $p = R^{1/(2(k-1))}$. It is easy to see that the left-hand side of (4.18) is positive and constant with respect to R while the right-hand one is proportional to $R^{-1/(2(k-1))}$. Left-hand side of (4.20) is constant with respect to R while the right-hand one is proportional to R. Left-hand side of (4.21) is positive and proportional to $R^{k/(2(k-1))}$ while the right-hand one bounded above by term proportional to $R^{1/(k-1)}$.

Let us now focus on (4.28). We write it in the form (4.31) where $\widetilde{a} = R^{s-(k-1)}a$, $\widetilde{b} \equiv 0$, $\widetilde{c} = c$ and $\widetilde{f} = Rf$. Since

$$\bigcup_{m>0} ((m-1)(k-1), (mk-1)(k-1)) = (1-k, \infty)$$

holds, for a fixed $s > 1-k$ we set $m > 0$ such that

$$s \in ((m-1)(k-1), (mk-1)(k-1)). \tag{4.32}$$

We fix $p = R^m$.

The left-hand side of (4.18) is positive and proportional to R^2 while the right-hand one is bounded above by term proportional to $R^{\max\{m+1-s/(k-1),1\}}$. Let us notice, that $m+1-s/(k-1) \in (-m(k-1)+2, 2)$. (4.20) is satisfied since its left-hand side is zero. Left-hand side of (4.21) is positive and proportional to R^{mk} while the right-hand one bounded above by term proportional to $R^{\max\{m,1+s/(k-1)\}}$. Let us notice, that $1+s/(k-1) \in (m, mk)$.

Now we deal with (4.29). We write it in the form (4.31) where $\widetilde{a} = R^{q-(k-1)}a$, $\widetilde{b} = b$, $\widetilde{c} = c$ and $\widetilde{f} = Rf$. We fix $p = R^{q/(k-1)+1/2}$.

The left-hand side of (4.18) is positive and proportional to R^2 while the right-hand one is bounded above by term proportional to $R^{3/2}$. Left-hand side of (4.20) is constant with respect to R while the right-hand one is proportional to R^q. Left-hand side of (4.21) is positive and proportional to $R^{qk/(k-1)+k/2}$ while the right-hand one bounded above by term proportional to $R^{\max\{q/(k-1)+1/2, 1+q/(k-1)\}}$.

If k is odd, then the verification of (4.23) is similar to (4.18). □

Example 4.14. By Corollary 4.13, the equation

$$\dot{z} = [R^4 e^{4it} + 120](z - Re^{-it})^5 + 1$$

has at least one 2π-periodic asymptotically unstable solution provided that R is big enough.

By Theorem 4.9, numerical estimations show that it is true for $R \geq 11$. Here $f = Re^{-it}$, $a = c \equiv 1$, $b \equiv 120$ and we fix $\alpha = 0.4$, $p = R(R\sin(\alpha) - 1.1)$.

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