Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder University Centre
Volume 39, 2012, 335–359

POINTWISE COMPARISON PRINCIPLE FOR CLAMPED TIMOSHENKO BEAM

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Abstract. We present the properties of three Green functions for:

1. general complex "clamped beam"

$$\text{(BC)}\qquad \begin{gathered} D_{\alpha,\beta}[y] \equiv y'''' - (\alpha^2+\beta^2)y'' + \alpha^2\beta^2 y = f, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{gathered}$$

2. Timoshenko clamped beam $D_{\alpha,\overline{\alpha}}[y] \equiv f$ with (BC).
3. Euler–Bernoulli clamped beam $D_{k(1+i),k(1-i)}[y] \equiv f$ with (BC).

In case 1. we represent solution via a Green operator expressed in terms of Kourensky type system of fundamental solutions for homogeneous case. This condense form is, up-to our knowledge, new even for the Euler–Bernoulli clamped beam and it allows to recognize the set of $\alpha's$ for which the Pointwise Comparison Principle for the Timoshenko beam holds. The presented approach to positivity of the Green function is much straightforward then ones known in the literature for the case 3 (see [12]).

1. Introduction

Consider the BVP for complex clamped beam

$$D_{\alpha,\beta}[y] \equiv y'''' - (\alpha^2+\beta^2)y'' + \alpha^2\beta^2 y = f, \tag{1.1}$$

$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34B27, 34B60, 34L40, 47E05, 70J10.

Key words and phrases. Timoshenko beam, Euler–Bernoulli clamped beam, Pointwise Comparison Principle, nonnegativity of Green function, fourth order ODE.

where $f \in L^1([0,1],\mathbb{C})$ and $\alpha,\beta \in \mathbb{C}\setminus\{0\}$, $\alpha^2 \neq \beta^2$. By a solution of (1.1) with boundary conditions (1.2) we mean a function

$$y \in V = W^{4,1}[0,1] \cap H_0^2[0,1]$$

satisfying (1.1) and (1.2) almost everywhere (a.e.). Actually the theory of the beam equation is quite well developed (see [1], [17]–[19], [9]). In particular, it is known that the solutions can be expressed with the use of the Green function. However in [N1] the a priori estimates are given in the integral form (nonlocal form) and there is a lack of pointwise inequalities. The reason of that is that for second order problem we have a maximum principle, while in the general situation it is actually unknown, with many counterexamples. The possibility to take advantage of the maximum principle is equivalent to nonnegativity of the Green functions. Some answers are in J. Schroeder [11]–[13], G. Sweers [14], H.-C. Grunau and G. Sweers [3], B. Kawohl and G. Sweers [6] and M. Ulm [15]. The list is far for being complete. For certain boundary conditions it is easy to obtain the nonnegativity of the Green kernel, but it is not the case of the problem we deal with. Our goal is to give an elementary analysis of properties of the Green function for Timoshenko beam

$$D_{\alpha,\overline{\alpha}}[y] = y'''' - (\alpha^2 + \overline{\alpha}^2)y'' + |\alpha|^4 y = f$$

with (1.2). However our approach is fair enough for general complex "clamped beam" $D_{\alpha,\beta}[y] = f$ with the boundary conditions (1.2). For $\alpha = k(1+i)$ we have the usual result for Euler–Bernoulli clamped beam

$$D_k[y] = y'''' + 4k^4 y = f$$

with boundary conditions (1.2).

In the complex case we derive a representation of the solution via a Green operator expressed in terms of Kourensky type system of fundamental solutions for homogeneous case. This condense form is, up-to our knowledge, new even for the Euler–Bernoulli clamped beam and it allows to recognize the set of $\alpha' s$ for which Pointwise Comparison Principle (PCP) for the Timoshenko clamped beam holds.

Definition 1.1 (PCP). We say that the operator $D_{\alpha,\overline{\alpha}}$ fulfills the Pointwise Comparison Principle (PCP) on V if for any $y,z \in V$ the inequality $D_{\alpha,\overline{\alpha}}[y] \leq D_{\alpha,\overline{\alpha}}[z]$ almost everywhere in $[0,1]$ implies that for all x is $y(x) \leq z(x)$.

Since the problem under consideration is linear the PCP is equivalent to Maximum Principle and this, in turn, is equivalent to nonnegativity of the Green function. Recall that Maximum Principle for higher order problem generally fails. Our main result gives sufficient and necessary conditions when they hold for Timoshenko beam.

Theorem 3.3 gives us a a condensed form of the Green function o considered BVP. If we write the determinants in expanded form we obtain the Green function in terms of hyperbolic trigonometric functions. Similar complicated forms appear in the paper by M. Ulm [15], who has given an elementary and straightforward analysis of the nonnegativity of the integral kernel.

The presented construction of the Green function for complex beam clearifies complicated formulas known in the literature for the Euler–Bernoulli case (see [12]). Additionally, we obtain that the Green kernel $\mathcal{G} = \mathcal{G}_{\alpha,\beta}$ to (1.1) is meromorphic in $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$.

The properties of Green functions are proved in Sections 4 and 5. In Section 2 we give some preliminary facts, while Section 3 is provided two methods of construction the solution operator of problem (1.1) with (1.2) in terms of the Green function. In our considerations some Maple symbolic calculations have been performed.

2. Preliminary notions and facts

For the corresponding homogeneous equation

$$y'''' - (\alpha^2 + \beta^2)y'' + \alpha^2\beta^2 y = 0 \tag{2.1}$$

there are many ways of choosing the system of FS. Following the idea of M. Kourensky [7] we take a system of FS

$$\begin{aligned} Y_1(x) &= \frac{\alpha^2\cosh(\alpha x) - \beta^2\cosh(\beta x)}{\alpha^2 - \beta^2}, \quad & Y_2(x) &= \frac{\alpha\sinh(\alpha x) - \beta\sinh(\beta x)}{\alpha^2 - \beta^2}, \\ Y_3(x) &= \frac{\cosh(\alpha x) - \cosh(\beta x)}{\alpha^2 - \beta^2}, & Y_4(x) &= \frac{\dfrac{\sinh(\alpha x)}{\alpha} - \dfrac{\sinh(\beta x)}{\beta}}{\alpha^2 - \beta^2}. \end{aligned} \tag{2.2}$$

We shall call this system to be complex principal, since it has similar properties to the principal systems considered by P. Hartman [5], A.Yu. Levin [8] and W.F. Trench [20]. This system has unit constant Wronskian and the following properties:

$$Y'_{i+1}(x) = Y_i(x), \quad i = 1, 2, 3$$
$$Y_1(0) = 1, \quad Y_2(0) = Y_3(0) = Y_4(0) = 0.$$

Remark 2.1. Note that functions $Y_4(x)$, $Y_3(x)$, $Y_2(x)$, $Y_1(x)$ are analytic with respect to α, β and x. In particular,

$$Y_4(x) = \frac{x^3}{6} + \ldots + \frac{x^{2n+1}}{(2n+1)!}((\alpha^{2n-2} + \ldots + \alpha^{2i}\beta^{2n-2i-2} + \ldots + \beta^{2n-2})) + \ldots$$

In what follows we shall need the following properties of complex principal system (2.2):

Proposition 2.2. *We have:*

(a)

$$Y_1'(x) = (\alpha^2+\beta^2)Y_2(x) - \alpha^2\beta^2 Y_4(x),$$
$$Y_1'''(x) = (\alpha^4+\alpha^2\beta^2+\beta^4)Y_2(x) - \alpha^2\beta^2(\alpha^2+\beta^2)Y_4(x), \tag{2.2}$$

(b)

$$\begin{bmatrix} \alpha^2\beta^2(2Y_4(x)Y_2(x) - Y_3(x)^2) - (\alpha^2+\beta^2)Y_2(x)^2 + Y_1(x)^2 \\ \alpha^2\beta^2 Y_4(x)^2 - (\alpha^2+\beta^2)Y_3(x)^2 - Y_2(x)^2 + 2Y_1(x)Y_3(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2.3}$$

(c)

$$\det\begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_3(x) & Y_3(x-t) & Y_2(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1) \end{bmatrix} - \det\begin{bmatrix} Y_3(1) & Y_2(1-t) & Y_2(1) \\ Y_4(x) & Y_3(x-t) & Y_3(x) \\ Y_4(1) & Y_3(1-t) & Y_3(1) \end{bmatrix} = -\det\begin{bmatrix} Y_2(1-t) & Y_2(x) & Y_2(1) \\ Y_3(1-t) & Y_3(x) & Y_3(1) \\ Y_4(1-t) & Y_4(x) & Y_4(1) \end{bmatrix},$$

(d)

$$\left(\begin{bmatrix} Y_4(x) \\ Y_3(x) \end{bmatrix}^T \begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{bmatrix} \begin{bmatrix} Y_4(1-t) \\ Y_3(1-t) \end{bmatrix} + \det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix} Y_4(s)\right) = \det\begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_4(x) & Y_4(s) & Y_3(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1) \end{bmatrix},$$

(e)

$$\det\begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_4(x) & 0 & Y_3(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1) \end{bmatrix} = \det\begin{bmatrix} Y_3(1) & Y_3(t) & Y_2(1) \\ Y_4(1-x) & Y_4(t-x) & Y_3(1-x) \\ Y_4(1) & Y_4(t) & Y_3(1) \end{bmatrix},$$

(f)

$$\frac{d}{dx}\left(\frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{Y_2(x)Y_4(x) - (Y_3(x))^2}\right) = -\left(\frac{Y_4(x)}{Y_2(x)Y_4(x) - (Y_3(x))^2}\right)^2.$$

Proof. (a) Since $Y_4(x)$ is a solution of (2.1) then

$$Y_4''''(x) - (\alpha^2+\beta^2)Y_4''(x) + \alpha^2\beta^2 Y_4(x) = 0.$$

But $Y_4''(x) = Y_2(x)$ and $Y_4''''(x) = Y_1'(x)$.

(b) Differentiating the left-hand side we get

$$\frac{d}{dx}\begin{bmatrix} \alpha^2\beta^2(2Y_4(x)Y_2(x) - Y_3(x)^2) - (\alpha^2+\beta^2)Y_2(x)^2 + Y_1(x)^2 \\ \alpha^2\beta^2 Y_4(x)^2 - (\alpha^2+\beta^2)Y_3(x)^2 - Y_2(x)^2 + 2Y_1(x)Y_3(x) \end{bmatrix} = 2(Y_1'(x) - (\alpha^2+\beta^2)Y_2(x) + \alpha^2\beta^2 Y_4(x)) \begin{bmatrix} Y_1(x) \\ Y_3(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} \alpha^2\beta^2(2Y_4(x)Y_2(x)-Y_3(x)^2)-(\alpha^2+\beta^2)Y_2(x)^2+Y_1(x)^2 \\ \alpha^2\beta^2Y_4(x)^2-(\alpha^2+\beta^2)Y_3(x)^2-Y_2(x)^2+2Y_1(x)Y_3(x) \end{bmatrix}$$

is a constant vector function equal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, since such is it value at $x=0$.

(c) It follows from algebraic identity

$$\det\begin{bmatrix} a & c & b \\ d & e & f \\ g & h & a \end{bmatrix} - \det\begin{bmatrix} a & k & b \\ l & e & d \\ g & c & a \end{bmatrix} + \det\begin{bmatrix} k & f & b \\ c & d & a \\ h & l & g \end{bmatrix} = 0,$$

what can be checked directly.

(d) It is a consequence of the formula $AA^{adj} = (\det A)I$, where $(A^{adj})^T$ is matrix of cofactors.

(e) By direct checking we have

$$\begin{aligned} Y_4(x-t) =& Y_1(t)Y_4(x) - Y_2(t)Y_3(x) + Y_3(t)Y_2(x) \\ & - Y_4(t)Y_1(x) - (\alpha^2+\beta^2)(Y_3(t)Y_4(x) - Y_4(t)Y_3(x)), \\ Y_3(x-t) =& Y_1(t)Y_3(x) - Y_2(t)Y_2(x) \\ & + Y_3(t)Y_1(x) + Y_4(t)Y_4(x)\alpha^2\beta^2 - Y_3(x)Y_3(t)(\alpha^2+\beta^2). \end{aligned}$$

Pluging that formulas one can check that

$$\begin{aligned} &\det\begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_4(x) & 0 & Y_3(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1) \end{bmatrix} - \det\begin{bmatrix} Y_3(1) & Y_3(t) & Y_2(1) \\ Y_4(1-x) & Y_4(t-x) & Y_3(1-x) \\ Y_4(1) & Y_4(t) & Y_3(1) \end{bmatrix} \\ &= (Y_3(t)Y_4(x) - Y_4(t)Y_3(x)) \\ &\quad \times (-Y_4(1)^2\alpha^2\beta^2 + Y_3(1)^2\alpha^2 + Y_3(1)^2\beta^2 + Y_2(1)^2 - 2Y_1(1)Y_3(1)). \end{aligned}$$

Now (b) makes the deal.

(f) It is equivalent to

$$L = (Y_2(x)Y_4(x) - (Y_3(x))^2)^2 \frac{d}{dx}\left(\frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{Y_2(x)Y_4(x) - (Y_3(x))^2}) + (Y_4(x)\right)^2 = 0.$$

But

$$\begin{aligned} L = -Y_4(x)\Bigg(& Y_1(x)^2Y_4(x) + Y_2(x)^3 - 2Y_1(x)Y_2(x)Y_3(x) \\ & + \frac{\partial Y_1(x)}{\partial x}(Y_3(x)^2 - Y_2(x)Y_4(x))\Bigg). \end{aligned}$$

Using (2.2) we obtain

$$\begin{aligned} L =& -Y_4(x)(\alpha^2\beta^2Y_4(x)(-Y_3(x)^2 + Y_2(x)Y_4(x)) \\ & - (\alpha^2+\beta^2)Y_2(x)^2Y_4(x) + (\alpha^2+\beta^2)Y_3(x)^2Y_2(x) \\ & + Y_1(x)^2Y_4(x) - 2Y_1(x)Y_2(x)Y_3(x) + Y_2(x)^3 - Y_4(x). \end{aligned}$$

Since from (2.3) is

$$-(\alpha^2+\beta^2)Y_2(x)^2 = 1-\alpha^2\beta^2(2Y_4(x)Y_2(x)-Y_3(x)^2)-Y_1(x)^2$$

then pluging we get

$$L = -Y_4(x)Y_2(x)(-\alpha^2\beta^2Y_4(x)^2+(\alpha^2+\beta^2)Y_3(x)^2+Y_2(x)^2-2Y_1(x)Y_3(x)).$$

Now the use of (2.3) again yields

$$(Y_2(x)Y_4(x)-(Y_3(x))^2)^2\frac{d}{dx}\bigg(\frac{Y_1(x)Y_4(x)-Y_2(x)Y_3(x)}{Y_2(x)Y_4(x)-(Y_3(x))^2})+(Y_4(x)\bigg)^2=0. \quad \square$$

3. Solution operators for the complex clamped beam problem

In this paper we present two ways of constructing the solutions of (1.1) with the boundary conditions (1.2). Both are done with the use of a Green function, however the second method is more suitable to prove the nonnegativity of the Green function.

3.1. Right inverse method. The construction of the Green function for the problem (1.1) with the boundary conditions (1.2) can be done in few ways. We propose one which, in a sense, is a generalization of the Hilbert resolvent formula. Observe first that the differential operator can be rewritten down in a form

$$D[y]=(y''-\alpha^2y)''-\beta^2(y''-\alpha^2y)=f.$$

This means that $D[y]=(T_{\beta^2}T_{\alpha^2})[y]$, where $T_\lambda[z]=z''-\lambda z$ for $\lambda=\alpha^2$ and $\lambda=\beta^2$, respectively. Therefore

$$y=R_{\alpha^2}R_{\beta^2}[f],$$

where R_λ is any right inverse to T_λ. In this case we have the following version of the Hilbert resolvent formula:

Proposition 3.1. *Let $z,w\in H^{2,1}[0,1]$ be functions satisfying*

(3.1) $$T_\lambda[z]=z''-\lambda z=f.$$

for $\lambda=\alpha^2$ and $\lambda=\beta^2$, respectively. Then the function

(3.2) $$y=\frac{z-w}{\alpha^2-\beta^2}$$

is a solution of (1.1).

Remark 3.2. In terms of resolvents the formula (3.2) can be written down as

$$R_{\alpha^2}R_{\beta^2}[f]=\frac{R_{\alpha^2}[f]-R_{\beta^2}[f]}{\alpha^2-\beta^2}.$$

It looks similar to the Hilbert resolvent identity, but in the Hilbert's case the domains are fixed, while in the presented above situation we can take any right inverse.

Proof. Taking $u = z - w$ we have

$$u'' = z'' - w'' = \alpha^2 z - \beta^2 w \in H^{2,1}[0,1]$$

and this implies that $u \in H^{4,1}[0,1]$. Hence

$$u'''' = \alpha^2 z'' - \beta^2 w'' = \alpha^2(\alpha^2 z + f) - \beta^2(\beta^2 w + f) = \alpha^4 z - \beta^4 w + (\alpha^2 - \beta^2) f$$

and

$$\begin{aligned} &u'''' - (\alpha^2+\beta^2)u'' + \alpha^2\beta^2 u \\ &= \alpha^4 z - \beta^4 w + (\alpha^2-\beta^2)f - (\alpha^2+\beta^2)(\alpha^2 z - \beta^2 w) + \alpha^2\beta^2(z-w) = (\alpha^2-\beta^2)f. \end{aligned}$$

Therefore $y = u/(\alpha^2 - \beta^2)$ is a solution of (1.1). □

For further considerations let us notice that the boundary conditions (1.2) can be translated as

$$z(0) = w(0), \quad z(1) = w(1), \quad z'(0) = w'(0) = A \quad \text{and} \quad z'(1) = w'(1) = B.$$

Now we can present the following construction of the Green function for (1.1) with the boundary conditions (1.2).

Theorem 3.3. *Let $f \in L^1([0,1],\mathbb{C})$ and $\alpha, \beta \in \mathbb{C}\setminus\{0\}$, $\alpha^2 \neq \beta^2$. Then the solution of* (1.1) *with the boundary conditions (1.2) is*

$$y(x) = \int_0^1 \mathcal{G}(x,t) f(t)\, dt,$$

where

$$\mathcal{G}(x,t) = \begin{cases} \dfrac{\det\begin{bmatrix} H(1,0) & H(0,1-t) & H(0,0) \\ H(1-x,0) & H(1-x,t) & H(x,0) \\ H(0,0) & H(0,t) & H(0,1)\end{bmatrix}}{\alpha\beta(\alpha^2-\beta^2)(\sinh\alpha\sinh\beta)\det\begin{bmatrix} H(1,0) & H(0,0) \\ H(0,0) & H(0,1)\end{bmatrix}} & \textit{for } 0 \le t \le x \le 1, \\[2ex] \dfrac{\det\begin{bmatrix} H(1,0) & H(0,1-t) & H(0,0) \\ H(1-x,0) & H(x,1-t) & H(x,0) \\ H(0,0) & H(0,t) & H(0,1)\end{bmatrix}}{\alpha\beta(\alpha^2-\beta^2)(\sinh\alpha\sinh\beta)\det\begin{bmatrix} H(1,0) & H(0,0) \\ H(0,0) & H(0,1)\end{bmatrix}} & \textit{for } 0 \le x \le t \le 1, \end{cases} \tag{3.3}$$

and

$$H(x,t) = \alpha \sinh\alpha \cosh(\beta t)\cosh(\beta x) - \beta\sinh\beta\cosh(\alpha t)\cosh(\alpha x).$$

Proof. We start with the equation

$$z'' - \alpha^2 z = f.$$

The general Sturm–Liouville theory (see [5]) says that for boundary conditions

$$z'(0) = z'(1) = 0$$

the solution is given by integral operator

$$z_0(x) = (T_\Phi f)(x) = \int_0^1 \Phi(x,t) f(t)\, dt,$$

where the Green function is

$$\Phi(x,t) = \frac{-1}{\alpha \sinh \alpha} \begin{cases} \cosh(\alpha t)\cosh(\alpha(1-x)) & \text{for } 0 \le t \le x \le 1, \\ \cosh(\alpha x)\cosh(\alpha(1-t)) & \text{for } 0 \le x \le t \le 1. \end{cases}$$

It can also be done directly. Indeed, differentiating twice the function

$$z_0(x) = -\frac{\cosh(\alpha(1-x))}{\alpha \sinh \alpha} \int_0^x \cosh(\alpha t) f(t)\, dt - \frac{\cosh(\alpha x)}{\alpha \sinh \alpha} \int_x^1 \cosh(\alpha(1-t)) f(t)\, dt$$

we have

$$\begin{aligned} z_0''(x) = {} & -\frac{\alpha \cosh(\alpha - x\alpha)}{\sinh \alpha} \int_0^x \cosh(t\alpha) f(t)\, dt \\ & - \frac{\alpha \cosh x\alpha}{\sinh \alpha} \int_x^1 \cosh(\alpha - t\alpha) f(t)\, dt + f(x) = \alpha^2 z_0(x) + f(x). \end{aligned}$$

Checking the boundary conditions is straightforward.

For the equation

$$z'' - \alpha^2 z = f \quad \text{with boundary conditions } z'(0) = A,\ z'(1) = B$$

we obtain the solution

$$\begin{aligned} z(x) &= z_0(x) - A\frac{\cosh(\alpha(1-x))}{\alpha \sinh \alpha} + B\frac{\cosh(\alpha x)}{\alpha \sinh \alpha} \\ &= \int_0^1 \Phi(x,t) f(t)\, dt + A\Phi(x,0) - B\Phi(x,1). \end{aligned}$$

Analogously, the problem

$$w'' - \beta^2 w = f \quad \text{with boundary conditions } w'(0) = A,\ w'(1) = B$$

possess the solution

$$w(x) = \int_0^1 \Psi(x,t) f(t)\, dt + A\Psi(x,0) - B\Psi(x,1),$$

where

$$\Psi(x,t) = \frac{-1}{\beta \sinh \beta} \begin{cases} \cosh(\beta t)\cosh(\beta(1-x)) & \text{for } 0 \le t \le x \le 1, \\ \cosh(\beta x)\cosh(\beta(1-t)) & \text{for } 0 \le x \le t \le 1. \end{cases}$$

By Proposition 2.2 we have got a family of solutions of

$$y'''' - (\alpha^2 + \beta^2) y'' + \alpha^2 \beta^2 y = f$$

with boundary condition $y'(0) = y'(1) = 0$ given by

$$y(x) = \frac{z(x) - w(x)}{\alpha^2 - \beta^2}$$

Now the boundary condition $y(0) = y(1) = 0$ lead to

$$\begin{bmatrix} A \\ B \end{bmatrix} = -\int_0^1 \begin{bmatrix} \dfrac{\det\begin{bmatrix} (\Phi(0,t)-\Psi(0,t)) & (\Phi(1,t)-\Psi(1,t)) \\ (\Phi(1,0)-\Psi(1,0)) & (\Phi(1,1)-\Psi(1,1)) \end{bmatrix}}{\det\begin{bmatrix} (\Phi(0,0)-\Psi(0,0)) & (\Phi(0,1)-\Psi(0,1)) \\ (\Phi(1,0)-\Psi(1,0)) & (\Phi(1,1)-\Psi(1,1)) \end{bmatrix}} \\ \dfrac{\det\begin{bmatrix} (\Phi(0,t)-\Psi(0,t)) & (\Phi(1,t)-\Psi(1,t)) \\ (\Phi(1,1)-\Psi(1,1)) & (\Phi(1,0)-\Psi(1,0)) \end{bmatrix}}{\det\begin{bmatrix} (\Phi(0,0)-\Psi(0,0)) & (\Phi(0,1)-\Psi(0,1)) \\ (\Phi(1,0)-\Psi(1,0)) & (\Phi(1,1)-\Psi(1,1)) \end{bmatrix}} \end{bmatrix} f(t)\,dt.$$

Hence

$$y(x) = \int_0^1 \mathcal{G}(x,t) f(t)\,dt,$$

where

$$\mathcal{G}(x,t) = \frac{\det\begin{bmatrix} (\Phi(0,0)-\Psi(0,0)) & (\Phi(0,t)-\Psi(0,t)) & (\Phi(0,1)-\Psi(0,1)) \\ (\Phi(x,0)-\Psi(x,0)) & (\Phi(x,t)-\Psi(x,t)) & (\Phi(x,1)-\Psi(x,1)) \\ (\Phi(1,0)-\Psi(1,0)) & (\Phi(1,t)-\Psi(1,t)) & (\Phi(1,1)-\Psi(1,1)) \end{bmatrix}}{(\alpha^2-\beta^2)\det\begin{bmatrix} (\Phi(0,0)-\Psi(0,0)) & (\Phi(0,1)-\Psi(0,1)) \\ (\Phi(1,0)-\Psi(1,0)) & (\Phi(1,1)-\Psi(1,1)) \end{bmatrix}}.$$

Denote now

$$H(x,t) = \alpha \sinh\alpha \cosh(\beta t)\cosh(\beta x) - \beta \sinh\beta \cosh(\alpha t)\cosh(\alpha x)$$

and observe that

$$\Phi(x,t) - \Psi(x,t) = \begin{cases} \dfrac{H(1-x,t)}{\alpha\beta \sinh\alpha \sinh\beta} & \text{for } 0 \le t \le x \le 1, \\ \dfrac{H(x,1-t)}{\alpha\beta \sinh\alpha \sinh\beta} & \text{for } 0 \le x \le t \le 1. \end{cases}$$

In particular,

$$\begin{aligned} \Phi(1,0) - \Psi(1,0) &= \Phi(0,1) - \Psi(0,1) = \frac{H(0,0)}{\alpha\beta \sinh\alpha \sinh\beta}, \\ \Phi(1,1) - \Psi(1,1) &= \Phi(0,0) - \Psi(0,0) = \frac{H(0,1)}{\alpha\beta \sinh\alpha \sinh\beta}. \end{aligned}$$

The latter easily leads to the required form of the Green function. □

3.2. Principal solution method. Theorem 3.3 gives us a condensed form of the Green function o considered BVP. If we write the determinants in expanded form we obtain the Green function in terms of hyperbolic trigonometric functions. Similar complicated forms appear in the paper by M. Ulm [15], who was concerned with nonnegativity of the integral kernel. His analysis is elementary and straightforward, however it is quite hard. Our formula (3.3) is more condensed, but it's analysis in even harder. Therefore for the direct analysis of some properties of the solution operator we need another form of it. The idea of the presented below considerations comes from a thorough analysis of the M. Kourensky paper [7] and relies on the principal solutions $Y_4, \dots, Y_1$ given by (2.2). It also gives similar formula for the Green function, but more suitable for establishing the nonnegativity.

Theorem 3.4. *Let* $f \in L^1([0,1],\mathbb{C})$ *and* $\alpha,\beta \in \mathbb{C}\setminus\{0\}$, $\alpha^2 \neq \beta^2$. *Then the solution of* (1.1) *with the boundary conditions* (1.2) *is*

$$y(x) = \int_0^1 \mathcal{G}(x,t) f(t)\,dt,$$

where

$$\mathcal{G}(x,t) = \begin{cases} \dfrac{\det\begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_4(x) & Y_4(x-t) & Y_3(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1)\end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1)\end{bmatrix}} & \text{for } 0 \le t \le x \le 1, \\[2ex] \dfrac{\det\begin{bmatrix} Y_3(1) & Y_3(t) & Y_2(1) \\ Y_4(1-x) & Y_4(t-x) & Y_3(1-x) \\ Y_4(1) & Y_4(t) & Y_3(1)\end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1)\end{bmatrix}} & \text{for } 0 \le x \le t \le 1, \end{cases} \tag{3.4}$$

$$= \begin{cases} \dfrac{\begin{bmatrix} Y_4(1-x) \\ Y_3(1-x)\end{bmatrix}^T \begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1)\end{bmatrix} \begin{bmatrix} Y_4(t) \\ Y_3(t)\end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1)\end{bmatrix}} & \text{for } 0 \le t \le x \le 1, \\[2ex] \dfrac{\begin{bmatrix} Y_4(x) \\ Y_3(x)\end{bmatrix}^T \begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1)\end{bmatrix} \begin{bmatrix} Y_4(1-t) \\ Y_3(1-t)\end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1)\end{bmatrix}} & \text{for } 0 \le x \le t \le 1. \end{cases}$$

Proof. Let us observe first that a PS of (1.1) is in convolution form

$$y_0(x) = (Y_4 * f)(x).$$

Indeed. For each function $\varphi \in C^1$ we have

$$(\varphi * f)'(x) = (\varphi' * f)(x) + \varphi(0) f(x).$$

Thus evaluating the derivatives we obtain

$$\begin{aligned} y_0'(x) &= (Y_3 * f)(x), & y_0''(x) &= (Y_2 * f)(x), \\ y_0'''(x) &= (Y_1 * f)(x), & y_0''''(x) &= (Y_1' * f)(x) + f(x). \end{aligned}$$

Hence, by (2.2), we get

$$\begin{aligned} y_0''''(x) &= (((\alpha^2+\beta^2)Y_2 - \alpha^2\beta^2 Y_4) * f)(x) + f(x) \\ &= (\alpha^2+\beta^2) y_0''(x) - \alpha^2\beta^2 y_0(x) + f(x). \end{aligned}$$

So the GS of (1.1) is

$$y(x) = AY_4(x) + BY_3(x) + CY_2(x) + DY_1(x) + (Y_4 * f)(x)$$

with

$$y'(x) = AY_3(x) + BY_2(x) + CY_1(x) + DY_1'(x) + (Y_3 * f)(x).$$

The boundary conditions at $x = 0$ give $C = D = 0$ and hence

$$\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = \begin{bmatrix} Y_4(x) & Y_3(x) \\ Y_3(x) & Y_2(x) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \left(\begin{bmatrix} Y_4 \\ Y_3 \end{bmatrix} * f \right)(x).$$

Now the boundary conditions at $x = 1$ yield the system

$$\begin{bmatrix} Y_4(1) & Y_3(1) \\ Y_3(1) & Y_2(1) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = -\left(\begin{bmatrix} Y_4 \\ Y_3 \end{bmatrix} * f \right)(1).$$

Solving we get

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{\begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{bmatrix} \left(\begin{bmatrix} Y_4 \\ Y_3 \end{bmatrix} * f \right)(1)}{\det \begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}}.$$

Thus

$$\begin{aligned}
\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} &= \frac{1}{\det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right]} \left(\left[\begin{smallmatrix} Y_4(x) & Y_3(x) \\ Y_3(x) & Y_2(x) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} Y_4 \\ Y_3 \end{smallmatrix}\right] * f \right)(1) \right. \\
&\quad \left. + \det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} Y_4 \\ Y_3 \end{smallmatrix}\right] * f \right)(x) \right) \\
&= \frac{1}{\det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right]} \left(\left[\begin{smallmatrix} Y_4(x) & Y_3(x) \\ Y_3(x) & Y_2(x) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{smallmatrix}\right] \int_0^1 \left[\begin{smallmatrix} Y_4(1-t) \\ Y_3(1-t) \end{smallmatrix}\right] f(t)\,dt \right. \\
&\quad \left. + \det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right] \int_0^x \left[\begin{smallmatrix} Y_4(x-t) \\ Y_3(x-t) \end{smallmatrix}\right] f(t)\,dt \right) \\
&= \frac{1}{\det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right]} \left(\int_0^x \left(\left[\begin{smallmatrix} Y_4(x) & Y_3(x) \\ Y_3(x) & Y_2(x) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_4(1-t) \\ Y_3(1-t) \end{smallmatrix}\right] \right. \right. \\
&\quad \left. + \det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_4(x-t) \\ Y_3(x-t) \end{smallmatrix}\right] \right) f(t)\,dt \\
&\quad \left. + \int_x^1 \left[\begin{smallmatrix} Y_4(x) & Y_3(x) \\ Y_3(x) & Y_2(x) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_4(1-t) \\ Y_3(1-t) \end{smallmatrix}\right] f(t)\,dt \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
y(x) = \frac{1}{\det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right]} &\left(\int_0^x \left(\left[\begin{smallmatrix} Y_4(x) \\ Y_3(x) \end{smallmatrix}\right]^T \left[\begin{smallmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_4(1-t) \\ Y_3(1-t) \end{smallmatrix}\right] \right. \right. \\
&\quad \left. + \det \left[\begin{smallmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{smallmatrix}\right] Y_4(x-t) \right) f(t)\,dt \\
&\quad \left. + \int_x^1 \left[\begin{smallmatrix} Y_4(x) \\ Y_3(x) \end{smallmatrix}\right]^T \left[\begin{smallmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{smallmatrix}\right] \left[\begin{smallmatrix} Y_4(1-t) \\ Y_3(1-t) \end{smallmatrix}\right] f(t)\,dt \right)
\end{aligned}$$

Applying Proposition 2.2(d) and (e) we therefore obtain

$$\begin{aligned} y(x) = {} & \frac{1}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} \bigg(\int_0^x \begin{bmatrix} Y_4(1-x) \\ Y_3(1-x) \end{bmatrix}^T \begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{bmatrix} \begin{bmatrix} Y_4(t) \\ Y_3(t) \end{bmatrix} f(t)\,dt \\ & + \int_x^1 \begin{bmatrix} Y_4(x) \\ Y_3(x) \end{bmatrix}^T \begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{bmatrix} \begin{bmatrix} Y_4(1-t) \\ Y_3(1-t) \end{bmatrix} f(t)\,dt \bigg) \\ = {} & \frac{1}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} \bigg(\int_0^x \det \begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_4(x) & Y_4(x-t) & Y_3(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1) \end{bmatrix} f(t)\,dt \\ & + \int_x^1 \det \begin{bmatrix} Y_3(1) & Y_3(t) & Y_2(1) \\ Y_4(1-x) & -Y_4(x-t) & Y_3(1-x) \\ Y_4(1) & Y_4(t) & Y_3(1) \end{bmatrix} f(t)\,dt \bigg) \end{aligned}$$

Thus the Green kernel is

$$\mathcal{G}(x,t) = \begin{cases} \dfrac{\det \begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_4(x) & Y_4(x-t) & Y_3(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1) \end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} & \text{for } 0 \le t \le x \le 1, \\[2ex] \dfrac{\begin{bmatrix} Y_3(1) & Y_3(t) & Y_2(1) \\ Y_4(1-x) & Y_4(t-x) & Y_3(1-x) \\ Y_4(1) & Y_4(t) & Y_3(1) \end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} & \text{for } 0 \le x \le t \le 1, \end{cases}$$

or

$$\mathcal{G}(x,t) = \begin{cases} \dfrac{\begin{bmatrix} Y_4(1-x) \\ Y_3(1-x) \end{bmatrix}^T \begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{bmatrix} \begin{bmatrix} Y_4(t) \\ Y_3(t) \end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} & \text{for } 0 \le t \le x \le 1, \\[2ex] \dfrac{\begin{bmatrix} Y_4(x) \\ Y_3(x) \end{bmatrix}^T \begin{bmatrix} Y_2(1) & -Y_3(1) \\ -Y_3(1) & Y_4(1) \end{bmatrix} \begin{bmatrix} Y_4(1-t) \\ Y_3(1-t) \end{bmatrix}}{\det\begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} & \text{for } 0 \le x \le t \le 1, \end{cases}$$

what completes the proof. □

4. Pointwise Comparison Principle

Theorem 3.4 gives the complex Green function for the problem (1.1) with the boundary conditions (1.2). However in applications we need to describe some situations when it is real or nonnegative.

4.1. Timoshenko beam. By Timoshenko beam we mean the case $\alpha = \overline{\beta} = a + ib$, $ab \neq 0$. This assumption implies that for $k = 1, \ldots, 4$ is

$$\overline{Y_k(x)} = Y_k(x)$$

and so all functions $Y_k(x)$ are real-valued. Furthermore, the functions:

$$W(x,t) = Y_3(x)Y_4(t) - Y_4(x)Y_3(t), \quad \frac{\partial W(x,x)}{\partial x} = Y_2(x)Y_4(x) - (Y_3(x))^2,$$
$$\frac{Y_3(x)}{Y_4(x)} \quad \text{and} \quad \frac{Y_2(x)}{Y_4(x)}$$

are real-valued as well. Additionally the function $\frac{Y_2(x)}{Y_4(x)}$ is also even, since both $Y_2(x)$ and $Y_4(x)$ are odd. Notice that, for $x \neq 0$, we have

$$\frac{\sinh^2(ax)}{(ax)^2} > 1 > \frac{\sin^2(bx)}{(bx)^2}$$

and hence

$$\frac{\partial W(x,x)}{\partial x} = Y_2(x)Y_4(x) - (Y_3(x))^2 = \frac{\dfrac{\sin^2(bx)}{(bx)^2} - \dfrac{\sinh^2(ax)}{(ax)^2}}{4x^2(a^2+b^2)} < 0.$$

Further properties of the Timoshenko beam will be provided later.

4.1.1. Zeros of $Y_4(x)$. In further analysis the crucial role play the set of zeros of $Y_4(x)$. Let $\alpha = \overline{\beta} = r(\cos\varphi + i\sin\varphi)$, where $\sin 2\varphi \neq 0$. Then

$$\begin{aligned} Y_4(x) &= \frac{\sin(rx\sin\varphi)\cosh(rx\cos\varphi)\cot\varphi - \cos(rx\sin\varphi)\sinh(rx\cos\varphi)}{2r^3\cos\varphi} \\ &= \frac{\cosh(rx\cos\varphi)\cos(rx\sin\varphi)}{2r^3\cos\varphi} T_\varphi(rx\sin\varphi), \end{aligned}$$

where $T_\varphi(t) = (\tan t)(\cot\varphi) - \tanh(t\cot\varphi)$. Observe also that $Y_4(x)$ is even with respect to φ and therefore we may assume that $\varphi \in (0,\pi/2)$. If $\cos(rx\sin\varphi) = 0$ then

$$|\sin(rx\sin\varphi)| = 1 \quad \text{and} \quad |Y_4(x)| = \frac{\cosh(rx\cos\varphi)}{2r^3|\sin\varphi|} > 0.$$

Therefore for zeros of $Y_4(x)$ we may assume that $\cos(rx\sin\varphi) \neq 0$. Notice that x is a zero of $Y_4(x)$ if and only if $t = rx\sin\varphi$ is a zero of $T_\varphi(t)$.

Lemma 4.1. *Fix $\varphi \in (0,\pi/2)$ and consider the function $T_\varphi(t)$ defined on $D = \bigcup_{n=0}^{\infty} (n\pi, \pi/2 + n\pi)$. Then $T_\varphi(t)$ possess, in each interval $(n\pi, \pi/2 + n\pi)$, $n = 1, 2, \ldots$, exactly one zero t_n. The zeros t_n are in the form $t_n = \varphi + n\pi - \varepsilon_n$, where $\varepsilon_n \in (0,\varphi) \subset (0,\pi/2)$ is such a sequence tending to 0 that*

$$\begin{aligned} \varepsilon_n = \exp(-(2\varphi + 2n\pi)\cot\varphi)\sin 2\varphi + \frac{\exp(-(4\varphi + 4n\pi)\cot\varphi)}{2}(4\sin 2\varphi + \sin 4\varphi) \\ + \frac{\exp(-(6\varphi + 6n\pi)\cot\varphi)}{24}(8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi) + R_n, \end{aligned}$$

where $|R_n| \le \exp(7 + (2\varphi - 8n\pi)\cot\varphi)$.

Proof. Both functions $(\tan t)(\cot\varphi)$ and $\tanh(t\cot\varphi)$ are increasing with $\lim_{t\to\infty} \tanh(t\cot\varphi) = 1$.

Also the function $T_\varphi(t)$ is increasing on each component of D, because $T'_\varphi(t) = (\cot\varphi)(\tan^2 t + \tanh^2(t\cot\varphi)) > 0$. So in the interval $(0,\pi/2)$ there is no zeros of $T_\varphi(t)$ because

$$(\tan t)(\cot\varphi)|_{t=0} = (\tanh(t\cot\varphi))|_{t=0} = 0.$$

For $t \in \bigcup_{n=1}^{\infty}(n\pi, \pi/2 + n\pi)$ we have the following values at the end points

$$(\tan t)(\cot\varphi)|_{t=(n\pi)^+} = 0, \quad (\tan t)(\cot\varphi)|_{t=(\pi/2+n\pi)^-} = \infty,$$
$$0 < \tanh(t\cot\varphi)|_{t=(n\pi)^+} < \tanh(t\cot\varphi)|_{t=(\pi/2+n\pi)^-} < 1.$$

Hence in every interval $(n\pi, \pi/2+n\pi)$, $n = 1, 2, \ldots$ there is exactly one zero t_n of $T_\varphi(t)$. But then $(\tan t_n)(\cot\varphi) = \tanh(t_n\cot\varphi) < 1$ and this yields $\tan t_n < \tan\varphi$. Hence each t_n can be represented in the form

$$t_n = \varphi + n\pi - \varepsilon_n,$$

where $\varepsilon_n \in (0,\varphi) \subset (0,\pi/2)$. Because $t_n \to \infty$ then we have

$$\lim_{n\to\infty}(\tan(\varphi - \varepsilon_n)) = \lim_{n\to\infty}(\tan t_n) = \lim_{n\to\infty}\{(\tan\varphi)\tanh(t_n\cot\varphi)\} = \tan\varphi.$$

Thus $\varepsilon_n \to 0$.

Observe now that $\tanh((\varphi + n\pi - \varepsilon_n)\cot\varphi) = (\tan(\varphi-\varepsilon_n))(\cot\varphi) \in (0,1)$.

Denoting $a_n = \exp(-(2\varphi + 2n\pi)\cot\varphi)$ we obtain

$$(\tan(\varphi - \varepsilon_n))(\cot\varphi) = \frac{\exp(-2\varepsilon_n\cot\varphi) - a_n}{\exp(-2\varepsilon_n\cot\varphi) + a_n}. \tag{4.1}$$

Notice that (4.1) leads to the formula:

$$a_n = \frac{\tan\varphi - \tan(\varphi-\varepsilon_n)}{\tan\varphi + \tan(\varphi - \varepsilon_n)}\exp(-2\varepsilon_n\cot\varphi).$$

In further analysis we shall examine the function

$$a = a(\varepsilon) = \frac{\tan\varphi - \tan(\varphi-\varepsilon)}{\tan\varphi + \tan(\varphi-\varepsilon)}\exp(-2\varepsilon\cot\varphi)$$

defined on the interval $(\varphi - \pi/2, \varphi)$. Thus

$$\tan(\varphi - \varepsilon) = (\tan\varphi)\frac{(1 - a(\varepsilon)\exp(2\varepsilon\cot\varphi))}{(1 + a(\varepsilon)\exp(2\varepsilon\cot\varphi))}$$

and

$$a'(\varepsilon) = \frac{(\exp(-\varepsilon\cot\varphi) - a(\varepsilon)\exp(\varepsilon\cot\varphi))^2}{\sin 2\varphi} > 0.$$

The latter yields the existence of the inverse $\varepsilon = \varepsilon(a)$ defined on $(-\exp((\pi - 2\varphi)\cot\varphi), \exp(-2\varphi\cot\varphi))$. Moreover, we have

$$\begin{aligned}
\varepsilon'(a) &= \frac{\sin 2\varphi}{(\exp(-\varepsilon\cot\varphi) - a\exp(\varepsilon\cot\varphi))^2},\\
\varepsilon''(0) &= 4\sin 2\varphi + \sin 4\varphi,\\
\varepsilon'''(0) &= \frac{1}{4}(8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon''''(a) = {} & \frac{2(\sin 2\varphi)}{(\exp(2\varphi\cot\varphi)-1)^{14}}\Bigg(12e^{12\varepsilon\cot\varphi}a^9 + 36e^{10\varepsilon\cot\varphi}(\cos 2\varphi - 2)a^8\\
& + 6e^{8\varepsilon\cot\varphi}(3\cos 4\varphi - 24\cos 2\varphi + 45)a^7\\
& + (e^{6\varepsilon\cot\varphi}(317\cos 2\varphi - 40\cos 4\varphi + 3\cos 6\varphi - 648)\\
& - 21e^{8\varepsilon\cot\varphi}(\sin^2 2\varphi))a^6\\
& + \Bigg(- 2e^{4\varepsilon\cot\varphi}(214\cos 2\varphi - 5\cos 4\varphi + 2\cos 6\varphi - 535)\\
& - \frac{1}{2}e^{6\varepsilon\cot\varphi}(21\cos 2\varphi + 84\cos 4\varphi - 21\cos 6\varphi - 4)\Bigg)a^5\\
& + (e^{2\varepsilon\cot\varphi}(265\cos 2\varphi + 64\cos 4\varphi - \cos 6\varphi - 1312)\\
& - 3e^{4\varepsilon\cot\varphi}(\sin^2 2\varphi)(4\cos 4\varphi - 22\cos 2\varphi + 79))a^4\\
& + (144\cos 2\varphi - 90\cos 4\varphi + 1242\\
& - 6e^{2\varepsilon\cot\varphi}(\sin^2 2\varphi)(10\cos 2\varphi + \cos 4\varphi - 61))a^3\\
& + \Bigg(- \frac{3}{2}(2\cos 2\varphi - 133\cos 4\varphi - 2\cos 6\varphi + 6\cos 8\varphi + 127)\\
& + e^{-2\varepsilon\cot\varphi}(8\cos 4\varphi - 481\cos 2\varphi + \cos 6\varphi - 920)\Bigg)a^2\\
& + (2e^{-4\varepsilon\cot\varphi}(214\cos 2\varphi + 31\cos 4\varphi + 2\cos 6\varphi + 239)\\
& + 6e^{-2\varepsilon\cot\varphi}(\sin^2 2\varphi)(25\cos 2\varphi - \cos 4\varphi + 47))a\\
& - e^{-6\varepsilon\cot\varphi}(137\cos 2\varphi + 32\cos 4\varphi + 3\cos 6\varphi + 120)\\
& - 3e^{-4\varepsilon\cot\varphi}(\sin^2 2\varphi)(34\cos 2\varphi + 4\cos 4\varphi + 37)\Bigg).
\end{aligned}$$

Restrict the domain of $\varepsilon(a)$ to the interval $|a| \leq \exp(-3\varphi\cot\varphi)$. Then

$$\exp(-\varepsilon\cot\varphi) - a\exp(\varepsilon\cot\varphi) \geq \exp(-\varphi\cot\varphi) - \exp(-2\varphi\cot\varphi) > 0.$$

Therefore

$$|\varepsilon''''(a)| < \frac{\exp(10 + 24\varphi\cot\varphi + 2\pi\cot\varphi)}{(\exp(2\varphi\cot\varphi) - 1)^{14}}.$$

and hence

$$\left|\frac{\varepsilon''''(a)}{24}\right| \leq \frac{\exp(7 + 24\varphi\cot\varphi + 2\pi\cot\varphi)}{(\exp(2\varphi\cot\varphi) - 1)^{14}}.$$

From the Taylor expansion formula we have for each $|a| \le \exp(-3\varphi \cot\varphi)$

$$\begin{aligned}\varepsilon(a) &= a\sin 2\varphi + \frac{a^2}{2}(4\sin 2\varphi + \sin 4\varphi)\\ &\quad + \frac{a^3}{24}(8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi) + R(a),\end{aligned}$$

where $|R(a)| \le |a|^4 \exp(7 + 24\varphi\cot\varphi + 2\pi\cot\varphi)/(\exp(2\varphi\cot\varphi) - 1)^{14}$. Therefore

$$\begin{aligned}\varepsilon_n &= \exp(-(2\varphi+2n\pi)\cot\varphi)\sin 2\varphi + \frac{\exp(-(4\varphi+4n\pi)\cot\varphi)}{2}(4\sin 2\varphi+\sin 4\varphi)\\ &\quad + \frac{\exp(-(6\varphi+6n\pi)\cot\varphi)}{24}(8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi) + R_n\end{aligned}$$

and

$$\begin{aligned}t_n \approx{}& \varphi + n\pi - \exp(-(2\varphi+2n\pi)\cot\varphi)\sin 2\varphi\\ &- \frac{\exp(-(4\varphi+4n\pi)\cot\varphi)}{2}(4\sin 2\varphi + \sin 4\varphi)\\ &- \frac{\exp(-(6\varphi+6n\pi)\cot\varphi)}{24}\\ &\times (8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi) + R_n,\end{aligned}$$

where $|R_n| \le \exp(7 + 16\varphi\cot\varphi + 2\pi(1-4n)\cot\varphi)/(\exp(2\varphi\cot\varphi) - 1)^{14}$. In particular

$$\begin{aligned}t_1 ={}& \varphi + \pi - \exp(-(2\varphi+2\pi)\cot\varphi)\sin 2\varphi\\ &- \frac{\exp(-(4\varphi+4\pi)\cot\varphi)}{2}(4\sin 2\varphi + \sin 4\varphi)\\ &- \frac{\exp(-(6\varphi+6\pi)\cot\varphi)}{24}(8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi) + R_1,\end{aligned}$$

where $|R_1| \le \exp(7 + 16\varphi\cot\varphi - 6\pi\cot\varphi)/(\exp(2\varphi\cot\varphi) - 1)^{14}$. □

Remark 4.2. If $\varphi = \pi/4$ then $t_1 \approx 3.9266$ with the accuracy $|R_1| \le 1.5052 \times 10^{-8}$ and for $\varphi = \pi/3$ we have $t_1 \approx 4.1818$ with the accuracy $|R_1| \le 2.0792 \times 10^{-3}$. But for $\varphi > 0.36022\pi$ the error $|R_1| > 0.1$.

Passing to the positive zeros of

$$Y_4(x) = \frac{\cosh(rx\cos\varphi)\cos(rx\sin\varphi)}{2r^3\cos\varphi} T_\varphi(rx\sin\varphi)$$

we have

$$x_n(\varphi) = \frac{\varphi + n\pi - \varepsilon_n}{r\sin\varphi},$$

where

$$\begin{aligned}\varepsilon_n &= \exp(-(2\varphi+2n\pi)\cot\varphi)\sin 2\varphi + \frac{\exp(-(4\varphi+4n\pi)\cot\varphi)}{2}(4\sin 2\varphi+\sin 4\varphi)\\ &\quad + \frac{\exp(-(6\varphi+6n\pi)\cot\varphi)}{24}(8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi) + R_n\end{aligned}$$

with $|R_n| \leq \exp(7 + (2\varphi - 8n\pi)\cot\varphi)$. In particular is

$$\begin{aligned} x_1(\varphi) = \frac{t_1}{r\sin\varphi} = \frac{1}{r\sin\varphi}\bigg(&\varphi + \pi - \exp(-(2\varphi+2\pi)\cot\varphi)\sin 2\varphi \\ &- \frac{\exp(-(4\varphi+4\pi)\cot\varphi)}{2}(4\sin 2\varphi + \sin 4\varphi) \\ &- \frac{\exp(-(6\varphi+6\pi)\cot\varphi)}{24} \\ &\times (8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi) - R_1\bigg), \end{aligned}$$

where $|R_1| \leq \exp(7 + (2\varphi - 8\pi)\cot\varphi)$.

Let us now observe that for the positivity of $Y_4(x)$ we need the assumption that $(0,1) \subset (0, x_1(\varphi))$. This holds for such $\alpha's$ that $x_1(\varphi) > 1$. Equivalently

$$\begin{aligned} |\alpha| + \frac{R_1}{\sin\varphi} < r_0(\varphi) = \frac{1}{\sin\varphi}\bigg(&\varphi + \pi - \exp(-(2\varphi+2\pi)\cot\varphi)\sin 2\varphi \\ &- \frac{\exp(-(4\varphi+4\pi)\cot\varphi)}{2}(4\sin 2\varphi + \sin 4\varphi) \\ &- \frac{\exp(-(6\varphi+6\pi)\cot\varphi)}{24} \\ &\times (8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi)\bigg). \end{aligned}$$

Graph of the function $r_0(\varphi)$ is presented in Figure 1.

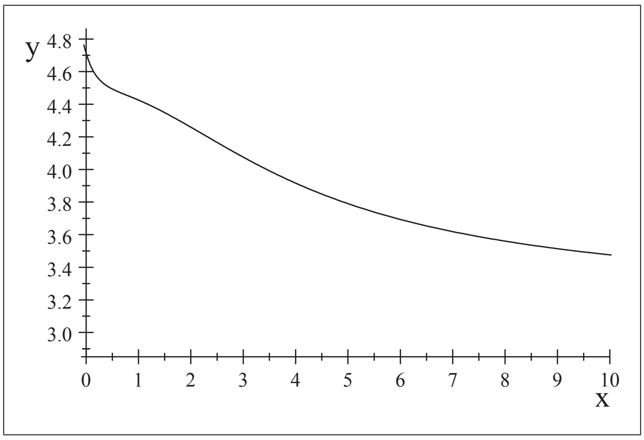

Figure 1. Graph of $r_0(\varphi)$.

Hence for $\varphi \in (0, 2\pi)$ the domain of positivity of the Green function for the clamped Timoshenko beam is between the lower and upper curves (see Figure 2).

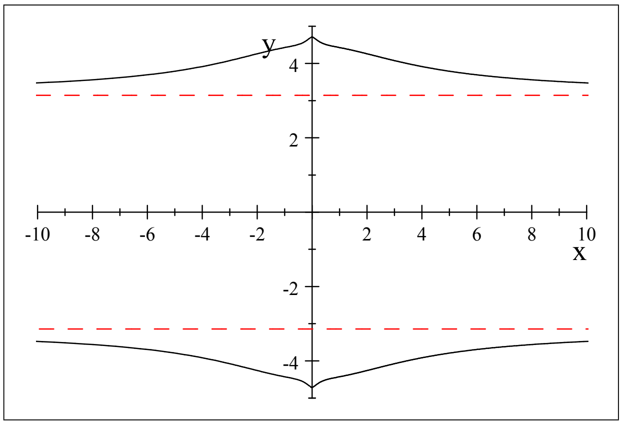

Figure 2. Asymptotes are dashed at $\pm\pi$.

Additionally for the reader wondering to make an idea of the phenomenon we make the graph of inversion

$$\frac{1}{r_0(\varphi)} = \frac{\sin\varphi}{M}$$

where

$$\begin{aligned} M = \Big(& \varphi + \pi - \exp(-(2\varphi + 2\pi)\cot\varphi)\sin 2\varphi \\ & - \frac{\exp(-(4\varphi + 4\pi)\cot\varphi)}{2}(4\sin 2\varphi + \sin 4\varphi) \\ & - \frac{\exp(-(6\varphi + 6\pi)\cot\varphi)}{24}(8\sin 6\varphi - 30\sin 4\varphi - 96\sin 2\varphi + 3\sin 8\varphi)\Big) \end{aligned}$$

and the admissible domain is out of the "apples" (see Figure 3).

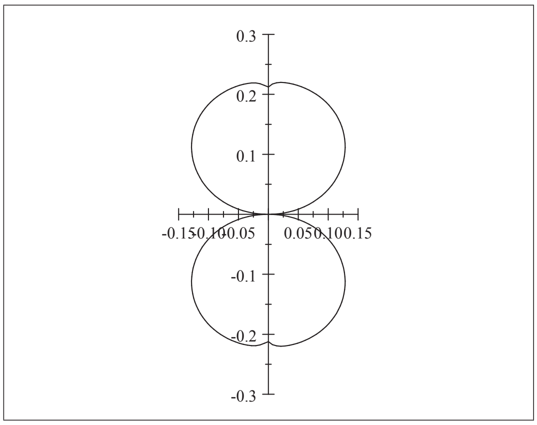

Figure 3. Graph of $\dfrac{1}{r_0(\varphi)}$.

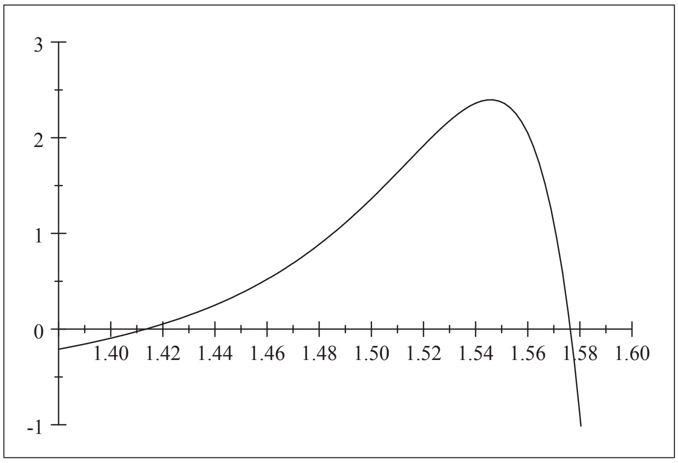

Figure 4. Graph of $\frac{dr_0}{d\varphi}$, where $\varphi_{\min} = 1.4134$ and $\varphi_{\max} = 1.5764$.

4.1.2. Properties of principal solutions. In further analysis we need some properties of principal solutions.

Proposition 4.3. *The functions*

$$\frac{Y_2(x)}{Y_4(x)}, \quad \frac{Y_3(x)}{Y_4(x)}, \quad \frac{\partial W(x,x)}{\partial x} = Y_2(x)Y_4(x) - Y_3(x)^2, \quad \frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{Y_2(x)Y_4(x) - Y_3(x)^2}$$

are strictly decreasing on $(0, x_1(\varphi))$. *Moreover, for all* $x \in (0, x_1(\varphi))$ *is*

$$\frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{Y_2(x)Y_4(x) - Y_3(x)^2} > 0.$$

Proof. We shall show then the derivatives of considered functions are negative.

(a) Differentiating the function $\frac{Y_2(x)}{Y_4(x)}$ on $(0, x_1(\varphi))$ we have

$$\frac{d}{dx}\Big(\frac{Y_2(x)}{Y_4(x)}\Big) = \frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{(Y_4(x))^2} = \frac{\dfrac{\sin 2bx}{2b} - \dfrac{\sinh 2ax}{2a}}{2|Y_4(x)|^2(a^2+b^2)}.$$

But for each $x \in \mathbb{R}$, by Mean-Value Theorem, there are $\theta_x, \vartheta_x \in (0,1)$ such that

$$\sin 2bx = 2bx\cos(2bx\theta_x) \quad \text{and} \quad \sinh 2ax = 2ax\cosh(2ax\vartheta_x).$$

Hence for any $x \in (0, x_1(\varphi))$ is

$$\frac{\sin 2bx}{2b} - \frac{\sinh 2ax}{2a} = (\cos(2bx\theta_x) - \cosh(2ax\vartheta_x))x < 0.$$

Thus for each $x \in (0, x_1(\varphi))$ is

$$\frac{d}{dx}\left(\frac{Y_2(x)}{Y_4(x)}\right) < 0.$$

(b) For the function $\frac{Y_3(x)}{Y_4(x)}$ on $(0, x_1(\varphi))$ is

$$\frac{d}{dx}\bigg(\frac{Y_3(x)}{Y_4(x)}\bigg) = \frac{Y_2(x)Y_4(x) - (Y_3(x))^2}{(Y_4(x))^2} = \frac{\dfrac{\partial W(x,x)}{\partial x}}{(Y_4(x))^2} < 0.$$

(c) For the function $Y_2(x)Y_4(x) - Y_3(x)^2 = \frac{\partial W(x,x)}{\partial x} < 0$ on $(0, x_1(\varphi))$ we have

$$\begin{aligned}\frac{d}{dx}((Y_2(x)Y_4(x) - Y_3(x)^2)) &= Y_1(x)Y_4(x) - Y_2(x)Y_3(x) \\ &= (Y_4(x))^2 \frac{d}{dx}\bigg(\frac{Y_2(x)}{Y_4(x)}\bigg) < 0.\end{aligned}$$

(d) Applying Proposition 2.2(f) we have

$$\frac{d}{dx}\bigg(\frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{Y_2(x)Y_4(x) - (Y_3(x))^2}\bigg) = -\bigg(\frac{Y_4(x)}{Y_2(x)Y_4(x) - (Y_3(x))^2}\bigg)^2 < 0.$$

To see that

$$\frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{Y_2(x)Y_4(x) - (Y_3(x))^2} > 0$$

observe that since $Y_2(x)Y_4(x) - Y_3(x)^2$ is negative and decreasing then the function $\ln((Y_3(x))^2 - Y_2(x)Y_4(x))$ is increasing. Hence

$$\begin{aligned}\frac{Y_1(x)Y_4(x) - Y_2(x)Y_3(x)}{Y_2(x)Y_4(x) - (Y_3(x))^2} &= \frac{\dfrac{d}{dx}((Y_3(x))^2 - Y_2(x)Y_4(x))}{(Y_3(x))^2 - Y_2(x)Y_4(x)} \\ &= \frac{d}{dx}(\ln((Y_3(x))^2 - Y_2(x)Y_4(x))) > 0.\end{aligned}$$

This completes the proof. □

Lemma 4.4. *Let $0 < t < s < x < x_1(\varphi)$. Then*

$$\det\begin{bmatrix} Y_2(t) & Y_2(s) & Y_2(x) \\ Y_3(t) & Y_3(s) & Y_3(x) \\ Y_4(t) & Y_4(s) & Y_4(x)\end{bmatrix} > 0.$$

Proof. Observe first that

$$\begin{aligned}\det\begin{bmatrix} Y_2(t) & Y_2(s) & Y_2(x) \\ Y_3(t) & Y_3(s) & Y_3(x) \\ Y_4(t) & Y_4(s) & Y_4(x)\end{bmatrix} &= Y_4(t)Y_4(s)Y_4(t)\bigg(\frac{Y_3(s)}{Y_4(s)} - \frac{Y_3(x)}{Y_4(x)}\bigg) \\ &\times \bigg(\frac{Y_3(t)}{Y_4(t)} - \frac{Y_3(s)}{Y_4(s)}\bigg)\left(\frac{\bigg(\dfrac{Y_2(t)}{Y_4(t)} - \dfrac{Y_2(s)}{Y_4(s)}\bigg)}{\bigg(\dfrac{Y_3(t)}{Y_4(t)} - \dfrac{Y_3(s)}{Y_4(s)}\bigg)} - \frac{\bigg(\dfrac{Y_2(s)}{Y_4(s)} - \dfrac{Y_2(x)}{Y_4(x)}\bigg)}{\bigg(\dfrac{Y_3(s)}{Y_4(s)} - \dfrac{Y_3(x)}{Y_4(x)}\bigg)}\right).\end{aligned} \tag{4.2}$$

From the Cauchy Mean-Value Theorem there are σ, ω with $0 < t < \sigma < s < \omega < x < x_1(\varphi)$ such that

$$\frac{\dfrac{Y_2(t)}{Y_4(t)} - \dfrac{Y_2(x)}{Y_4(x)}}{\dfrac{Y_3(t)}{Y_4(t)} - \dfrac{Y_3(x)}{Y_4(x)}} = \frac{\left(\dfrac{Y_2}{Y_4}\right)'(\sigma)}{\left(\dfrac{Y_3}{Y_4}\right)'(\sigma)} = \left(\frac{Y_1Y_4 - Y_2Y_3}{Y_2Y_4 - (Y_3)^2}\right)(\sigma)$$

and

$$\frac{\dfrac{Y_2(x)}{Y_4(x)} - \dfrac{Y_2(1)}{Y_4(1)}}{\dfrac{Y_3(x)}{Y_4(x)} - \dfrac{Y_3(1)}{Y_4(1)}} = \frac{\left(\dfrac{Y_2}{Y_4}\right)'(\omega)}{\left(\dfrac{Y_3}{Y_4}\right)'(\omega)} = \left(\frac{Y_1Y_4 - Y_2Y_3}{Y_2Y_4 - (Y_3)^2}\right)(\omega).$$

But the functions $\frac{Y_1Y_4 - Y_2Y_3}{Y_2Y_4 - (Y_3)^2}$ and $\frac{Y_3(x)}{Y_4(x)}$ are decreasing. Therefore

$$\frac{\dfrac{Y_2(t)}{Y_4(t)} - \dfrac{Y_2(x)}{Y_4(x)}}{\dfrac{Y_3(t)}{Y_4(t)} - \dfrac{Y_3(x)}{Y_4(x)}} - \frac{\dfrac{Y_2(x)}{Y_4(x)} - \dfrac{Y_2(1)}{Y_4(1)}}{\dfrac{Y_3(x)}{Y_4(x)} - \dfrac{Y_3(1)}{Y_4(1)}} = \left(\frac{Y_1Y_4 - Y_2Y_3}{Y_2Y_4 - (Y_3)^2}\right)(\sigma) - \left(\frac{Y_1Y_4 - Y_2Y_3}{Y_2Y_4 - (Y_3)^2}\right)(\omega) > 0$$

and

$$\frac{Y_3(s)}{Y_4(s)} - \frac{Y_3(x)}{Y_4(x)} > 0, \qquad \frac{Y_3(t)}{Y_4(t)} - \frac{Y_3(s)}{Y_4(s)} > 0.$$

Therefore all factors in (4.2) are positive, what yields our claim. □

4.2. Nonnegativity of the Green kernel. We are now ready to demonstrate the positivity of the Green function. It can be done with the use of the formula (3.4), when the integral kernel is given by

$$\mathcal{G}(x,t) = \begin{cases} \dfrac{\det \begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1) \\ Y_4(x) & Y_4(x-t) & Y_3(x) \\ Y_4(1) & Y_4(1-t) & Y_3(1) \end{bmatrix}}{\det \begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} & \text{for } 0 \le t \le x \le 1, \\[2ex] \dfrac{\det \begin{bmatrix} Y_3(1) & Y_4(1-x) & Y_4(1) \\ Y_3(t) & Y_4(t-x) & Y_4(t) \\ Y_2(1) & Y_3(1-x) & Y_3(1) \end{bmatrix}}{\det \begin{bmatrix} Y_3(1) & Y_4(1) \\ Y_2(1) & Y_3(1) \end{bmatrix}} & \text{for } 0 \le x \le t \le 1. \end{cases}$$

We shall show the following

Theorem 4.5. *Assume that $\alpha = \overline{\beta} = a+ib = r(\cos\varphi + i\sin\varphi)$, where $ab \neq 0$ and $x_1(\varphi) > 1$. Then for each $x, t \in (0,1)$ we have $\mathcal{G}(x,t) > 0$.*

Proof. We may restrict our considerations to the quarter $1/2 < x < 1$ and $1 - x < t < x$ since the Green function is symmetric with respect to both diagonals of the square $[0,1]\times[0,1]$. We shall show first that

$$\frac{\partial \mathcal{G}(x,t)}{\partial v}<0, \quad \text{where } v=\bigg[\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\bigg].$$

Using Proposition 2.2(c) we proceed as follows

$$\bigg(\det\begin{bmatrix} Y_3(1) & Y_4(1)\\ Y_2(1) & Y_3(1)\end{bmatrix}\bigg)\frac{\partial \mathcal{G}(x,t)}{\partial v}\sqrt{2}=\det\begin{bmatrix} Y_3(1) & Y_4(1)\\ Y_2(1) & Y_3(1)\end{bmatrix}\bigg(\frac{\partial \mathcal{G}(x,t)}{\partial x}+\frac{\partial \mathcal{G}(x,t)}{\partial t}\bigg)$$

$$=\det\begin{bmatrix} Y_3(1) & Y_3(1-t) & Y_2(1)\\ Y_3(x) & Y_3(x-t) & Y_2(x)\\ Y_4(1) & Y_4(1-t) & Y_3(1)\end{bmatrix}-\det\begin{bmatrix} Y_3(1) & Y_2(1-t) & Y_2(1)\\ Y_4(x) & Y_3(x-t) & Y_3(x)\\ Y_4(1) & Y_3(1-t) & Y_3(1)\end{bmatrix}$$
$$=\det\begin{bmatrix} Y_2(1-t) & Y_2(x) & Y_2(1)\\ Y_3(1-t) & Y_3(x) & Y_3(1)\\ Y_4(1-t) & Y_4(x) & Y_4(1)\end{bmatrix}.$$

But we have $1-t<x<1$, hence from Lemmas 4.1 and 4.4 we conclude that

$$\frac{\partial \mathcal{G}(x,t)}{\partial v}<0,$$

what gives our claim.

Negativity of $\frac{\partial \mathcal{G}(x,t)}{\partial v}$ means that the function $\mathcal{G}(x,t)$ is decreasing on each line $x-t=s$, where $0<s<1$. Hence for $x\in(1/2,1)$ and $t=x-s$ is

$$\mathcal{G}(x,t)=\mathcal{G}(x,x-s)>\mathcal{G}(1,1-s)=\frac{\det\begin{bmatrix} Y_2(s) & Y_2(1) & Y_2(1)\\ Y_3(s) & Y_3(1) & Y_3(1)\\ Y_4(s) & Y_4(1) & Y_4(1)\end{bmatrix}}{\det\begin{bmatrix} Y_4(1) & Y_3(1)\\ Y_3(1) & Y_2(1)\end{bmatrix}}=0,$$

what was to be proved. □

We are now in the position to present our main results concerning the clamped Timoshenko beam:

Theorem 4.6 (PCP). *Assume that $\alpha=\overline{\beta}=a+ib=r(\cos\varphi+i\sin\varphi)$, where $ab\neq 0$ and $x_1(\varphi)>1$. Then for every α such that $|\alpha|<x_1(\varphi)/\sin(\varphi)$ the Pointwise Comparison Principle holds for solutions of clamped Timoshenko beam problem*

$$y''''-(\alpha^2+\overline{\alpha}^2)y''+|\alpha|^4y=f, \tag{4.3}$$
$$y(0)=y(1)=y'(0)=y'(1)=0. \tag{4.4}$$

Proof. The formulas for solutions are the contents of Theorems 3.3 and 3.4. For $|\alpha|<x_1(\varphi)/\sin(\varphi)$ the function $\mathcal{G}_{\mathsf{T}}(x,t)\geq 0$. Thus for $f\in L^1([0,1])$ with

$f \geq 0$ we have

$$y(x) = \int_0^1 \mathcal{G}_{\mathsf{T}}(x,t) f(t)\,dt \geq 0,$$

what shows that $\mathcal{G}_{\mathsf{T}}(x,t) \geq 0$ for all $x, t \in [0,1]$. □

5. Clamped Euler–Bernoulli beam

A particular case of the problem considered in our Theorem PCP, presented in the previous section, is the following BVP

$$y'''' + 4k^4 y = f(x), \tag{5.1}$$

$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{5.2}$$

where $k > 0$ is given. The case $k = 0$ was examined by T. Boggio [2] in 1905 and $k = 1/\sqrt{2}$ by W. Gunsenheimer [4] in 1994. The general case was covered by M. Ulm [15] in 1999 and Sweers with collaborators (see, [14], [3], [6]). In our case

$$\alpha = (1 - i)k, \quad \beta = \overline{\alpha} = (1 + i)k$$

and the fundamental solutions are

$$Y_4(x) = \frac{\cosh kx \sin kx - \sinh kx \cos kx}{4k^3}, \qquad Y_3(x) = \frac{\sinh kx \sin kx}{2k^2},$$
$$Y_2(x) = \frac{\cosh kx \sin kx + \sinh kx \cos kx}{2k}, \qquad Y_1(x) = \cosh kx \cos kx.$$

Following J. Schröder [11] denote by $\varkappa = t_1 \approx 3.9266$ the first positive zero of the equation $\tanh x - \tan x = 0$. In our case the first positive zero of $Y_4(x)$ is $x_1 = x_1(\pi/4) = \varkappa/k$. Because of the boundary conditions we need to assume that $x_1 = \varkappa/k \geq 1$, which holds for $0 < k \leq \varkappa$. Therefore we have the following (PCP) result for the clamped Euler–Bernoulli beam:

Theorem 5.1. *For each $0 < k \leq \varkappa$ the solution of the problem* (5.1) *with* (5.2) *is given by formula*

$$y(x) = \int_0^1 \mathcal{G}_{\mathsf{EB}}(x,t) f(t)\,dt,$$

where the Green function $\mathcal{G}_{\mathsf{EB}}(x,t)$ is for $0 \leq t \leq x \leq 1$ given by

$$\frac{\det\begin{bmatrix} 2\sinh k \sin k & \frac{2\sinh(k(1-t))\sin(k(1-t))}{1} & \begin{pmatrix} \cosh k \sin k \\ +\sinh k \cos k \end{pmatrix} \\ \begin{pmatrix} \cosh kx \sin kx \\ -\sinh kx \cos kx \end{pmatrix} & \begin{pmatrix} \frac{\cosh(k(x-t))\sin(k(x-t))}{1} \\ -\frac{\sinh(k(x-t))\cos(k(x-t))}{1} \end{pmatrix} & \sinh kx \sin kx \\ \begin{pmatrix} \cosh k \sin k \\ -\sinh k \cos k \end{pmatrix} & \begin{pmatrix} \frac{\cosh(k(1-t))\sin(k(1-t))}{1} \\ -\frac{\sinh(k(1-t))\cos(k(1-t))}{1} \end{pmatrix} & \sinh k \sin k \end{bmatrix}}{4k^3 \det\begin{bmatrix} 2\sinh k \sin k & \cosh k \sin k + \sinh k \cos k \\ \cosh k \sin k - \sinh k \cos k & \sinh k \sin k \end{bmatrix}}$$

and for $0 \le x \le t \le 1$ *by*

$$\frac{\det\begin{bmatrix} \sin k \sinh k & \begin{pmatrix} \frac{\cos(k(1-x))\sinh(k(1-x))}{1} \\ -\frac{\sin(k(1-x))\cosh(k(1-x))}{1} \end{pmatrix} & \begin{pmatrix} \sin k \cosh k \\ -\cos k \sinh k \end{pmatrix} \\ \sinh kt \sin kt & \begin{pmatrix} \frac{\cosh(k(t-x))\sin(k(t-x))}{1} \\ -\frac{\sinh(k(t-x))\cos(k(t-x))}{1} \end{pmatrix} & \begin{pmatrix} \frac{\cos(k(1-t))\sinh(k(1-t))}{1} \\ -\frac{\sin(k(1-t))\cosh(k(1-t))}{1} \end{pmatrix} \\ \begin{pmatrix} \frac{\cos k \sinh k}{1} \\ +\frac{\sin k \cosh k}{1} \end{pmatrix} & \frac{2\sin(k(1-x))\sinh(k(1-x))}{1} & 2\sin k \sinh k \end{bmatrix}}{4k^3 \det\begin{bmatrix} 2\sinh k \sin k & \cosh k \sin k + \sinh k \cos k \\ \cosh k \sin k - \sinh k \cos k & \sinh k \sin k \end{bmatrix}}.$$

Furthermore $\mathcal{G}_{\mathsf{EB}}(x,t) > 0$ *for all* $x, t \in (0,1)$.

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Manuscript received May 13, 2011

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