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EXISTENCE OF PERIODIC TRAVELING-WAVE SOLUTIONS FOR A NONLINEAR SCHRÖDINGER SYSTEM: A TOPOLOGICAL APPROACH

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Abstract. In this paper, the existence of periodic traveling-wave solutions for a nonlinear Schrödinger system is established using the topological degree theory for positive operators. The method guarantees existence of periodic solutions in a parameter region in the period and phase speed plane.

1. Introduction

The nonlinear Schrödinger (NLS) equation

$$iu_t + \triangle u \pm |u|^2 u = 0, \tag{1.1}$$

where u is a function of $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, arises in many situations. The equation describes evolution of small amplitude, slowly varying wave packets in a nonlinear medum [2]. Indeed, it has been derived in such diverse fields as waves in deep water [13], plasma physics [14], nonlinear fiber optics [7], [8], magneto-static spin waves [15], and many others. Similarly, the $m-$coupled nonlinear Schrödinger (CNLS) system

$$i\,\frac{\partial}{\partial t}\,u_j + \triangle u_j + \sum_{k=1}^{m} a_{jk}|u_k|^2 u_j = 0, \quad x \in \mathbb{R}^N,\ j = 1,\dots,m, \tag{1.2}$$

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where u_j are complex-valued functions of $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $a_{jk} = a_{kj}$ are real numbers, arises physically under conditions similar to those described by (1.1). CNLS models physical systems in which the field has more than one component. For example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. The CNLS system also arises in the Hartree–Fock theory for a two-component Bose–Einstein condensate, i.e. a binary mixture of Bose–Einstein condensates in two distinct hyperfine states. Readers are referred to [2], [7], [8], [13], [14] for the derivation and applications of this system.

In this paper, consideration is given to the system (1.2) where $x \in \mathbb{R}$, a_{jk} are certain positive numbers such that the matrix $A = (a_{jk})$ is non-singular. A novel approach is employed to establish the existence of periodic traveling-wave solutions for the system (1.2), namely the topological degree theory for positive operators. The theory was introduced in a series of works by Krasnosel'skiĭ in [9], [10] and has been used successfully to establish existence of solutions for certain models, see for example, [1], [3], [4]. In this manuscript, this approach is applied to show the existence of solutions $(u_1(x,t), \dots, u_m(x,t))$ of (1.2) of the form

$$u_j(x,t) = \phi_j(x-\theta t)e^{i(\omega_j - \theta^2/4)t + i\theta x/2},$$

with

$$\phi_j(x-\theta t) = \sum_{n=-\infty}^{\infty} \phi_{jn} e^{i(n\pi/l)(x-\theta t)}, \tag{1.3}$$

$j = 1, \dots, m$ where l, ω_j and θ connote the half-period, phase speed and physical speed, respectively. It is proved that

(i) when $m = 2$, for any $\omega_1, \omega_2 > 0$;
(ii) when $m \geq 3$, for any $\omega_j = \omega > 0$;

and for any l large enough, there exist infinitely smooth non-trivial solutions of the system (1.2) in the form of (1.3). Notice that even though we refer to the above as periodic solutions, $\mathbf{u}(x,t) = (u_1(x,t), \dots, u_m(x,t))$ are in general quasi-periodic functions of x and t.

It should be pointed out that exact periodic *stationary* solutions of non-trivial phase to the system (1.2) have been computed explicitly in [5]. In fact, it was shown that there are two distinct types of solutions of the form $u_j(x,t) = r_j(x)e^{i(-\omega_j t + \theta_j(x))}$ to the system

$$i\frac{\partial}{\partial t}u_j + \frac{1}{2\mu_j}\frac{\partial^2}{\partial x^2}u_j + V_j(x)u_j + \sum_{k=1}^{m} a_{jk}|u_k|^2 u_j = 0, \quad x \in \mathbb{R},\ j = 1, \dots, m,$$

where $V_j(x)$ is the external potential taking the form of the square of the Jacobi elliptic sine function and μ_j are positive parameters.

2. Preliminaries and statement of results

In this section, we recall definitions that will be used and give a brief review of the topological degree theory for positive operators.

For $1 \le q < \infty$ and Ω an open set in $\mathbb{R}$, let $L^q(\Omega)$ be the usual Banach space of real or complex-valued, Lebesgue measurable functions defined on Ω with the norm

$$\|f\|^q_{L^q(\Omega)} = \int_\Omega |f|^q \, dx,$$

and $L^\infty(\Omega)$ be the space of measurable, essentially bounded functions with the norm

$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |f(x)|.$$

When it introduces no confusion, $L^q(\Omega)$ is simply written as L^q. Let $\mathbb{C}$ denote the complex field and l_q be the Banach space

$$l_q \equiv \left\{ \mathbf{u} = \{u_n\}_{n=-\infty}^{\infty} : u_n \in \mathbb{C}, \ \sum_{n=-\infty}^{\infty} |u_n|^q < \infty \right\}$$

with the norm $\|\mathbf{u}\|^q_q = \sum_{n=-\infty}^{\infty} |u_n|^q$, whereas l_∞ is defined as

$$l_\infty \equiv \left\{ \mathbf{u} = \{u_n\}_{n=-\infty}^{\infty} : u_n \in \mathbb{C}, \ \sup_{-\infty<n<\infty} |u_n|^p < \infty \right\}$$

with its usual norm $\|\mathbf{u}\|_\infty = \sup_{-\infty<n<\infty} |u_n|$. For $1 \le q \le \infty$, the Banach space of k-times product $\underbrace{l_q \times \ldots \times l_q}_{k}$ is denoted by $\mathcal{L}_{q,k}$ and is equipped with the norm

$$\|(\mathbf{u}_1, \ldots, \mathbf{u}_k)\|^q_{\mathcal{L}_{q,k}} = \sum_{n=-\infty}^{\infty} (|u_{1n}|^q + \ldots + |u_{kn}|^q) = \|\mathbf{u}_1\|^q_q + \ldots + \|\mathbf{u}_k\|^q_q.$$

The following elementary facts from analysis are recalled. Any $\mathbf{f} = \{f_n\}_{n=-\infty}^{\infty} \in l_2$ defines a periodic function f of period $2l$, where

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{in\pi x/l}. \tag{2.1}$$

Vice versa, if $f \in L^2(-l, l)$, then f can be expanded almost everywhere as a series in the form (2.1), with $f_n = (1/2l) \int_{-l}^{l} f(x) e^{-i(n\pi x/l)} \, dx$. In this sense, one can identify $f \in L^2(-l, l)$ with the sequence of its Fourier coefficients $\mathbf{f} = \{f_n\}_{n=-\infty}^{\infty}$. Moreover, $\|f\|_{L^2} = \sqrt{2l}\|\mathbf{f}\|_2$. In general, let $1 \le q \le 2$ and q' be the real number such that $1/q + 1/q' = 1$, then $l_q \subset L^{q'}(-l, l)$ in the sense that for any $\mathbf{a} = \{a_n\} \in l_q$, $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{-i(n\pi x/l)} \in L^{q'}(-l, l)$ and $\|f\|_{L^{q'}} \le (2l)^{1/q'} \|\mathbf{a}\|_q$.

For any $q, q' \geq 1$ satisfying $1/q + 1/q' = 1$, the convolution of $\mathbf{u} \in l_q$ and $\mathbf{v} \in l_{q'}$ is defined as

$$\mathbf{u} \times \mathbf{v} = \{(\mathbf{u} \times \mathbf{v})_n\}_{n=-\infty}^{\infty} \quad \text{where } (\mathbf{u} \times \mathbf{v})_n = \sum_{k=-\infty}^{\infty} u_{n-k} v_k.$$

Notice that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u} \in l_\infty$ and $\|\mathbf{u} \times \mathbf{v}\|_\infty \leq \|\mathbf{u}\|_q \|\mathbf{v}\|_{q'}$. In general, for $1 \leq q_1, \ldots, q_N \leq \infty$ satisfying $1/q_1 + \ldots + 1/q_N = N - 1$ and $\mathbf{u}_1 = \{u_{1n}\} \in l_{q_1}, \ldots, \mathbf{u}_N = \{u_{Nn}\} \in l_{q_N}$, then

$$\mathbf{w} = \{w_n\} = \mathbf{u}_1 \times \ldots \times \mathbf{u}_N \in l_\infty,$$

where

$$w_n = \sum_{k_1 + \ldots + k_N = n} u_{1k_1} \ldots u_{Nk_N} \quad \text{and} \quad \|\mathbf{w}\|_\infty \leq \|\mathbf{u}_1\|_{q_1} \ldots \|\mathbf{u}_N\|_{q_N}.$$

For the convenience of readers, a brief review of the topological degree theory for positive operators on Banach spaces is given here and we refer the readers to the works of Krasnosel'skiĭ [9], [10], Granas [6] and Benjamin *et al* [1] for details.

Let X be a Banach space equipped with the norm $\|\cdot\|_X$. We define a closed subset $K \subset X$ as a cone, if the following conditions are satisfied:

(i) $\lambda K \equiv \{\lambda f : f \in K\} \subset K$ for all $\lambda \geq 0$,
(ii) $K + K \equiv \{f + g : f, g \in K\} \subset K$,
(iii) $K \cap \{-K\} \equiv K \cap \{-f : f \in K\} = \{0\}$.

For any $0 < r < R < \infty$, denote

$$B_r = \{f \in X : \|f\|_X < r\}, \qquad \partial B_r = \{f \in X : \|f\|_X = r\},$$

$$K_r = K \cap B_r, \qquad \partial K_r = K \cap \partial B_r,$$
$$K_r^R = \{f \in K : r < \|f\|_X < R\}.$$

An operator $\mathcal{A}$ defined on K is said to be positive if $\mathcal{A}K \subset K$. A positive operator $\mathcal{A}$ is compact if $\mathcal{A}(K_r)$ has a compact closure. Note that the operator $\mathcal{A}$ is not necessarily linear. In fact, for the remaining of our paper, $\mathcal{A}$ will be nonlinear.

A triple $(K, \mathcal{A}, U)$ is called admissible if

(i) K is a convex subset of X,
(ii) $U \subset K$ is open in the relative topology on K,
(iii) $\mathcal{A}: K \to K$ is continuous and $\mathcal{A}(U)$ is a subset of a compact set in K, and
(iv) $\mathcal{A}$ has no fixed point on ∂U, the boundary of open set U in the relative topology on K.

Denote the set of all admissible triples by $\mathcal{T}$. Let $(K, \mathcal{A}, U) \in \mathcal{T}$ and $\mathcal{A}$ be a constant mapping on K, namely, there is a point $a \in K$ such that $\mathcal{A}u = a$ for every $u \in K$. The fixed point index of the positive operator $\mathcal{A}$ on U is defined as

$$i(K, \mathcal{A}, U) = \begin{cases} 1 & \text{if } a \in U, \\ 0 & \text{if } a \notin U. \end{cases}$$

We mention here, among the many properties of $i(K, \mathcal{A}, U)$, the three that will be of use in our current problem.

(a) (Homotopy invariance) If two triples $(K, \mathcal{A}, U)$ and $(K, \mathcal{B}, U) \in \mathcal{T}$ and $\mathcal{A}$ is homotopic to $\mathcal{B}$ on U, then $i(K, \mathcal{A}, U) = i(K, \mathcal{B}, U)$.

(b) (Fixed point property) If $(K, \mathcal{A}, U) \in \mathcal{T}$ and $i(K, \mathcal{A}, U) \neq 0$, then $\mathcal{A}$ has at least one fixed point in U,

(c) (Additivity) If $(K, \mathcal{A}, U) \in \mathcal{T}$ and $U_1, \dots, U_n$ is a collection of mutually disjoint open subsets of U such that $\mathcal{A}u \neq u$ for all $u \in U \setminus \bigcup_{j=1}^{n} U_j$, then

$$i(K, \mathcal{A}, U) = \sum_{j=1}^{n} i(K, \mathcal{A}, U_i).$$

The following three lemmas are taken directly from [1] in which K is a cone, the operator $\mathcal{A}$ is positive, continuous and compact on K.

Lemma 2.1. *Suppose that* $0 < \rho < \infty$ *and that either*

(a) $\mathcal{A}x - x \notin K$ *for all* $x \in \partial K_\rho$, *or*

(b) $t\mathcal{A}x \neq x$ *for all* $x \in \partial K_\rho$ *and all* $t \in [0, 1]$.

Then $i(K, \mathcal{A}, K_\rho) = 1$.

Lemma 2.2. *Suppose that* $0 < \rho < \infty$ *and that either*

(c) $x - \mathcal{A}x \notin K$ *for all* $x \in \partial K_\rho$, *or*

(d) *there exists a non-zero* $\widetilde{x} \in K$ *such that* $x - \mathcal{A}x \neq \lambda\widetilde{x}$ *for all* $x \in \partial K_\rho$ *and all* $\lambda \geq 0$.

Then $i(K, \mathcal{A}, K_\rho) = 0$.

Lemma 2.3. *Let* $(K, \mathcal{A}, U)$ *be admissible. If there exists a non-zero* $\widetilde{x} \in K$ *such that* $x - \mathcal{A}x \neq \lambda\widetilde{x}$ *for all* $x \in \partial U$ *and all* $\lambda \geq 0$, *then* $i(K, \mathcal{A}, U) = 0$.

The following theorem is an immediate consequence of the first two lemmas.

Theorem 2.4. *Suppose that either* (a) *or* (b) *holds for an* r *satisfying* $0 < r < \infty$ *and that either* (c) *or* (d) *holds for an* R *satisfying* $r < R < \infty$. *Then* $\mathcal{A}$ *has at least one fixed point in* $K_r^R \equiv \{f \in K, r < \|f\|_X < R\}$. *Moreover,* $i(K, \mathcal{A}, K_r^R) = -1$.

The theory described above will be utilized to establish the existence of periodic traveling-wave solutions for (1.2) as follows. By substituting (1.3) into system (1.2) and equating the Fourier coefficients, an infinite system is derived which can be posed as a fixed point problem on a certain cone. Using the theory above, the index of the operator associated with this fixed point problem is shown to be non-zero (hence, there must exist at least one solution in the cone). The analysis is complicated a little bit by the fact that there are several trivial (constant) solutions lying in the cone. By choosing the half-period l large enough, however, one can exclude these trivial solutions. The case $m = 2$ is studied in detail first to set foundations for the general case. The statement is as follows.

THEOREM 2.5. *For $a_{11}, a_{12}, a_{22} > 0$ such that $a_{12}^2 - a_{11}a_{22} > 0$ and phase speeds $\omega_1, \omega_2 > 0$ such that $a_{12}\omega_2 - a_{22}\omega_1 > 0$, $a_{12}\omega_1 - a_{11}\omega_2 > 0$, if the half-period l is chosen large enough, there exist infinitely smooth periodic traveling-wave solutions of the form* (1.3) *for the system* (1.2)*. The more detailed properties of such solutions will be discussed in Theorem* 3.8.

It will be made clear then how the theory can be extended to include the general case (1.2) as well. The problem one has at hand is the exclusion of the trivial fixed points. This could be a daunting task as one must consider all the sub-cases when the system (1.2) collapses to lower-order ones. (For a related discussion on this issue, see for example [11], [12].) Because of this, the special case of $\omega_j = \omega$ for all $j = 1, \dots, m$, $a_{jk} = a_{kj} = b$ when $j \neq k$, and $a_{jj} = a$ is considered since the problem of collapsing to lower-order systems of (1.2) can be handled in a straightforward manner and the trivial fixed points can be written down explicitly. The statement for this part is as follows.

THEOREM 2.6. *Let $b > a > 0$ and $\omega > 0$. If the half-period l is chosen large enough, there exist infinitely smooth periodic traveling-wave solutions of the form* (1.3) *for the system* (1.2)*. The more detailed properties of such solutions will be discussed in Theorem* 4.8.

3. The case $m = 2$

In this section, we study first the case when $m = 2$, that is, the $2-$coupled system

$$\begin{cases} iu_t + u_{xx} + (a_{11}|u|^2 + a_{12}|v|^2)u = 0, \\ iv_t + v_{xx} + (a_{21}|u|^2 + a_{22}|v|^2)v = 0. \end{cases}$$

For reason of clarity of notation, let $a_{12} = a_{21} = b$, $a_{11} = a$ and $a_{22} = c$. Hence, the system becomes

$$\begin{cases} iu_t + u_{xx} + (a|u|^2 + b|v|^2)u = 0, \\ iv_t + v_{xx} + (b|u|^2 + c|v|^2)v = 0. \end{cases} \tag{3.1}$$

Let $u(x,t) = \phi(x-\theta t)e^{i(\omega_1-\theta^2/4)t+i\theta x/2}$ and $v(x,t) = \psi(x-\theta t)e^{i(\omega_2-\theta^2/4)t+i\theta x/2}$ where $\phi, \psi \in \mathbb{R}$ be the traveling-wave solution. Substituting these expressions into (3.1) yields

$$\begin{cases} -\omega_1\phi + \phi'' + (a\phi^2 + b\psi^2)\phi = 0, \\ -\omega_2\psi + \psi'' + (b\phi^2 + c\psi^2)\psi = 0, \end{cases} \tag{3.2}$$

where the primes denote the derivatives with respect to the moving frame $\xi = x - \theta t$. Substituting (1.3) into (3.2) and equating the Fourier coefficients yield the following system

$$\begin{cases} -\omega_1\phi_n - (n\pi/l)^2\phi_n + a(\Phi\times\Phi\times\Phi)_n + b(\Phi\times\Psi\times\Psi)_n = 0, \\ -\omega_2\psi_n - (n\pi/l)^2\psi_n + b(\Phi\times\Phi\times\Psi)_n + c(\Psi\times\Psi\times\Psi)_n = 0. \end{cases} \tag{3.3}$$

The system (3.3) can be put into a more convenient matrix form

$$T_n \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} = \begin{bmatrix} a(\Phi\times\Phi\times\Phi)_n + b(\Phi\times\Psi\times\Psi)_n \\ b(\Phi\times\Phi\times\Psi)_n + c(\Psi\times\Psi\times\Psi)_n \end{bmatrix}, \tag{3.4}$$

where

$$T_n = \begin{bmatrix} \omega_1 + (n\pi/l)^2 & 0 \\ 0 & \omega_2 + (n\pi/l)^2 \end{bmatrix}. \tag{3.5}$$

Notice that for any phase speeds $\omega_1, \omega_2 > 0$, the matrix T_n is invertible for all n with

$$T_n^{-1} = \frac{1}{(\omega_1 + (n\pi/l)^2)(\omega_2 + (n\pi/l)^2)} \begin{bmatrix} \omega_2 + (n\pi/l)^2 & 0 \\ 0 & \omega_1 + (n\pi/l)^2 \end{bmatrix}. \tag{3.6}$$

The l_∞-norm of T_n^{-1} is defined as

$$\|T_n^{-1}\|_\infty = \max\left\{\frac{1}{\omega_1 + (n\pi/l)^2}, \frac{1}{\omega_2 + (n\pi/l)^2}\right\}. \tag{3.7}$$

To set up the problem as a fixed point problem, a set $K \subset \mathcal{L}_{3/2,2}$ is defined by

$$K = \{(\mathbf{u},\mathbf{v}) \in \mathcal{L}_{3/2,2} : (u_n, v_n) = (u_{-n}, v_{-n}), \\ u_0 \geq u_1 \geq \ldots \geq 0,\ v_0 \geq v_1 \geq \ldots \geq 0\}.$$

One can easily verify that K is indeed a cone in $\mathcal{L}_{3/2,2}$ equipped with the norm

$$\|(\mathbf{u},\mathbf{v})\|_{\mathcal{L}_{3/2,2}}^{3/2} = \sum_{n=-\infty}^{\infty} (|u_n|^{3/2} + |v_n|^{3/2}) = \|\mathbf{u}\|_{3/2}^{3/2} + \|\mathbf{v}\|_{3/2}^{3/2}.$$

An operator $\mathcal{A}$ on K is now defined as follows: for any $\Gamma \equiv (\Phi, \Psi) = \{(\phi_n, \psi_n)\} \in K$, $\mathcal{A}\Gamma = \{(\mathcal{A}\Gamma)_n\}$, where

$$(\mathcal{A}\Gamma)_n = T_n^{-1} \begin{bmatrix} a(\Phi\times\Phi\times\Phi)_n + b(\Phi\times\Psi\times\Psi)_n \\ b(\Phi\times\Phi\times\Psi)_n + c(\Psi\times\Psi\times\Psi)_n \end{bmatrix}. \tag{3.8}$$

Thus (3.4) can be written in the form $\Gamma = \mathcal{A}\Gamma$ and the fixed points of operator $\mathcal{A}$ in the cone K are solutions of (3.4).

LEMMA 3.1. *The operator $\mathcal{A}$ is continuous, positive and compact on the cone K.*

PROOF. (a) $\mathcal{A}$ is a positive operator on K; i.e. $\mathcal{A}$ maps K into itself.

For any $\Gamma = (\Phi, \Psi) \in K$, let

$$\begin{aligned}
\tau_n :&\equiv a(\Phi \times \Phi \times \Phi)_n + b(\Phi \times \Psi \times \Psi)_n \\
&= a \sum_{j,k=-\infty}^{\infty} \phi_k \phi_{j-k} \phi_{n-j} + b \sum_{j,k=-\infty}^{\infty} \psi_k \psi_{j-k} \phi_{n-j}; \\
\eta_n :&\equiv b(\Phi \times \Phi \times \Psi)_n + c(\Psi \times \Psi \times \Psi)_n \\
&= b \sum_{j,k=-\infty}^{\infty} \phi_k \phi_{j-k} \psi_{n-j} + c \sum_{j,k=-\infty}^{\infty} \psi_k \psi_{j-k} \psi_{n-j}.
\end{aligned}$$

Using the facts that $\phi_{-n} = \phi_n$ and $\psi_{-n} = \psi_n$, it is easy to verify that for all $n \geq 0$, $\tau_{-n} = \tau_n$ and $\eta_{-n} = \eta_n$. Notice next that

$$\begin{aligned}
&a \sum_{j,k=-\infty}^{\infty} \phi_k \phi_{j-k} \phi_{n-j} - a \sum_{j,k=-\infty}^{\infty} \phi_k \phi_{j-k} \phi_{n+1-j} \\
&\quad = a \sum_{j,k=0}^{\infty} \big(\phi_k \phi_{j-k}(\phi_{n-j} - \phi_{n+1-j}) + \phi_k \phi_{j+1-k}(\phi_{n+1+j} - \phi_{n+2+j}) \\
&\qquad + \phi_{k+1}\phi_{j+1+k}(\phi_{n-j} - \phi_{n+1-j}) + \phi_{k+1}\phi_{j+2+k}(\phi_{n+1+j} - \phi_{n+2+j})\big) \geq 0.
\end{aligned}$$

Likewise, one can see that

$$b \sum_{j,k=-\infty}^{\infty} \psi_k \psi_{j-k} \phi_{n-j} - b \sum_{j,k=-\infty}^{\infty} \psi_k \psi_{j-k} \phi_{n+1-j} \geq 0.$$

Therefore, τ_n is a decreasing function of $|n|$ and

$$0 \leq \tau_n \leq \tau_0 \leq \max\{a, b\}(\|\Phi\|_{3/2}^3 + \|\Psi\|_{3/2}^2 \|\Phi\|_{3/2}). \tag{3.9}$$

Similar arguments show that η_n is also a decreasing function of $|n|$ and

$$0 \leq \eta_n \leq \eta_0 \leq \max\{b, c\}(\|\Psi\|_{3/2}^3 + \|\Phi\|_{3/2}^2 \|\Psi\|_{3/2}). \tag{3.10}$$

Since each entry $T_n^{-1}(i,j)$ of T_n^{-1} is positive, even in n, decreasing with respect to $|n|$ and

$$\|\mathcal{A}\Gamma\|_{\mathcal{L}_{3/2,2}}^{3/2} \leq \sum_{n=-\infty}^{\infty} \|T_n^{-1}\|_\infty^{3/2} (|\tau_n|^{3/2} + |\eta_n|^{3/2}) < \infty,$$

it follows immediately that $\mathcal{A}K \subset K$.

(b) $\mathcal{A}$ is continuous.

Let $\Gamma = (\Phi, \Psi)$ and $\overline{\Gamma} = (\overline{\Phi}, \overline{\Psi})$ be two arbitrary elements in K. For all n, the difference $(\mathcal{A}\Gamma)_n - (\mathcal{A}\overline{\Gamma})_n$ can be bounded component-wise, namely,

(i) $|(\Phi \times \Phi \times \Phi)_n - (\overline{\Phi} \times \overline{\Phi} \times \overline{\Phi})_n| \le C_1 \|\Phi - \overline{\Phi}\|_{3/2}$;

(ii) $|(\Phi \times \Phi \times \Psi)_n - (\overline{\Phi} \times \overline{\Phi} \times \overline{\Psi})_n| \le C_2(\|\Phi - \overline{\Phi}\|_{3/2} + \|\Psi - \overline{\Psi}\|_{3/2})$;

(iii) $|(\Psi \times \Psi \times \Phi)_n - (\overline{\Psi} \times \overline{\Psi} \times \overline{\Phi})_n| \le C_3(\|\Psi - \overline{\Psi}\|_{3/2} + \|\Phi - \overline{\Phi}\|_{3/2})$;

(iv) $|(\Psi \times \Psi \times \Psi)_n - (\overline{\Psi} \times \overline{\Psi} \times \overline{\Psi})_n| \le C_4 \|\Psi - \overline{\Psi}\|_{3/2}$

where $C_i = C_i(\|\Phi\|_{3/2}, \|\Psi\|_{3/2}, \|\overline{\Phi}\|_{3/2}, \|\overline{\Psi}\|_{3/2})$. Hence, it follows that

$$\begin{aligned}\|\mathcal{A}\Gamma - \mathcal{A}\overline{\Gamma}\|^{3/2}_{\mathcal{L}_{3/2,2}} \le \sum_{n=-\infty}^{\infty} \|T_n^{-1}\|^{3/2}_{\infty} C_5(\|\Phi - \overline{\Phi}\|_{3/2} + \|\Psi - \overline{\Psi}\|_{3/2})^{3/2} \\ \le C_6 \|\Gamma - \overline{\Gamma}\|^{3/2}_{\mathcal{L}_{3/2,2}}.\end{aligned}$$

The operator $\mathcal{A}$ is now readily seen to be continuous from K into itself.

(c) $\mathcal{A}$ is compact.

Consider a bounded set M in $\mathcal{L}_{3/2,2}$, say $M \subset \{\Gamma = (\Phi, \Psi) \in \mathcal{L}_{3/2,2} : \|\Gamma\|_{\mathcal{L}_{3/2,2}} \le B\}$. For each N, a cut-off operator $\mathcal{A}_N$ is defined as follows:

$$(\mathcal{A}_N \Gamma)_n = \begin{cases} (\mathcal{A}\Gamma)_n, & \text{for } -N \le n \le N, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{A}_N$ is a compact operator having a rank of $(2N+1)$ as $\mathcal{A}$ is continuous. For $\Gamma \in M$,

$$|(\mathcal{A}\Gamma)_n| \le \|T_n^{-1}\|_\infty \max\{a, b, c\} \begin{bmatrix} \|\Phi\|^3_{3/2} + \|\Psi\|^2_{3/2}\|\Phi\|_{3/2} \\ \|\Phi\|^2_{3/2}\|\Psi\|_{3/2} + \|\Psi\|^3_{3/2} \end{bmatrix}. \tag{3.11}$$

Thus, one can conclude that

$$\|\mathcal{A}_n\Gamma - \mathcal{A}\Gamma\|^{3/2}_{\mathcal{L}_{3/2,2}} \le C \sum_{|n| \ge N} \|T_n^{-1}\|^{3/2}_\infty.$$

Consequently, $\sup\limits_{\Gamma \in M} \|\mathcal{A}_N\Gamma - \mathcal{A}\Gamma\|_{\mathcal{L}_{3/2,2}} \to 0$ as $N \to \infty$. Thus, $\mathcal{A}$ is compact as it is the uniform limit of compact operators on bounded sets. $\square$

We now turn our attention to the fixed points of $\mathcal{A}$. It is clear that (3.4) possesses some constant solutions which are not of interest to us. A fixed point $(\mathbf{p}, \mathbf{q}) = \{(p_n, q_n)\}$ of (3.4) is said to be trivial if $p_n = q_n = 0$ for all $n \ne 0$. Notice that a trivial solution with $p_0 = q_0 = 0$ corresponds to the origin while $p_0 \ne 0$ (or $q_0 \ne 0$, or both $p_0 \ne 0$ and $q_0 \ne 0$) corresponds to a non-zero constant solution. The next proposition is straightforward to verify.

PROPOSITION 3.2. *The operator $\mathcal{A}$ has nine trivial fixed points. However, only the following four fixed points are inside K: the origin and $(\mathbf{p}_i^*, \mathbf{q}_i^*)$ where*

$$\mathbf{p}_i^* = (\ldots, 0, p_{i0}, 0, \ldots) \quad \textit{and} \quad \mathbf{q}_i^* = (\ldots, 0, q_{i0}, 0, \ldots)$$

with

$$\begin{aligned} (p_{10}, q_{10}) &= \left(0, \sqrt{\frac{\omega_2}{c}}\right); \qquad (p_{20}, q_{20}) = \left(\sqrt{\frac{\omega_1}{a}}, 0\right); \\ (p_{30}, q_{30}) &= \left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}, \sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\right). \end{aligned}$$

Let

$$\begin{aligned} \beta_1 = \min\Bigg\{ &\left\| \left(0, \sqrt{\frac{\omega_2}{c}}\right) \right\|_{\mathcal{L}_{3/2,2}}, \left\| \left(\sqrt{\frac{\omega_1}{a}}, 0\right) \right\|_{\mathcal{L}_{3/2,2}}, \\ &\left\| \left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}, \sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\right) \right\|_{\mathcal{L}_{3/2,2}} \Bigg\} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \beta_2 = \max\Bigg\{ &\left\| \left(0, \sqrt{\frac{\omega_2}{c}}\right) \right\|_{\mathcal{L}_{3/2,2}}, \left\| \left(\sqrt{\frac{\omega_1}{a}}, 0\right) \right\|_{\mathcal{L}_{3/2,2}}, \\ &\left\| \left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}, \sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\right) \right\|_{\mathcal{L}_{3/2,2}} \Bigg\}; \end{aligned} \tag{3.13}$$

and set

$$\gamma = (\max\{a, b, c\})^{1/2} \left(\sum_{n=-\infty}^{\infty} \|T_n^{-1}\|_{\infty}^{3/2} \right)^{1/3}. \tag{3.14}$$

PROPOSITION 3.3. *For all r satisfying $0 < r < r_0 \equiv \min\{1/(2\gamma), \beta_1\}$, one must have $\Gamma \neq t\mathcal{A}\Gamma$ for all $\Gamma \in \partial K_r$ and all $t \in [0, 1]$.*

PROOF. Suppose there exists a $\Gamma \in \partial K_r$ and a $t \in [0, 1]$ such that $\Gamma = t\mathcal{A}\Gamma$. Then, using (3.11) on all $(\mathcal{A}\Gamma)_n$ yields

$$\begin{aligned} \|\Gamma\|_{\mathcal{L}_{3/2,2}}^{3/2} = r^{3/2} = t^{3/2} &\sum_{n=-\infty}^{\infty} |(\mathcal{A}\Gamma)_n|^{3/2} \\ &\leq 2^{5/2} r^{9/2} \sum_{n=-\infty}^{\infty} \|T_n^{-1}\|_{\infty}^{3/2} \max\{a, b, c\}^{3/2} \end{aligned}$$

which implies that $r \geq 1/(2\gamma)$, a contradiction. $\square$

PROPOSITION 3.4. *For any*

$$R > R_0 \equiv 4\beta_2 \left(\sum_{n=-\infty}^{\infty} \max\left\{ \omega_1 \|T_n^{-1}\|_{\infty}, \omega_2 \|T_n^{-1}\|_{\infty}, \frac{1}{1 + n^2} \right\}^{3/2} \right)^{2/3},$$

there exists a nonzero $\widetilde{\Gamma} \in K$ *such that* $\Gamma - \mathcal{A}\Gamma \neq \lambda\widetilde{\Gamma}$, *for all* $\Gamma \in \partial K_R$ *and all* $\lambda \geq 0$.

PROOF. Let $\widetilde{\Gamma} = \{(\widetilde{\Phi}, \widetilde{\Psi})_n\}$ be given by

$$\begin{bmatrix} \widetilde{\phi}_n \\ \widetilde{\psi}_n \end{bmatrix} = \frac{1}{1+n^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is clear that $\widetilde{\Gamma} \neq 0$ and $\widetilde{\Gamma} \in K$. Now, suppose to the contrary that there exist a $\Gamma = (\Phi, \Psi) \in \partial K_R$ and a $\lambda \geq 0$ such that for all n,

$$\begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega_1 + (n\pi/l)^2}[a(\Phi \times \Phi \times \Phi)_n + b(\Phi \times \Psi \times \Psi)_n] \\ \dfrac{1}{\omega_2 + (n\pi/l)^2}[b(\Phi \times \Phi \times \Psi)_n + c(\Psi \times \Psi \times \Psi)_n] \end{bmatrix} + \lambda\widetilde{\Gamma}. \tag{3.15}$$

In particular,

$$\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega_1}(a(\Phi \times \Phi \times \Phi)_0 + b(\Phi \times \Psi \times \Psi)_0) \\ \dfrac{1}{\omega_2}(b(\Phi \times \Phi \times \Psi)_0 + c(\Psi \times \Psi \times \Psi)_0) \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.16}$$

One can verify easily that

$$\begin{aligned} &(\Phi \times \Phi \times \Phi)_0 \geq \phi_0^3; && (\Psi \times \Psi \times \Phi)_0 \geq \psi_0^2\phi_0; \\ &(\Phi \times \Phi \times \Psi)_0 \geq \phi_0^2\psi_0; && (\Psi \times \Psi \times \Psi)_0 \geq \psi_0^3. \end{aligned}$$

Consequently, it is drawn from (3.16) the following bounds: $0 \leq \phi_0 \leq \sqrt{\omega_1/a}$, $0 \leq \psi_0 \leq \sqrt{\omega_2/c}$, $0 \leq \lambda \leq \min\{\sqrt{\omega_1/a}, \sqrt{\omega_2/c}\}$, and

$$\begin{aligned} \tau_0 &= a(\Phi \times \Phi \times \Phi)_0 + b(\Phi \times \Psi \times \Psi)_0 \leq \omega_1\phi_0 \leq \omega_1\sqrt{\frac{\omega_1}{a}}; \\ \eta_0 &= b(\Phi \times \Phi \times \Psi)_0 + c(\Psi \times \Psi \times \Psi)_0 \leq \omega_2\psi_0 \leq \omega_2\sqrt{\frac{\omega_2}{c}}. \end{aligned}$$

Notice that while it is true that one also has the bounds $0 \leq \phi_0 \leq \sqrt{\omega_2/b}$, $0 \leq \psi_0 \leq \sqrt{\omega_1/b}$, those bounds are less helpful in this case. It is also worth noting that ϕ_0 and ψ_0 cannot be zero simultaneously as $\Gamma = (\Phi, \Psi) \in \partial K_R$, even though each can vanish individually. One can see now from (3.15) and the fact that τ_n and η_n are decreasing in $|n|$ that

$$\phi_n \leq \sqrt{\frac{\omega_1}{a}}\left(\omega_1\|T_n^{-1}\|_\infty + \frac{1}{1+n^2}\right), \qquad \psi_n \leq \sqrt{\frac{\omega_2}{c}}\left(\omega_2\|T_n^{-1}\|_\infty + \frac{1}{1+n^2}\right).$$

Therefore,

$$\begin{aligned} R^{3/2} &\leq \beta_2^{3/2} \sum_{n=-\infty}^{\infty} \left\{ \left(\omega_1\|T_n^{-1}\|_\infty + \frac{1}{1+n^2}\right)^{3/2} + \left(\omega_2\|T_n^{-1}\|_\infty + \frac{1}{1+n^2}\right)^{3/2} \right\} \\ &\leq 2^{5/2}\beta_2^{3/2} \sum_{n=-\infty}^{\infty} \max\left\{\omega_1\|T_n^{-1}\|_\infty, \omega_2\|T_n^{-1}\|_\infty, \frac{1}{1+n^2}\right\}^{3/2}. \end{aligned}$$

Thus, it is deduced that

$$R \leq 4\beta_2 \left(\sum_{n=-\infty}^{\infty} \max \left\{ \omega_1 \|T_n^{-1}\|_\infty, \omega_2 \|T_n^{-1}\|_\infty, \frac{1}{1+n^2} \right\}^{3/2} \right)^{2/3},$$

a contradiction to the assumption on R. □

REMARK 3.5. Notice that $R > \beta_2$ since

$$\left(\sum_{n=-\infty}^{\infty} \left[\frac{1}{1+n^2} \right]^{3/2} \right)^{2/3} > 1.$$

THEOREM 3.6. *Let r and R be defined as above. Then the fixed point index of $\mathcal{A}$ on $K_r^R = \{\Gamma \in K : r < \|\Gamma\|_{\mathcal{L}_{3/2,2}} < R\}$ is $i(K, \mathcal{A}, K_r^R) = -1$.*

PROOF. This follows immediately from Theorem 2.4 and Propositions 3.3 and 3.4. □

An immediate consequence of Theorem 3.6 is that there must be at least one fixed point of $\mathcal{A}$ in K_r^R. The analysis is not yet complete, however, since the three constant solutions $(\mathbf{p}_i^*, \mathbf{q}_i^*)$ where

$$(p_{10}, q_{10}) = \left(0, \sqrt{\frac{\omega_2}{c}}\right); \qquad (p_{20}, q_{20}) = \left(\sqrt{\frac{\omega_1}{a}}, 0\right);$$
$$(p_{30}, q_{30}) = \left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}, \sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\right)$$

could be the only fixed points in K_r^R (the origin does not belong to K_r^R.) This case is excluded through the following lemma.

LEMMA 3.7. *If for $i = 1, 2, 3$, $(\mathbf{p}_i^*, \mathbf{q}_i^*)$ are the only fixed points of $\mathcal{A}$ in K_r^R, then when the half-period $l > 0$ is chosen large enough,*

$$\sum_{i=1}^{3} i(K, \mathcal{A}, K_\varepsilon(\mathbf{p}_i^*, \mathbf{q}_i^*)) = 0.$$

PROOF. For $i = 1, 2, 3$, let $\varepsilon_i = \varepsilon_i(l) > 0$ be arbitrarily fixed, sufficiently small numbers whose values will be determined later. Let

$$K_{\varepsilon_i}(\mathbf{p}_i^*, \mathbf{q}_i^*) = \{\Gamma = (\Phi, \Psi) \in K : \|(\Phi, \Psi) - (\mathbf{p}_i^*, \mathbf{q}_i^*)\|_{\mathcal{L}_{3/2,2}} < \varepsilon_i\},$$

$$\partial K_{\varepsilon_i}(\mathbf{p}_i^*, \mathbf{q}_i^*) = \{\Gamma = (\Phi, \Psi) \in K : \|(\Phi, \Psi) - (\mathbf{p}_i^*, \mathbf{q}_i^*)\|_{\mathcal{L}_{3/2,2}} = \varepsilon_i\}.$$

The ε_i will be chosen small enough so that $\overline{K_{\varepsilon_i}}$ is in K_r^R and $\{K_{\varepsilon_i}\}$ forms a collection of mutually disjoint open subsets of K_R^r. Notice then that if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, then $\overline{K_\varepsilon}(\mathbf{p}_i^*, \mathbf{q}_i^*) \subset K_r^R$ and $\{K_\varepsilon(\mathbf{p}_i^*, \mathbf{q}_i^*)\}$ forms a collection of mutually disjoint open subsets of K_r^R. Therefore, the lemma is proved, owing to Lemma 2.4 and the additivity of the fixed point index, if one can show that $(I - \mathcal{A})\partial K_\varepsilon(\mathbf{p}_i^*, \mathbf{q}_i^*)$ omits the ray $\{\lambda\widetilde{\Gamma} : \lambda \geq 0\}$, where $\widetilde{\Gamma} \in K$ is defined as in

Proposition 3.4. So suppose that for each $i = 1, 2, 3$, there are a $\Gamma = (\Phi, \Psi) \in \partial K_{\varepsilon_i}(\mathbf{p}_i^*, \mathbf{q}_i^*)$ and a $\lambda \geq 0$ such that $\Gamma - \mathcal{A}\Gamma = \lambda\widetilde{\Gamma}$. Then for all $n \in \mathbb{Z}$,

$$\begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega_1 + (n\pi/l)^2}[a(\Phi \times \Phi \times \Phi)_n + b(\Psi \times \Psi \times \Phi)_n] \\ \dfrac{1}{\omega_2 + (n\pi/l)^2}[b(\Phi \times \Phi \times \Psi)_n + c(\Psi \times \Psi \times \Psi)_n] \end{bmatrix} + \lambda\widetilde{\Gamma}. \tag{3.17}$$

In particular, for $n = 1$ and let $A = 1/(\omega_1 + (\pi/l)^2)$ and $B = 1/(\omega_2 + (\pi/l)^2)$, one has

$$\begin{aligned} \phi_1 &= A\big(\Phi \times (a\Phi \times \Phi + b\Psi \times \Psi)\big)_1 + \frac{\lambda}{2}; \\ \psi_1 &= B\big(\Psi \times (b\Phi \times \Phi + c\Psi \times \Psi)\big)_1 + \frac{\lambda}{2}; \end{aligned} \tag{3.18}$$

from which one can conclude

$$\begin{aligned} \phi_1 &\geq A(3a\phi_0^2\phi_1 + 2b\phi_0\psi_0\psi_1 + b\psi_0^2\phi_1) + \frac{\lambda}{2}; \\ \psi_1 &\geq B(3c\psi_0^2\psi_1 + 2b\phi_0\psi_0\phi_1 + b\phi_0^2\psi_1) + \frac{\lambda}{2}. \end{aligned} \tag{3.19}$$

It is claimed next that when the half-period l is large enough, the $\varepsilon_i(l)$ can be chosen sufficiently small so that $\phi_n = \psi_n = 0$ for all $n \neq 0$. One needs to consider the following three cases separately.

Case 1. The fixed point $(\mathbf{p}_1^*, \mathbf{q}_1^*)$ where $(p_{10}, q_{10}) = (0, \sqrt{\omega_2/c})$.

Since $\Gamma \in \partial K_{\varepsilon_1}(\mathbf{p}_1^*, \mathbf{q}_1^*)$, it can be written as $\Gamma = (\Phi, \Psi) = (\mathbf{p}_1^*, \mathbf{q}_1^*) + \varepsilon_1(\widetilde{\Phi}, \widetilde{\Psi})$ where $\|(\widetilde{\Phi}, \widetilde{\Psi})\|_{\mathcal{L}_{\frac{3}{2},2}} = 1$. Notice that $\varepsilon_1(\widetilde{\Phi}, \widetilde{\Psi}) = (\Phi, \Psi) - (\mathbf{p}_1^*, \mathbf{q}_1^*)$, therefore for $n \geq 1$

$$\begin{cases} \widetilde{\phi}_n = \phi_n/\varepsilon_1 \geq 0, \\ \widetilde{\psi}_n = \psi_n/\varepsilon_1 \geq 0, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{\phi}_n \geq \widetilde{\phi}_{n+1}, \\ \widetilde{\psi}_n \geq \widetilde{\psi}_{n+1}. \end{cases}$$

In terms of the new variables $(\widetilde{\Phi}, \widetilde{\Psi})$, (3.19) yields

$$\begin{aligned} \varepsilon_1\widetilde{\phi}_1 \geq \frac{Ab\omega_2}{c}\varepsilon_1\widetilde{\phi}_1 &+ 2Ab\left(\sqrt{\frac{\omega_2}{c}} + \varepsilon_1\widetilde{\psi}_0\right)\varepsilon_1\widetilde{\phi}_0\varepsilon_1\widetilde{\psi}_1 \\ &+ 2Ab\sqrt{\frac{\omega_2}{c}}\varepsilon_1\widetilde{\psi}_0\varepsilon_1\widetilde{\phi}_1 + \frac{\lambda}{2}; \end{aligned} \tag{3.20}$$

$$\begin{aligned} \varepsilon_1\widetilde{\psi}_1 \geq\, & 3B\omega_2\varepsilon_1\widetilde{\psi}_1 + 6Bc\sqrt{\frac{\omega_2}{c}}\varepsilon_1\widetilde{\psi}_0\varepsilon_1\widetilde{\psi}_1 \\ &+ 2Bb\left(\sqrt{\frac{\omega_2}{c}} + \varepsilon_1\widetilde{\psi}_0\right)\varepsilon_1\widetilde{\phi}_0\varepsilon_1\widetilde{\phi}_1 + \frac{\lambda}{2}. \end{aligned} \tag{3.21}$$

First, choose l large enough so that both the following hold

$$\frac{Ab\omega_2}{c} > 1 \quad \text{and} \quad 3B\omega_2 > 1.$$

The explicit condition for l is

$$l^2 > \max\left\{\frac{\pi^2 c}{b\omega_2 - c\omega_1}, \frac{\pi^2}{2\omega_2}\right\} :\equiv L_1. \tag{3.22}$$

The number $\varepsilon_1 = \varepsilon_1(l)$ can now be chosen small enough so that

$$\begin{aligned}&\frac{Ab\omega_2}{c}\varepsilon_1\widetilde{\phi}_1 + 2Ab\left(\sqrt{\frac{\omega_2}{c}} + \varepsilon_1\widetilde{\psi}_0\right)\varepsilon_1\widetilde{\phi}_0\varepsilon_1\widetilde{\psi}_1 + 2Ab\sqrt{\frac{\omega_2}{c}}\varepsilon_1\widetilde{\psi}_0\varepsilon_1\widetilde{\phi}_1 > \varepsilon_1\widetilde{\phi}_1;\\ &3B\omega_2\varepsilon_1\widetilde{\psi}_1 + 6Bc\sqrt{\frac{\omega_2}{c}}\varepsilon_1\widetilde{\psi}_0\varepsilon_1\widetilde{\psi}_1 + 2Bb\left(\sqrt{\frac{\omega_2}{c}} + \varepsilon_1\widetilde{\psi}_0\right)\varepsilon_1\widetilde{\phi}_0\varepsilon_1\widetilde{\phi}_1 > \varepsilon_1\widetilde{\psi}_1.\end{aligned} \tag{3.23}$$

It follows immediately from (3.20), (3.21) and (3.23) that $\lambda = 0$ and $\widetilde{\phi}_1 = \widetilde{\psi}_1 = 0$, hence $\phi_n = \psi_n = 0$ for all $n \neq 0$.

Case 2. The fixed point $(\mathbf{p}_2^*, \mathbf{q}_2^*)$ where $(p_{20}, q_{20}) = (\sqrt{\omega_1/a}, 0)$.

Suppose that there are a $\Gamma = (\Phi, \Psi) \in \partial K_{\varepsilon_2}(\mathbf{p}_2^*, \mathbf{q}_2^*)$ and a $\lambda \geq 0$ such that $\Gamma - \mathcal{A}\Gamma = \lambda\widetilde{\Gamma}$. Since $\Gamma \in \partial K_{\varepsilon_2}(\mathbf{p}_2^*, \mathbf{q}_2^*)$, it can be written as $\Gamma = (\Phi, \Psi) = (\mathbf{p}_2^*, \mathbf{q}_2^*) + \varepsilon_2(\widetilde{\Phi}, \widetilde{\Psi})$ where $\|(\widetilde{\Phi}, \widetilde{\Psi})\|_{\mathcal{L}_{3/2,2}} = 1$. In terms of the new variables $(\widetilde{\Phi}, \widetilde{\Psi})$, one has

$$\begin{aligned}\varepsilon_2\widetilde{\phi}_1 \geq\, &3A\omega_1\varepsilon_2\widetilde{\phi}_1 + 6Aa\sqrt{\frac{\omega_1}{a}}\varepsilon_2\widetilde{\phi}_0\varepsilon_2\widetilde{\phi}_1\\ &+ 2Ab\left(\sqrt{\frac{\omega_1}{a}} + \varepsilon_2\widetilde{\phi}_0\right)\varepsilon_2\widetilde{\psi}_0\varepsilon_2\widetilde{\psi}_1 + \frac{\lambda}{2};\end{aligned} \tag{3.24}$$

$$\begin{aligned}\varepsilon_2\widetilde{\psi}_1 \geq\, &\frac{Bb\omega_1}{a}\varepsilon_2\widetilde{\psi}_1 + 2Bb\left(\sqrt{\frac{\omega_1}{a}} + \varepsilon_2\widetilde{\phi}_0\right)\varepsilon_2\widetilde{\psi}_0\varepsilon_2\widetilde{\phi}_1\\ &+ 2Bb\sqrt{\frac{\omega_1}{a}}\varepsilon_2\widetilde{\phi}_0\varepsilon_2\widetilde{\psi}_1 + \frac{\lambda}{2}.\end{aligned} \tag{3.25}$$

Now, choose l large enough so that both the following hold $3A\omega_1 > 1$ and $Bb\omega_1/a > 1$. The explicit condition for l is

$$l^2 > \max\left\{\frac{\pi^2 a}{b\omega_1 - a\omega_2}, \frac{\pi^2}{2\omega_1}\right\} :\equiv L_2. \tag{3.26}$$

The number $\varepsilon_2 = \varepsilon_2(l)$ can now be chosen small enough so that

$$\begin{aligned}&3A\omega_1\varepsilon_2\widetilde{\phi}_1 + 6Aa\sqrt{\frac{\omega_1}{a}}\,\varepsilon_2\widetilde{\phi}_0\varepsilon_2\widetilde{\phi}_1 + 2Ab\left(\sqrt{\frac{\omega_1}{a}} + \varepsilon_2\widetilde{\phi}_0\right)\varepsilon_2\widetilde{\psi}_0\varepsilon_2\widetilde{\psi}_1 > \varepsilon_2\widetilde{\phi}_1;\\ &\frac{Bb\omega_1}{a}\varepsilon_2\widetilde{\psi}_1 + 2Bb\left(\sqrt{\frac{\omega_1}{a}} + \varepsilon_2\widetilde{\phi}_0\right)\varepsilon_2\widetilde{\psi}_0\varepsilon_2\widetilde{\phi}_1 + 2Bb\sqrt{\frac{\omega_1}{a}}\varepsilon_2\widetilde{\phi}_0\varepsilon_2\widetilde{\psi}_1 > \varepsilon_2\widetilde{\psi}_1.\end{aligned} \tag{3.27}$$

It follows from (3.24), (3.25) and (3.27) that $\lambda = 0$ and $\widetilde{\phi}_1 = \widetilde{\psi}_1 = 0$, hence $\phi_n = \psi_n = 0$ for all $n \neq 0$.

Case 3. The fixed point $(\mathbf{p}_3^*, \mathbf{q}_3^*)$ where

$$(p_{30}, q_{30}) = \left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}, \sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\right).$$

Again, suppose that there are a $\Gamma = (\Phi, \Psi) \in \partial K_{\varepsilon_3}(\mathbf{p}_3^*, \mathbf{q}_3^*)$ and a $\lambda \geq 0$ such that $\Gamma - \mathcal{A}\Gamma = \lambda\widetilde{\Gamma}$. Since $\Gamma \in \partial K_{\varepsilon_3}(\mathbf{p}_3^*, \mathbf{q}_3^*)$, it can be written as $\Gamma = (\Phi, \Psi) = (\mathbf{p}_3^*, \mathbf{q}_3^*) + \varepsilon_3(\widetilde{\Phi}, \widetilde{\Psi})$ where $\|(\widetilde{\Phi}, \widetilde{\Psi})\|_{\mathcal{L}_{3/2,2}} = 1$. In terms of the new variables $(\widetilde{\Phi}, \widetilde{\Psi})$, one has

$$
\begin{aligned}
&\frac{A\big(2ab\omega_2 + \omega_1(b^2 - 3ac)\big)}{b^2 - ac}\varepsilon_3\widetilde{\phi}_1 \\
&+ \left(6Aa\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}\widetilde{\phi}_0 + 2Ab\sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\widetilde{\psi}_0\right)\varepsilon_3^2\widetilde{\phi}_1 \\
&+ 2Ab\left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}} + \varepsilon_3\widetilde{\phi}_0\right)\left(\sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}} + \varepsilon_3\widetilde{\psi}_0\right)\varepsilon_3\widetilde{\psi}_1 \leq \varepsilon_3\widetilde{\phi}_1;
\end{aligned} \tag{3.28}
$$

and

$$
\begin{aligned}
&\frac{B\big(2bc\omega_1 + \omega_2(b^2 - 3ac)\big)}{b^2 - ac}\varepsilon_3\widetilde{\psi}_1 \\
&+ \left(6Bc\sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\widetilde{\psi}_0 + 2Bb\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}\widetilde{\phi}_0\right)\varepsilon_3^2\widetilde{\psi}_1 \\
&+ 2Bb\left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}} + \varepsilon_3\widetilde{\phi}_0\right)\left(\sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}} + \varepsilon_3\widetilde{\psi}_0\right)\varepsilon_3\widetilde{\phi}_1 \leq \varepsilon_3\widetilde{\psi}_1.
\end{aligned} \tag{3.29}
$$

First, choose l large enough so that both the following hold

$$
\frac{A(2ab\omega_2 + b^2\omega_1 - 3ac\omega_1)}{(b^2 - ac)} > 1 \quad \text{and} \quad \frac{B(2bc\omega_1 + b^2\omega_2 - 3ac\omega_2)}{(b^2 - ac)} > 1.
$$

The explicit condition for l is

$$
l^2 > \max\left\{\frac{\pi^2(b^2 - ac)}{2a(b\omega_2 - c\omega_1)}, \frac{\pi^2(b^2 - ac)}{2c(b\omega_1 - a\omega_2)}\right\} :\equiv L_3. \tag{3.30}
$$

The number $\varepsilon_3 = \varepsilon_3(l)$ can now be chosen small enough so that

$$
\begin{aligned}
\varepsilon_3(\widetilde{\phi}_1 + \widetilde{\psi}_1) < {} & \frac{A\big(2ab\omega_2 + \omega_1(b^2 - 3ac)\big)}{b^2 - ac}\varepsilon_3\widetilde{\phi}_1 \\
&+ \frac{B\big(2bc\omega_1 + \omega_2(b^2 - 3ac)\big)}{b^2 - ac}\varepsilon_3\widetilde{\psi}_1 \\
&+ \left(6Aa\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}\widetilde{\phi}_0 + 2Ab\sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\widetilde{\psi}_0\right)\varepsilon_3^2\widetilde{\phi}_1 \\
&+ \left(6Bc\sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\widetilde{\psi}_0 + 2Bb\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}\widetilde{\phi}_0\right)\varepsilon_3^2\widetilde{\psi}_1 \\
&+ 2b\left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}} + \varepsilon_3\widetilde{\phi}_0\right) \\
&\cdot \left(\sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}} + \varepsilon_3\widetilde{\psi}_0\right)(B\varepsilon_3\widetilde{\phi}_1 + A\varepsilon_3\widetilde{\psi}_1).
\end{aligned} \tag{3.31}
$$

It follows from (3.28), (3.29) and (3.31) that $\lambda = 0$ and $\phi_n = \psi_n = 0$ for all $n \neq 0$. Thus, the claim is proved. Next, let

$$l^2 > \max\{L_1, L_2, L_3\}, \tag{3.32}$$

and redefines $\widetilde{\varepsilon}_1(l), \widetilde{\varepsilon}_2(l), \widetilde{\varepsilon}_3(l)$ if necessary so that for

$$\varepsilon(l) = \min\{\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2, \widetilde{\varepsilon}_3\}, \tag{3.33}$$

the expressions (3.23), (3.27) and (3.31) hold. Under these conditions, $\lambda = 0$ and for all $n \neq 0$, $\phi_n = \psi_n = 0$.

Using (3.17) for $n = 0$, one sees that

$$\begin{cases} \omega_1 \phi_0 = a\phi_0^3 + b\phi_0\psi_0^2, \\ \omega_2 \psi_0 = b\phi_0^2\psi_0 + c\psi_0^3. \end{cases} \tag{3.34}$$

The only solutions for (3.34) are

$$\left(0, \sqrt{\frac{\omega_2}{c}}\right), \quad \left(\sqrt{\frac{\omega_1}{a}}, 0\right), \quad \left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}, \sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}}\right).$$

But then this contradicts with the facts that $\{K_\varepsilon(\mathbf{p}_i^*, \mathbf{q}_i^*)\}$ forms a collection of mutually disjoint open subsets of K_r^R and $\Gamma = (\Phi, \Psi) \in \partial K_\varepsilon(\mathbf{p}_i^*, \mathbf{q}_i^*)$, $i = 1, 2, 3$ for small ε when the half-period l is chosen large enough as in (3.32). The proof of Lemma 3.7 is hence concluded. □

Theorem 3.8. *For $a, b, c > 0$ such that $b^2 - ac > 0$ and phase speeds $\omega_1, \omega_2 > 0$ such that $b\omega_2 - c\omega_1 > 0$, $b\omega_1 - a\omega_2 > 0$, if the half-period l is chosen large enough as in (3.32), then the operator $\mathcal{A}$ must have at least one non-trivial fixed point $\overline{\Gamma} = (\overline{\Phi}, \overline{\Psi})$ in the cone segment K_R^r. Moreover,*

(a) *either both sequences of components $\overline{\phi}_n, \overline{\psi}_n > 0$ for every $n \in \mathbb{Z}$ and one obtains a vector solution $(\overline{\Phi}, \overline{\Psi})$, or one of the sequences $\overline{\phi}_n, \overline{\psi}_n$ vanishes for every $n \in \mathbb{Z}$ while the other remains strictly positive for every $n \in \mathbb{Z}$, in which case one obtains semi-trivial solutions $(\overline{\Phi}, 0)$ or $(0, \overline{\Psi})$;*

(b) *for any $\sigma \geq 0$, the sequences $\{|n|^\sigma \overline{\phi}_n\}$ and $\{|n|^\sigma \overline{\psi}_n\}$ are in l_1. Therefore, the non-trivial fixed point solutions are infinitely smooth.*

Proof. The existence of a non-trivial fixed point follows immediately from Theorem 3.6 and Lemma 3.7. It is left to establish (a) and (b). Recall that the solution $\overline{\Gamma} = (\overline{\Phi}, \overline{\Psi})$ must satisfy for all $n \in \mathbb{Z}$,

$$\begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega_1 + (n\pi/l)^2}[a(\Phi \times \Phi \times \Phi)_n + b(\Psi \times \Psi \times \Phi)_n] \\ \dfrac{1}{\omega_2 + (n\pi/l)^2}[b(\Phi \times \Phi \times \Psi)_n + c(\Psi \times \Psi \times \Psi)_n] \end{bmatrix}. \tag{3.35}$$

In order to establish (a), we will show that there are only three possibilities: either

(i) $\overline{\phi}_0 = 0$, in which case $\overline{\psi}_n > 0$ for all $n \in \mathbb{Z}$; or
(ii) $\overline{\psi}_0 = 0$, in which case $\overline{\phi}_n > 0$ for all $n \in \mathbb{Z}$; or
(iii) $\overline{\phi}_0 > 0$ and $\overline{\psi}_0 > 0$, in which case both $\overline{\phi}_n, \overline{\psi}_n > 0$ for all $n \in \mathbb{Z}$.

Case 1. Suppose $\overline{\phi}_0 = 0$. Let N be the smallest non-negative integer such that $\overline{\psi}_N = 0$. Notice that $N > 1$ because if $\overline{\psi}_1 = 0$ then this will lead to the trivial solution $(0, \sqrt{\omega_2/c})$. It follows from (3.35) that

$$0 = \frac{c}{\omega_2 + (N\pi/l)^2} \sum_{j,k=\infty}^{\infty} \overline{\psi}_k \overline{\psi}_{j-k} \overline{\psi}_{N-j} \geq \frac{c}{\omega_2 + (N\pi/l)^2} \overline{\psi}_{N-1} \overline{\psi}_1 \overline{\psi}_0 > 0$$

which is a contradiction. Thus, it must be the case that $\overline{\psi}_n > 0$ for all $n \in \mathbb{Z}$ and the non-trivial fixed point is $(0, \overline{\Psi})$.

Case 2. Suppose $\overline{\psi}_0 = 0$. Using exact same argument, one concludes that $\overline{\phi}_n > 0$ for all $n \in \mathbb{Z}$ in which case the non-trivial fixed point is $(\overline{\Phi}, 0)$.

Case 3. Suppose $\overline{\phi}_0 > 0$ and $\overline{\psi}_0 > 0$. Let N be the smallest non-negative integer such that either $\overline{\phi}_n$ or $\overline{\psi}_n$ is zero. Notice that if $\overline{\phi}_1 = 0$ then since

$$0 \geq \frac{2b\overline{\phi}_0 \overline{\psi}_0 \overline{\psi}_1}{\omega_1 + (\pi/l)^2}$$

this implies that $\overline{\psi}_1 = 0$ which leads to the trivial solution

$$\left(\sqrt{\frac{b\omega_2 - c\omega_1}{b^2 - ac}}, \sqrt{\frac{b\omega_1 - a\omega_2}{b^2 - ac}} \right).$$

Similar situation occurs if $\overline{\psi}_1 = 0$. Hence $N > 1$. Without loss of generality, assume that $\overline{\phi}_N = 0$ (exact argument holds when $\overline{\psi}_N = 0$). It follows from (3.35) that

$$\begin{aligned} 0 = \frac{1}{\omega_1 + (N\pi/l)^2} &\left(a \sum_{j,k=-\infty}^{\infty} \overline{\phi}_k \overline{\phi}_{j-k} \overline{\phi}_{N-j} + b \sum_{j,k=-\infty}^{\infty} \overline{\psi}_k \overline{\psi}_{j-k} \overline{\phi}_{N-j} \right) \\ &> \frac{3a\overline{\phi}_0^2 \overline{\phi}_1}{\omega_1 + (N\pi/l)^2} > 0 \end{aligned}$$

which is a contradiction. Thus, it must be the case that $\overline{\phi}_n, \overline{\psi}_n > 0$ for all $n \in \mathbb{Z}$ and the non-trivial fixed point is $(\overline{\Phi}, \overline{\Psi})$. Thus (a) is proved.

Since

$$\begin{aligned} (\overline{\Phi} \times \overline{\Phi} \times \overline{\Phi})_n &\leq \|\overline{\Phi}\|_{3/2}^{3/2} < \infty, & (\overline{\Psi} \times \overline{\Psi} \times \overline{\Phi})_n &\leq \|\overline{\Psi}\|_{3/2}^2 \|\overline{\Phi}\|_{3/2} < \infty; \\ (\overline{\Psi} \times \overline{\Psi} \times \overline{\Psi})_n &\leq \|\overline{\Psi}\|_{3/2}^{3/2} < \infty, & (\overline{\Phi} \times \overline{\Phi} \times \overline{\Psi})_n &\leq \|\overline{\Phi}\|_{3/2}^2 \|\overline{\Psi}\|_{3/2} < \infty; \end{aligned}$$

it follows that for $|n| \geq 1$, there exists a constant $C_0 > 0$ independent of n satisfying

$$\overline{\phi}_n \leq C\|T_n^{-1}\|_\infty \leq \frac{C_0}{n^2} \quad \text{and} \quad \overline{\psi}_n \leq C\|T_n^{-1}\|_\infty \leq \frac{C_0}{n^2}. \tag{3.36}$$

Therefore, $\{\overline{\phi}_n\}$ and $\{\overline{\psi}_n\}$ are in l_1.

One deduces from (3.36) that $|n|\overline{\phi}_n \leq C_0/|n|$, $|n|\overline{\psi}_n \leq C_0/|n|$ and consequently,

$$\sum_{n=-\infty}^{\infty} (1+|n|)^{3/2}\overline{\phi}_n^{3/2} \leq C, \qquad \sum_{n=-\infty}^{\infty} (1+|n|)^{3/2}\overline{\psi}_n^{3/2} \leq C.$$

Notice next that for any $j, k, n \in \mathbb{Z}$,

$$1+|n| \leq (1+|n-j-k|)(1+|j|)(1+|k|).$$

Thus, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in l_{3/2}$ one has

$$\begin{aligned}
(\mathbf{u}\times\mathbf{v}\times\mathbf{w})_n &= \sum_{j,k=-\infty}^{\infty} \frac{(1+|k|)u_k(1+|j-k|)v_{j-k}(1+|n-j|)w_{n-j}}{(1+|k|)(1+|j-k|)(1+|n-j|)} \\
&\leq \frac{1}{(1+|n|)} \sum_{j,k=-\infty}^{\infty} (1+|k|)u_k(1+|j-k|)v_{j-k}(1+|n-j|)w_{n-j} \leq \frac{C}{(1+|n|)}.
\end{aligned}$$

It then follows that for $|n| \geq 1$,

$$\begin{aligned}
\overline{\phi}_n &\leq \|T_n^{-1}\|_\infty (a(\overline{\Phi}\times\overline{\Phi}\times\overline{\Phi})_n + b(\overline{\Psi}\times\overline{\Psi}\times\overline{\Phi})_n) \leq \frac{C}{|n|^3}, \\
\overline{\psi}_n &\leq \|T_n^{-1}\|_\infty (b(\overline{\Phi}\times\overline{\Phi}\times\overline{\Psi})_n + c(\overline{\Psi}\times\overline{\Psi}\times\overline{\Psi})_n) \leq \frac{C}{|n|^3}.
\end{aligned}$$

Thus, the sequences $\{|n|\overline{\phi}_n\}$ and $\{|n|\overline{\psi}_n\}$ are in l_1. It is readily seen by continuing this bootstrapping argument that $\forall\sigma \geq 0$, the sequences $\{|n|^\sigma\overline{\phi}_n\}$ and $\{|n|^\sigma\overline{\psi}_n\}$ are in l_1. □

4. The general case

It is clear that the above theory can be extended to include the general case (1.2) as well. The issue one has at hand is how to exclude the trivial fixed points of the operator $\mathcal{A}$. This could be a daunting task as one must consider all the sub-cases when the system (1.2) collapses to a lower-order ones. (Readers are referred to [11], [12] for a related discussion on this issue.) In this section, the special case of $\omega_j = \omega$ for all $j = 1, 2, \ldots, m$, $a_{jk} = a_{kj} = b$ when $j \neq k$, and $a_{jj} = a$ is considered since the problem of collapsing to lower-order systems of (1.2) can be handled in a straightforward manner and the trivial fixed points can be written down explicitly.

The system now takes the form

$$iu_{jt} + u_{jxx} + a|u_j|^2 u_j + \sum_{k=1,\, k\neq j}^{m} b|u_k|^2 u_j = 0. \tag{4.1}$$

Substituting (1.3) into (4.1) and equating the Fourier coefficients yield the following system

$$-\omega\phi_{jn} - \left(\frac{n\pi}{l}\right)^2 \phi_{jn} + a(\Phi_j \times \Phi_j \times \Phi_j)_n + b\left(\Phi_j \times \sum_{k=1,\, k\neq j}^{m} (\Phi_k \times \Phi_k)\right)_n = 0. \tag{4.2}$$

The system (4.2) can be put into a more convenient matrix form

$$T_n[\phi_{jn}] = \left[a(\Phi_j \times \Phi_j \times \Phi_j)_n + b\left(\Phi_j \times \sum_{k=1,\, k\neq j}^{m} (\Phi_k \times \Phi_k)\right)_n\right], \tag{4.3}$$

where the $m \times m$ matrix T_n is given by

$$T_n = \left(\omega + \left(\frac{n\pi}{l}\right)^2\right) I \tag{4.4}$$

with I being the $m \times m$ identity matrix. Notice that

$$\|T_n^{-1}\|_\infty = \frac{1}{\omega + (n\pi/l)^2}. \tag{4.5}$$

For $j = 1, \ldots, m$, define a set $K \subset \mathcal{L}_{3/2,m}$ as follows

$$K = \{(\mathbf{u}_1, \ldots, \mathbf{u}_m) \in \mathcal{L}_{3/2,m} : \\ (u_{1n}, \ldots, u_{mn}) = (u_{-1n}, \ldots, u_{-mn}),\ u_{j0} \geq u_{j1} \geq \ldots \geq 0\}.$$

One can easily verify that K is indeed a cone in $\mathcal{L}_{3/2,m}$ equipped with the norm

$$\|(\mathbf{u}, \mathbf{v})\|_{\mathcal{L}_{3/2,m}}^{3/2} = \sum_{n=-\infty}^{\infty} (|u_{1n}|^{3/2} + \ldots + |u_{mn}|^{3/2}) = \|\mathbf{u}_1\|_{3/2}^{3/2} + \ldots + \|\mathbf{u}_m\|_{3/2}^{3/2}.$$

An operator $\mathcal{B}$ on K is now defined as follows: for any $\Gamma \equiv (\Phi_1, \ldots, \Phi_m) = \{(\phi_{1n}, \ldots \phi_{mn})\} \in K$, $\mathcal{B}\Gamma = \{(\mathcal{B}\Gamma)_n\}$, where

$$(\mathcal{B}\Gamma)_n = T_n^{-1}\left[a(\Phi_j \times \Phi_j \times \Phi_j)_n + b\left(\Phi_j \times \sum_{k=1,\, k\neq j}^{m} (\Phi_k \times \Phi_k)\right)_n\right]. \tag{4.6}$$

Thus (4.3) can be written in the form $\Gamma = \mathcal{B}\Gamma$ and the fixed points of operator $\mathcal{B}$ in the cone K are solutions of (4.3). The proof of the next Lemma follows from the same argument used in Lemma 3.1 hence is omitted.

LEMMA 4.1. *The operator $\mathcal{B}$ is continuous, positive and compact on the cone K.*

It is now important to make clear of what it means for a solution to be called a trivial fixed point. A fixed point $(\mathbf{p}_1, \ldots, \mathbf{p}_m) = \{(p_{1n}, \ldots, p_{mn})\}$ of (4.3) is said to be trivial if $p_{1n} = \ldots = p_{mn} = 0$ for all $n \neq 0$. We will make no distinction between the permutations of the k-nonvanishing trivial solutions; for example, all the 1-nonvanishing trivial solutions $(A, 0, \ldots, 0)$, $(0, A, 0, \ldots, 0), \ldots, (0, \ldots, 0, A)$ belong to the same class. Again, a trivial solution with $p_{10} = \ldots = p_{m0} = 0$ corresponds to the origin. The next Proposition is straight forward to verify.

PROPOSITION 4.2. *Up to permutations, the operator $\mathcal{B}$ has $(m+1)$ trivial fixed points that lie inside K : the origin and $(\mathbf{p}_1^*, \ldots, \mathbf{p}_m^*)$ where $(p_{10}^*, \ldots, p_{m0}^*)$ takes on the following form:*

(a) 1-*nonvanishing*: $\left(\sqrt{\dfrac{\omega}{a}}, 0, \ldots, 0\right)$;

(b) k-*nonvanishing*: $\Bigg(\underbrace{\sqrt{\dfrac{\omega}{a+(k-1)b}}, \ldots, \sqrt{\dfrac{\omega}{a+(k-1)b}}}_{k}, 0, \ldots, 0\Bigg)$;

(c) m-*nonvanishing* (*or vector solution*):

$$\left(\sqrt{\frac{\omega}{a+(m-1)b}}, \ldots, \sqrt{\frac{\omega}{a+(m-1)b}}\right).$$

Let

$$\begin{aligned}(4.7)\quad \widetilde{\beta}_1 = \min\Bigg\{ &\left\|\left(\sqrt{\frac{\omega}{a}}, 0, \ldots, 0\right)\right\|_{\mathcal{L}_{3/2,m}}, \\ &\left\|\Bigg(\underbrace{\sqrt{\frac{\omega}{a+(k-1)b}}, \ldots, \sqrt{\frac{\omega}{a+(k-1)b}}}_{k}, 0, \ldots, 0\Bigg)\right\|_{\mathcal{L}_{3/2,m}}, \\ &\left\|\left(\sqrt{\frac{\omega}{a+(m-1)b}}, \ldots, \sqrt{\frac{\omega}{a+(m-1)b}}\right)\right\|_{\mathcal{L}_{3/2,m}}\Bigg\};\end{aligned}$$

$$\begin{aligned}(4.8)\quad \widetilde{\beta}_2 = \max\Bigg\{ &\left\|\left(\sqrt{\frac{\omega}{a}}, 0, \ldots, 0\right)\right\|_{\mathcal{L}_{3/2,m}}, \\ &\left\|\Bigg(\underbrace{\sqrt{\frac{\omega}{a+(k-1)b}}, \ldots, \sqrt{\frac{\omega}{a+(k-1)b}}}_{k}, 0, \ldots, 0\Bigg)\right\|_{\mathcal{L}_{3/2,m}}, \\ &\left\|\left(\sqrt{\frac{\omega}{a+(m-1)b}}, \ldots, \sqrt{\frac{\omega}{a+(m-1)b}}\right)\right\|_{\mathcal{L}_{3/2,m}}\Bigg\};\end{aligned}$$

and set

$$\widetilde{\gamma} = \big(\max\{a,b\}\big)^{1/2}\bigg(\sum_{n=-\infty}^{\infty} \|T_n^{-1}\|_\infty^{3/2}\bigg)^{1/3}.$$

Proposition 4.3. *For all r satisfying $0 < r < r_0 \equiv \min\{1/(m\gamma), \widetilde{\beta}_1\}$, one must have $\Gamma \neq t\mathcal{B}\Gamma$ for all $\Gamma \in \partial K_r$ and all $t \in [0,1]$.*

Proof. Suppose there exists a $\Gamma \in \partial K_r$ and a $t \in [0,1]$ such that $\Gamma = t\mathcal{B}\Gamma$. Then, using (3.11) on all $(\mathcal{B}\Gamma)_n$ yields

$$\|\Gamma\|^{3/2}_{\mathcal{L}_{3/2,m}} = r^{3/2} = t^{3/2}\sum_{n=-\infty}^{\infty}|(\mathcal{B}\Gamma)_n|^{3/2} \le m^{5/2}r^{9/2}\sum_{n=-\infty}^{\infty}\|T_n^{-1}\|_\infty^{3/2}\max\{a,b\}^{3/2}$$

which implies that $r \geq 1/(m\gamma)$, a contradiction. □

Proposition 4.4. *For any*

$$R > R_0 \equiv 2\widetilde{\beta}_2\bigg(m\sum_{n=-\infty}^{\infty}\max\bigg\{\omega\|T_n^{-1}\|_\infty, \frac{1}{1+n^2}\bigg\}^{3/2}\bigg)^{2/3},$$

there exists a nonzero $\widetilde{\Gamma} \in K$ such that $\Gamma - \mathcal{B}\Gamma \neq \lambda\widetilde{\Gamma}$, for all $\Gamma \in \partial K_R$ and all $\lambda \geq 0$.

Proof. Let $\widetilde{\Gamma} = \{(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)_n\}$ be given by

$$\begin{bmatrix}\widetilde{\phi}_{1n}\\ \vdots \\ \widetilde{\phi}_{mn}\end{bmatrix} = \frac{1}{1+n^2}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}.$$

It is clear that $\widetilde{\Gamma} \neq 0$ and $\widetilde{\Gamma} \in K$. Now, suppose to the contrary that there exist a $\Gamma = (\Phi_1, \dots, \Phi_m) \in \partial K_R$ and a $\lambda \geq 0$ such that for all n,

$$\text{(4.9)}\quad [\phi_{1j}] = \frac{1}{\omega + (n\pi/l)^2}\bigg[a(\Phi_j\times\Phi_j\times\Phi_j)_n + b\bigg(\Phi_j\times\sum_{k=1,\,k\neq j}^{m}(\Phi_k\times\Phi_k)\bigg)_n\bigg] + \lambda\widetilde{\Gamma}.$$

In particular,

$$\text{(4.10)}\qquad [\phi_{j0}] = \frac{1}{\omega}\bigg[a(\Phi_j\times\Phi_j\times\Phi_j)_0 + b\bigg(\Phi_j\times\sum_{k=1,\,k\neq j}^{m}(\Phi_k\times\Phi_k)\bigg)_0\bigg] + \lambda[1].$$

One can verify easily that $(\Phi_j\times\Phi_j\times\Phi_j)_0 \geq \phi_{j0}^3$. Consequently, it is drawn from (4.10) the following bounds: $0 \le \phi_{j0} \le \sqrt{\omega/a}$, $0 \le \lambda \le \sqrt{\omega/a}$, and

$$\tau_{j0} = a(\Phi_j\times\Phi_j\times\Phi_j)_0 + b\bigg(\Phi_j\times\sum_{k=1,\,k\neq j}^{m}(\Phi_k\times\Phi_k)\bigg)_0 \le \omega\phi_{j0} \le \omega\sqrt{\frac{\omega}{a}}.$$

It is worth noting that all the terms ϕ_{j0} cannot be zero simultaneously as $\Gamma = (\Phi_1, \dots, \Phi_m) \in \partial K_R$, even though each can vanish individually. One can see now from (4.9) and the fact that τ_{jn} is decreasing in $|n|$ that

$$\phi_{jn} \le \sqrt{\frac{\omega}{a}}\left(\omega\|T_n^{-1}\|_\infty + \frac{1}{1+n^2}\right).$$

Therefore,

$$\begin{aligned} R^{3/2} &\le \widetilde{\beta}_2^{3/2} m \sum_{n=-\infty}^{\infty} \left(\omega\|T_n^{-1}\|_\infty + \frac{1}{1+n^2}\right)^{3/2} \\ &\le 2^{3/2}\widetilde{\beta}_2^{3/2} m \sum_{n=-\infty}^{\infty} \max\left\{\omega\|T_n^{-1}\|_\infty, \frac{1}{1+n^2}\right\}^{3/2}. \end{aligned}$$

Thus, it is deduced that

$$R \le 2\widetilde{\beta}_2\left(m\sum_{n=-\infty}^{\infty} \max\left\{\omega\|T_n^{-1}\|_\infty, \frac{1}{1+n^2}\right\}^{3/2}\right)^{2/3},$$

a contradiction to the assumption on R. □

REMARK 4.5. Notice that $R > \beta_2$ since

$$\left(\sum_{n=-\infty}^{\infty}\left[\frac{1}{1+n^2}\right]^{3/2}\right)^{2/3} > 1.$$

The next theorem follows from Theorem 2.4 and Propositions 4.3 and 4.4.

THEOREM 4.6. *Let r and R be defined as above. Then the fixed point index of $\mathcal{B}$ on $K_r^R = \{\Gamma \in K : r < \|\Gamma\|_{\mathcal{L}_{3/2,m}} < R\}$ is $i(K, \mathcal{B}, K_r^R) = -1$.*

Thus, there must be at least one fixed point of $\mathcal{B}$ in K_r^R. The analysis is not yet complete, however, since the above m constant periodic solutions could be the only fixed points in K_r^R (the origin does not belong to K_r^R.)

LEMMA 4.7. *If for $j = 1, \dots, m$, $(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$ are the only fixed points of $\mathcal{B}$ in K_r^R, then for the half-period $l > 0$ chosen large enough,*

$$\sum i(K, \mathcal{B}, K_\varepsilon(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)) = 0.$$

PROOF. For $j = 1, \dots, m$, let $\varepsilon_j = \varepsilon_j(l) > 0$ be arbitrarily fixed, sufficiently small numbers whose values will be determined later. Let

$$\begin{aligned} K_{\varepsilon_j}(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*) = \{\Gamma = (\Phi_1, \dots, \Phi_m) \in K :& \\ \|(\Phi_1, \dots, \Phi_m) - (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)\|_{\mathcal{L}_{3/2,m}} < \varepsilon_j\},& \end{aligned}$$

$$\begin{aligned} \partial K_{\varepsilon_j}(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*) = \{\Gamma = (\Phi_1, \dots, \Phi_m) \in K :& \\ \|(\Phi_1, \dots, \Phi_m) - (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)\|_{\mathcal{L}_{3/2,m}} = \varepsilon_j\}.& \end{aligned}$$

The ε_j will be chosen small enough so that $\overline{K_{\varepsilon_j}}$ is in K_r^R and $\{K_{\varepsilon_j}\}$ forms a collection of mutually disjoint open subsets of K_R^r.

Notice then that if $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_m\}$, then $\overline{K_\varepsilon}(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*) \subset K_r^R$ and $\{K_\varepsilon(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)\}$ forms a collection of mutually disjoint open subsets of K_r^R. Therefore, the lemma is proved, owing to Lemma 2.4 and the additivity of the fixed point index, if one can show that $(I - \mathcal{B})\partial K_\varepsilon(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$ omits the ray $\{\lambda\widetilde{\Gamma} : \lambda \geq 0\}$, where $\widetilde{\Gamma} \in K$ is defined as in Proposition 4.4. So suppose that for each $j = 1, \dots, m$, there are a $\Gamma = (\Phi_1, \dots, \Phi_m) \in \partial K_{\varepsilon_j}(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$ and a $\lambda \geq 0$ such that $\Gamma - \mathcal{B}\Gamma = \lambda\widetilde{\Gamma}$. Then, for all $n \in \mathbb{Z}$,

$$(4.11) \quad [\phi_{jn}] = \frac{1}{\omega + (n\pi/l)^2}\bigg[a(\Phi_j \times \Phi_j \times \Phi_j)_n + b\bigg(\Phi_j \times \sum_{k=1,\, k\neq j}^{m} (\Phi_k \times \Phi_k)\bigg)_n\bigg] + \lambda\widetilde{\Gamma}.$$

In particular, for $n = 1$, one has

$$(4.12) \qquad \phi_{j1} \geq A\bigg[3a\phi_{j0}^2\phi_{j1} + b\sum_{k=1,\, k\neq j}^{m}(2\phi_{j0}\phi_{k0}\phi_{k1} + \phi_{j1}\phi_{k0}^2)\bigg] + \frac{\lambda}{2}$$

where $A = 1/(\omega + (\pi/l)^2)$. It is claimed next that when the half-period l is large enough, the $\varepsilon_j(l)$ can be chosen sufficiently small so that $\phi_{jn} = 0$ for all $n \neq 0$ and all $j = 1, \dots, m$. One needs to consider the following cases separately.

Case 1. 1-nonvanishing fixed point $(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$ where $(p_{10}^*, \dots, p_{m0}^*) = (\sqrt{\omega/a}, 0, \dots, 0)$.

Since $\Gamma \in \partial K_{\varepsilon_1}(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$, it can be written as $\Gamma = (\Phi_1, \dots, \Phi_m) = (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*) + \varepsilon_1(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)$ where $\|(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)\|_{\mathcal{L}_{3/2,m}} = 1$. Note that

$$\varepsilon_1(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m) = (\Phi_1, \dots, \Phi_m) - (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*),$$

therefore, for $n \geq 1$, $\widetilde{\phi}_{jn} = \phi_{jn}/\varepsilon_1 \geq 0$, and $\widetilde{\phi}_{jn} \geq \widetilde{\phi}_{j(n+1)}$.

In terms of the new variables $(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)$, (4.12) yields

$$\varepsilon_1\widetilde{\phi}_{11} \geq 3A\omega\varepsilon_1\widetilde{\phi}_{11} + 6Aa\sqrt{\frac{\omega}{a}}\varepsilon_1\widetilde{\phi}_{10}\varepsilon_1\widetilde{\phi}_{11} + Ab\sum_{k=2}^{m}2\bigg(\sqrt{\frac{\omega}{a}} + \varepsilon_1\widetilde{\phi}_{10}\bigg)\varepsilon_1\widetilde{\phi}_{k0}\varepsilon_1\widetilde{\phi}_{k1} + \frac{\lambda}{2};$$

and for $j = 2, \dots, m$

$$(4.13) \qquad \varepsilon_1\widetilde{\phi}_{j1} \geq \frac{Ab\omega}{a}\varepsilon_1\widetilde{\phi}_{j1} + 2Ab\sqrt{\frac{\omega}{a}}\varepsilon_1\widetilde{\phi}_{10}\varepsilon_1\widetilde{\phi}_{j1} + 2Ab\sum_{k=1,k\neq j}^{m}\varepsilon_1\widetilde{\phi}_{j0}\varepsilon_1\widetilde{\phi}_{k0}\varepsilon_1\widetilde{\phi}_{k1} + \frac{\lambda}{2}.$$

First, choose l large enough such that both following conditions hold: $3A\omega > 1$ and $Ab\omega/a > 1$. The explicit condition for l is

$$l^2 > \max\left\{\frac{\pi^2 a}{(b-a)\omega}, \frac{\pi^2}{2\omega}\right\} :\equiv L_1. \tag{4.14}$$

The number $\varepsilon_1 = \varepsilon_1(l)$ can now be chosen small enough so that there hold both

$$\begin{aligned} 3A\omega\varepsilon_1\widetilde{\phi}_{11} + 6Aa\sqrt{\frac{\omega}{a}}\varepsilon_1\widetilde{\phi}_{10}\varepsilon_1\widetilde{\phi}_{11} \\ + Ab\sum_{k=2}^{m} 2\left(\sqrt{\frac{\omega}{a}} + \varepsilon_1\widetilde{\phi}_{10}\right)\varepsilon_1\widetilde{\phi}_{k0}\varepsilon_1\widetilde{\phi}_{k1} > \varepsilon_1\widetilde{\phi}_{11}; \end{aligned} \tag{4.15}$$

$$\frac{Ab\omega}{a}\varepsilon_1\widetilde{\phi}_{j1} + 2Ab\sqrt{\frac{\omega}{a}}\varepsilon_1\widetilde{\phi}_{10}\varepsilon_1\widetilde{\phi}_{j1} + 2Ab\sum_{k=1,\, k\neq j}^{m}\varepsilon_1\widetilde{\phi}_{j0}\varepsilon_1\widetilde{\phi}_{k0}\varepsilon_1\widetilde{\phi}_{k1} > \varepsilon_1\widetilde{\phi}_{j1}. \tag{4.16}$$

It follows immediately from (4.13), (4.15) and (4.16) that $\lambda = 0$ and $\widetilde{\phi}_{j1} = 0$, hence $\phi_{jn} = 0$ for all $n \neq 0$ and all $j = 1, \dots, m$.

Case 2. k-nonvanishing fixed point $(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$ where

$$(p_{10}^*, \dots, p_{m0}^*) = \Bigg(\underbrace{\sqrt{\frac{\omega}{a+(k-1)b}}, \dots, \sqrt{\frac{\omega}{a+(k-1)b}}}_{k}, 0, \dots, 0\Bigg).$$

Since $\Gamma \in \partial K_{\varepsilon_2}(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$, it can be written as

$$\Gamma = (\Phi_1, \dots, \Phi_m) = (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*) + \varepsilon_2(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)$$

where $\|(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)\|_{\mathcal{L}_{3/2,m}} = 1$. Note that

$$\varepsilon_2(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m) = (\Phi_1, \dots, \Phi_m) - (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*),$$

therefore, for $n \geq 1$, $\widetilde{\phi}_{jn} = \phi_{jn}/\varepsilon_2 \geq 0$ and $\widetilde{\phi}_{jn} \geq \widetilde{\phi}_{j(n+1)}$.

In terms of the new variables $(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)$, (4.12) yields for $j = 1, \dots, k$

$$\varepsilon_2\widetilde{\phi}_{j1} \geq \frac{A\omega(3a+bk-b)}{a+(k-1)b}\,\varepsilon_2\widetilde{\phi}_{j1} - C_1\varepsilon_2^2 + \frac{\lambda}{2}; \tag{4.17}$$

and for $j = k+1, \dots, m$

$$\varepsilon_2\widetilde{\phi}_{j1} \geq \frac{Abk\omega}{a+(k-1)b}\,\varepsilon_2\widetilde{\phi}_{j1} - C_2\varepsilon_2^2 + \frac{\lambda}{2}. \tag{4.18}$$

First, choose l large enough so that both the following conditions hold

$$\frac{A\omega(3a+bk-b)}{a+(k-1)b} > 1 \quad \text{and} \quad \frac{Abk\omega}{a+(k-1)b} > 1.$$

The explicit condition for l is

$$l^2 > \max\left\{\frac{\pi^2[a+(k-1)b]}{(b-a)\omega}, \frac{\pi^2[a+(k-1)b]}{2a\omega}\right\} :\equiv L_2. \tag{4.19}$$

The number $\varepsilon_2 = \varepsilon_2(l)$ can now be chosen small enough so that there hold both

$$\begin{aligned} &\frac{A\omega(3a+bk-b)}{a+(k-1)b}\,\varepsilon_2\widetilde{\phi}_{j1} - C_1\varepsilon_2^2 > \varepsilon_2\widetilde{\phi}_{j1} \quad \text{for } j=1,\dots,k; \\ &\frac{Abk\omega}{a+(k-1)b}\,\varepsilon_2\widetilde{\phi}_{j1} - C_2\varepsilon_2^2 > \varepsilon_2\widetilde{\phi}_{j1} \quad \text{for } j=k+1,\dots,m. \end{aligned} \tag{4.20}$$

It follows immediately from (4.17), (4.18) and (4.20) that $\lambda = 0$ and $\widetilde{\phi}_{j1} = 0$, hence $\phi_{jn} = 0$ for all $n \neq 0$ and all $j = 1, \dots, m$.

Case 3. m-nonvanishing (or vector) fixed point $(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$ where

$$(p_{10}^*, \dots, p_{m0}^*) = \left(\sqrt{\frac{\omega}{a+(m-1)b}}, \dots, \sqrt{\frac{\omega}{a+(m-1)b}}\right).$$

As before, since $\Gamma \in \partial K_{\varepsilon_3}(\mathbf{p}_1^*, \dots, \mathbf{p}_m^*)$, it can be written as

$$\Gamma = (\Phi_1, \dots, \Phi_m) = (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*) + \varepsilon_3(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)$$

where $\|(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)\|_{\mathcal{L}_{3/2,m}} = 1$. Note that

$$\varepsilon_3(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m) = (\Phi_1, \dots, \Phi_m) - (\mathbf{p}_1^*, \dots, \mathbf{p}_m^*),$$

therefore, for $n \geq 1$ $\widetilde{\phi}_{jn} = \phi_{jn}/\varepsilon_3 \geq 0$ and $\widetilde{\phi}_{jn} \geq \widetilde{\phi}_{j(n+1)}$.

In terms of the new variables $(\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m)$, (4.12) yields for $j = 1, \dots, m$

$$\varepsilon_3\widetilde{\phi}_{j1} \geq \frac{A\omega(3a+bm-b)}{a+(m-1)b}\,\varepsilon_3\widetilde{\phi}_{j1} - C_3\varepsilon_3^2 + \frac{\lambda}{2}. \tag{4.21}$$

Now, one can choose l large enough so that

$$\frac{A\omega(3a+bm-b)}{a+(m-1)b} > 1.$$

The explicit condition for l is

$$l^2 > \frac{\pi^2[a+(m-1)b]}{2a\omega} :\equiv L_3. \tag{4.22}$$

The number $\varepsilon_3 = \varepsilon_3(l)$ can now be chosen small enough so that there holds

$$\frac{A\omega(3a+bm-b)}{a+(m-1)b}\,\varepsilon_3\widetilde{\phi}_{j1} - C_3\varepsilon_3^2 > \varepsilon_3\widetilde{\phi}_{j1}. \tag{4.23}$$

It follows immediately from (4.21) and (4.23) that $\lambda = 0$ and $\widetilde{\phi}_{j1} = 0$, hence $\phi_{jn} = 0$ for all $n \neq 0$ and all $j = 1, \dots, m$. Thus, the claim is proved.

Next, let

$$l^2 > \max\{L_1, L_2, L_3\}, \tag{4.24}$$

and redefine $\widetilde{\varepsilon}_1(l), \widetilde{\varepsilon}_2(l), \widetilde{\varepsilon}_3(l)$ if necessary so that for

$$\varepsilon(l) = \min\{\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2, \widetilde{\varepsilon}_3\}, \tag{4.25}$$

the expressions (4.15), (4.16), (4.20) and (4.23) hold. Under these conditions, $\lambda = 0$ and for all $n \neq 0$ and all $j = 1, \ldots, m$, one has $\phi_{jn} = 0$.

Using (4.11) for $n = 0$, one sees that

$$\omega\phi_{j0} = a\phi_{j0}^3 + b\phi_{j0} \sum_{k=1,k\neq j}^{m} \phi_{k0}^2. \tag{4.26}$$

Up to permutation, the only solutions for (4.26) are the origin and $(\mathbf{p}_1^*, \ldots, \mathbf{p}_m^*)$ where $(p_{10}^*, \ldots, p_{m0}^*)$ takes on the following form:

(a) 1-nonvanishing: $\left(\sqrt{\frac{\omega}{a}}, 0, \ldots, 0\right)$;

(b) k-nonvanishing: $\Big(\underbrace{\sqrt{\frac{\omega}{a+(k-1)b}}, \ldots, \sqrt{\frac{\omega}{a+(k-1)b}}}_{k}, 0, \ldots, 0\Big)$;

(c) m-nonvanishing (or vector solution):

$$\left(\sqrt{\frac{\omega}{a+(m-1)b}}, \ldots, \sqrt{\frac{\omega}{a+(m-1)b}}\right).$$

But then this contradicts with the facts that $\{K_\varepsilon(\mathbf{p}_1^*, \ldots, \mathbf{p}_m^*)\}$ forms a collection of mutually disjoint open subsets of K_r^R and

$$\Gamma = (\Phi_1, \ldots, \Phi_m) \in \partial K_\varepsilon(\mathbf{p}_1^*, \ldots, \mathbf{p}_m^*)$$

for small ε when the half-period l is chosen large enough as in (4.24). The proof of Lemma 4.7 is hence concluded. □

Straightforward calculations similar to the proof of Theorem 3.8 gives the following.

THEOREM 4.8. *Let $b > a > 0$ and $\omega > 0$. If the half-period l is chosen large enough as in* (4.24)*, then the operator $\mathcal{B}$ must have at least one non-trivial fixed point $\overline{\Gamma} = (\overline{\Phi}_1, \ldots, \overline{\Phi}_m)$ in the cone segment K_R^r. Moreover,*

(a) *either all the sequences of components $\overline{\phi}_{jn} > 0$ for every $n \in \mathbb{Z}$ and all $j = 1, \ldots, m$ in which case one obtains a vector solution $(\overline{\Phi}_1, \ldots, \overline{\Phi}_m)$, or it must be the case that $(m-k)$ of the sequences $\overline{\phi}_{jn}$ vanish for every $n \in \mathbb{Z}$, $k = 1, \ldots, m-1$, while the others remain strictly positive for every $n \in \mathbb{Z}$ in which case one obtains, up to permutation, semi-trivial solutions (k-nonvanishing) $(\overline{\Phi}_1, \ldots, \overline{\Phi}_k, 0, \ldots, 0)$;*

(b) *for any $\sigma \geq 0$, the sequences $\{|n|^\sigma \overline{\phi}_{jn}\}$ are in l_1. Therefore, the non-trivial fixed point solutions are infinitely smooth.*

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