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A SECOND ORDER DIFFERENTIAL INCLUSION WITH PROXIMAL NORMAL CONE IN BANACH SPACES

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Abstract. In the present paper we mainly consider the second order evolution inclusion with proximal normal cone:

$$(*)\qquad \begin{cases} -\ddot{x}(t) \in N_{K(t)}(\dot{x}(t)) + F(t, x(t), \dot{x}(t)), & \text{a.e.} \\ \dot{x}(t) \in K(t), \\ x(0) = x_0, \quad \dot{x}(0) = u_0, \end{cases}$$

where $t \in I = [0,T]$, E is a separable reflexive Banach space, $K(t)$ a ball compact and r-prox-regular subset of E, $N_{K(t)}(\,\cdot\,)$ the proximal normal cone of $K(t)$ and F an u.s.c. set-valued mapping with nonempty closed convex values. First, we prove the existence of solutions of $(*)$. After, we give an other existence result of $(*)$ when $K(t)$ is replaced by $K(x(t))$.

1. Introduction

The existence of solutions for the sweeping processes has been introduced and thoroughly studied in the 70's by Moreau in [23], in the setting where all the sets are assumed to be convex. Recently in [10], the authors proved the existence of solutions of the perturbed sweeping process,

$$(1.1)\qquad -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t)), \quad \text{a.e. } t \in [0,T];\ x(0) = x_0 \in K(0),$$

where $K\colon [0,T] \rightrightarrows H$ is a set-valued mapping which has not necessarily convex values and $F\colon [0,T] \times H \rightrightarrows H$ is an upper semicontinuous set-valued mapping

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with convex compact values. Then the main concept, which appeared to get around the convexity of the sets $K(t)$, is the notion of "uniform prox-regularity". This property is very well-adapted to the resolution of (1.1).

Several papers (for example [2], [3], [4], [10], ...) have been devoted to the study of the properties of prox-regular (or proximally smooth) sets. Using these properties, the authors in [1], [7], [9], [13], [16], [17], [19], [20] and [26] proved some existence results of the first and second order sweeping processes.

Note that all the results recalled in the last paragraph above have been obtained in finite dimensional or Hilbert spaces. Newly, some extensions of convex sweeping processes from Hilbert spaces to Banach spaces are proved in [5], [6], [8] and [22].

Our aim in this paper is to prove, in a separable reflexive and uniformly smooth Banach space E, the existence results for the following problem

$$(\mathcal{P}_F) \qquad \begin{cases} -\ddot{x}(t) \in N_{K(t)}(\dot{x}(t)) + F(t, x(t), \dot{x}(t)), & \text{a.e. } t \in [0,T],\\ \dot{x}(t) \in K(t), & \text{for all } t \in [0,T],\\ x(0) = x_0, \quad \dot{x}(0) = u_0, \end{cases}$$

where $K\colon [0,T] \rightrightarrows E$ is a nonempty ball compact and r-prox-regular valued set-valued mapping, $N_{K(t)}(\,\cdot\,)$ the proximal normal cone $K(t)$ and $F\colon [0,T] \times E \times E \rightrightarrows E$ an upper semicontinuous set-valued mapping with nonempty closed convex values. Moreover, we extend this result to the case where the set-valued mapping K depends on the state variable, that is, we give an existence theorem for the problem:

$$(\mathcal{P}'_F) \qquad \begin{cases} -\ddot{x}(t) \in N_{K(x(t))}(\dot{x}(t)) + F(t, x(t), \dot{x}(t)), & \text{a.e. } t \in [0,T],\\ \dot{x}(t) \in K(x(t)), & \text{for all } t \in [0,T],\\ x(0) = x_0, \quad \dot{x}(0) = u_0. \end{cases}$$

Note that our first theorem extend to the second order the result proved in [5].

The organization of this paper is as follows. Section 2 is devoted to some definitions and notation needed in the paper and Section 3 is reserved to the main results.

2. Notation and preliminaries

In this section we recall the main definitions and notations used throughout the paper.

Let $(E, \|\cdot\|)$ be a separable Banach space, E' its topological dual and $\langle\,\cdot\,,\,\cdot\,\rangle$ their duality product. $\overline{\mathbf{B}}_E(0,r)$ is the closed ball of E of center 0 and radius r, $\overline{\mathbf{B}}_E$ the closed unit ball and $\mathbf{S}_E$ is the unit sphere of E.

Let $\mathbf{C}_E([0,T])$ $(T > 0)$ be the Banach space of all continuous mappings $u\colon [0,T] \to E$, endowed with the sup-norm, and $\mathbf{C}^1_E([0,T])$ be the Banach space of all continuous mappings $u\colon [0,T] \to E$ with continuous derivative, equipped with the norm

$$\|u\|_{\mathbf{C}^1} = \max\Big\{ \max_{t\in[0,T]} \|u(t)\|, \max_{t\in[0,T]} \|\dot{u}(t)\|\Big\}.$$

We denote by $\mathcal{L}([0,T])$ the σ-algebra of Lebesgue measurable subsets of $[0,T]$. $(\mathbf{L}^1_E([0,T]), \|\cdot\|_1)$ is the Banach space of Lebesgue–Bochner integrable E-valued mappings and $(\mathbf{L}^\infty_E([0,T]), \|\cdot\|_\infty)$ is the Banach space of essentially bounded E-valued mappings.

We said that a mapping $u\colon [0,T] \to E$ is absolutely continuous if there is a mapping $v \in \mathbf{L}^1_E([0,T])$ such that

$$u(t) = u(0) + \int_0^t v(s)\,ds, \quad \text{for all } t \in [0,T],$$

in this case $v = \dot{u}$ almost everywhere.

For $A \subset E$, $\mathrm{co}(A)$ denotes the convex hull of A and $\overline{\mathrm{co}}(A)$ its closed convex hull. We have the following characterization.

Theorem 2.1. *Let K be a nonempty subset of E. Then,*

$$\overline{\mathrm{co}}(K) = \{x \in E \mid \langle x', x\rangle \leq \delta^*(x', K) \text{ for all } x' \in E'\},$$

where $\delta^*(x', K)$ denotes the support function associated with K, i.e.

$$\delta^*(x', K) = \sup_{y\in K} \langle x', y\rangle.$$

It is well know that the support function of an upper semicontinuous set-valued mapping is upper semicontinuous.

For closed subsets A and B of E, the Hausdorff distance between A and B is defined by

$$\mathcal{H}(A,B) = \sup(e(A,B), e(B,A)),$$

where $e(A,B) = \sup_{a\in A} d(a,B)$ stands for the excess of A over B and $d(a,B) = \inf_{x\in B} \|a - x\|$.

We recall that for a closed convex subset A of E, one has

$$d(x,A) = \sup_{x'\in\overline{\mathbf{B}}_{E'}} (\langle x', x\rangle - \delta^*(x', A)). \tag{2.1}$$

Definition 2.2. A subset $A \subset E$ is said to be ball-compact if for all closed ball $\overline{\mathbf{B}} = \overline{\mathbf{B}}(x,R)$ of E, the set $\overline{\mathbf{B}} \cap A$ is compact. Obviously, a ball-compact subset A is closed.

Definition 2.3. Let A be a closed subset of E. Then the set-valued projection operator P_A is defined by

$$P_A(x) = \{y \in E \mid \|x - y\| = d(x,A)\} \quad \text{for all } x \in E.$$

Definition 2.4. Let A be a closed subset of E and $x \in A$, we denote by $N_A(x)$ the proximal normal cone of A at x, defined by

$$N_A(x) = \{v \in E,\ \exists s > 0,\ x \in P_A(x + sv)\}.$$

Definition 2.5. For every point $x \in A$ and $r > 0$, we define $\Gamma^r(x, A)$ as the set of good directions v to project at the scale r from $x + rv$ to x, that is,

$$\Gamma^r(A, x) = \{v \in E,\ x \in P_A(x + rv)\}.$$

Remark 2.6. For all $x \in A$, we obviously have by definition of the proximal normal cone

$$N_A(x) = \bigcup_{r>0} \Gamma^r(A, x).$$

We refer to [3, Lemma 2.1] for the following geometric lemma.

Lemma 2.7. *Let E be a Banach space and K be a closed subset of E. Then for $x \in A$ and $v \in \Gamma^r(A, x)$, we have $\lambda v \in \Gamma^r(A, x)$ for all $\lambda \in (0, 1)$. Therefore, if we assume that E is uniformly convex then for all $\lambda \in (0, 1)$, we have $x \in P_A(x + \lambda rv)$.*

We now come to the main notion of prox-regularity. It was initially introduced by H. Federer ([21]) in spaces of finite dimension under the name of positively reached sets. Then, it was extended in Hilbert spaces by A. Canino in [11] and A.S. Shapiro in [25]. After, this notion was studied by F.H. Clarke, R.J. Stern and P.R. Wolenski in [14] (see also [15]) and by R.A. Poliquin, R.T. Rockafellar and L. Thibault in [24]. Few years later, F. Bernard, L. Thibault and N. Zlateva have defined this notion in Banach spaces (see [2], [3] and [4]).

Definition 2.8. Let A be a closed subset of E and $r > 0$. The set A is said to be r-prox-regular if for all $x \in A$ and $v \in N_A(x) \setminus \{0\}$

$$\mathbf{B}\bigg(x + r\frac{v}{\|v\|}, r\bigg) \cap A = \emptyset.$$

The following Proposition give some important consequences of the prox-regularity needed in the sequel. For the proof and more details we refer the reader to [24].

Proposition 2.9. *Let $r \in (0, +\infty]$ and let A be a nonempty closed and r-prox-regular subset of E. Then we have the following*:

(a) *For all $x \in E$ with $d(x, A) < r$, the projection of x onto A is well-defined and continous, that is, $P_A(x)$ is single-valued;*
(b) *if $u = P_A(x)$ then, $u = P_A(u + r(x - u)/\|x - u\|)$.*

Now we recall some useful definitions, due to the geometric theory of Banach spaces (we refer the reader to [18] for these concepts and more details).

Definition 2.10. The Banach space $(E, \|\cdot\|)$ is said to be uniformly smooth if his norm is uniformly Fréchet differentiable away of 0, it means that for any two unit vectors $x_0, h \in E$, the limit

$$\lim_{t \downarrow 0} \frac{\|x_0 + th\| - \|x_0\|}{t}$$

exists uniformly with respect to $h, x_0 \in \mathbf{S}_E$.

As we know that the norm could be non differentiable at the origin 0, we study the function $x \mapsto \|x\|^p$ for an exponent $p > 1$.

Proposition 2.11. *Let E be a uniformly smooth Banach space and $p \in (1, \infty)$ be an exponent. The function $x \mapsto \|x\|^p$ is $\mathbf{C}^1$ over the whole space E.*

Definition 2.12. For E a uniformly smooth Banach space and $p \in (1, \infty)$, we denote

$$J_p(x) := \frac{1}{p}(\nabla \|\cdot\|^p)(x) \in E'.$$

Definition 2.13. Let I be an interval of $\mathbb{R}$. A separable reflexive uniformly smooth Banach space E is said to be "I-smoothly weakly compact" for an exponent $p \in (1, \infty)$ if for all bounded sequence $(x_n)_n$ of $\mathbf{L}^\infty_E(I)$, we can extract a subsequence $(y_n)_n$ weakly converging to a point $y \in \mathbf{L}^\infty_E(I)$ such that for all $z \in \mathbf{L}^\infty_E(I)$ and $\phi \in \mathbf{L}^1_{\mathbb{R}}(I)$,

$$\lim_{n \to \infty} \int_I \langle J_p(z(t) + y_n(t)) - J_p(y_n(t)), y_n(t) \rangle \phi(t)\, dt \\ = \int_I \langle J_p(z(t) + y(t)) - J_p(y(t)), y(t) \rangle \phi(t)\, dt.$$

Remark 2.14. It is easy to check that the notion of "I-smoothly weakly compactness" does not depend on the time-interval I.

Proposition 2.15. *All separable Hilbert space H is I-smoothly weakly compact for $p = 2$.*

The following proposition describes a useful property of weak continuity of the projection operator. For the proof we refer the reader to [5].

Proposition 2.16. *Let $(E, \|\cdot\|)$ be a separable, reflexive and uniformly smooth Banach space. Let $C_n, C \colon I \rightrightarrows E$ be set-valued mappings taking nonempty closed values and satisfying*

$$\sup_{t \in I} \mathcal{H}(C_n(t), C(t)) \underset{n \to \infty}{\longrightarrow} 0.$$

We assume that for an exponent $p \in [2, \infty)$ and a bounded sequence $(v_n)_n$ of $\mathbf{L}^\infty_E(I)$, we can extract a subsequence $(v_{k(n)})_n$ weakly converging to a point

$v \in \mathbf{L}^\infty_E(I)$ such that for all $z \in \mathbf{L}^\infty_E(I)$ and $\phi \in \mathbf{L}^1_{\mathbb{R}}(I)$,

$$\limsup_{n\to\infty} \int_I \langle J_p(z(t)+v_{k(n)}(t)) - J_p(v_{k(n)}(t)), v_{k(n)}(t)\rangle \phi(t)\,dt = \int_I \langle J_p(z(t)+v(t)) - J_p(v(t)), v(t)\rangle \phi(t)\,dt.$$

Then the projection $P_{C(\,\cdot\,)}$ is weakly continuous in $\mathbf{L}^\infty_E(I)$ (relatively to the directions given by the sequence $(v_n)_n$) in the following sense: for all $r > 0$ and for any bounded sequence $(u_n)_n$ of $\mathbf{L}^\infty_E(I)$ satisfying

$$\begin{cases} u_n \to u & \text{in } \mathbf{L}^\infty_E(I), \\ u_n(t) \in P_{C_n(t)}(u_n(t) + rv_n(t)) & \text{a.e. } t \in I, \end{cases}$$

one has $u(t) \in P_{C(t)}(u(t) + rv(t))$ for almost every $t \in I$.

3. Main results

Now, we are able to prove our main existence theorems.

Theorem 3.1. *Let $I = [0,T]$ $(T > 0)$ and E be a separable, reflexive, uniformly smooth Banach space, which is I-smoothly weakly compact for an exponent $p \in [2,\infty)$. Let $F\colon I \times E \times E \rightrightarrows E$ be an upper semicontinuous set-valued mapping with nonempty closed convex values. We assume that there exists a constant $m > 0$ such that*

$$F(t,x,u) \subset m\overline{\mathbf{B}}_E, \quad \text{for all } (t,x,u) \in I \times E \times E. \tag{3.1}$$

Let $r > 0$ and $K\colon [0,T] \rightrightarrows E$ be a set-valued mapping taking nonempty ball-compact and r-prox-regular values. We assume that $K(\,\cdot\,)$ moves in a Lipschitz way, that is, there exists a constant $k > 0$ such that for all $s,t \in I$,

$$\mathcal{H}(K(t),K(s)) \le k|t-s|. \tag{3.2}$$

Then for all $x_0 \in E$ and $u_0 \in K(0)$, the differential inclusion

$$(\mathcal{P}_F) \qquad \begin{cases} u(0) = u_0, \\ x(t) = x_0 + \displaystyle\int_0^t u(s)\,ds & \text{for all } t \in I, \\ u(t) \in K(t) & \text{for all } t \in I, \\ -\dot u(t) \in N_{K(t)}(u(t)) + F(t,x(t),u(t)) & \text{a.e. } t \in I, \end{cases}$$

has Lipschitz solutions $u,x\colon I \to E$. Moreover, we have $\|\dot u(t)\| \le 2m + k$ for almost every $t \in I$. In other words, the differential inclusion

$$(\mathcal{P}_F) \qquad \begin{cases} -\ddot x(t) \in N_{K(t)}(\dot x(t)) + F(t,x(t),\dot x(t)) & \text{a.e. } t \in I, \\ \dot x(t) \in K(t) & \text{for all } t \in I, \\ x(0) = x_0, \quad \dot x(0) = u_0, \end{cases}$$

has at least a Lipschitz solution $x(\cdot) \in \mathbf{C}^1_E(I)$.

Proof. *Step 1.* Let $n_0 \in \mathbb{N}^*$ such that

$$\frac{T}{n_0}(m+k) \le \frac{r}{2}. \tag{3.3}$$

Consider the partition $\{t_{n,0}, I_{n,i}\}$, $0 \le i \le n$ of the time-interval $I = [0,T]$ defined by $I_{n,i} = (t_{n,i}, t_{n,i+1}]$ for $0 \le i \le n-1$, $t_{n,i} = ih$ for $0 \le i \le n$ and $h = T/n$. For every $n \ge n_0$ we define the following approximating mappings on each interval $I_{n,i}$ as follows (this approximation is inspired from [5])

$$\begin{cases} u_n(t) = u_{n,i} + \left(\dfrac{t}{h} - i\right)(u_{n,i+1} - u_{n,i}), \\ x_n(t) = x_0 + \displaystyle\int_0^t u_n(s)\, ds, \\ x_{n,i} = x_n(t_{n,i}), \\ K_n(t) = K(t_{n,i+1}), \end{cases} \tag{3.4}$$

and since F has nonempty values, we choose a point,

$$z_{n,i} \in F(t_{n,i}, x_{n,i}, u_{n,i}) \tag{3.5}$$

and we define the mapping z_n from I to E by $z_n(t) = z_{n,i}$, for all $t \in I_{n,i}$, where $u_{n,0} = u_0$; $x_{n,0} = x_0$, and for all $0 \le i \le n-1$ the point $u_{n,i+1}$ is given by

$$u_{n,i+1} = P_{K(t_{n,i+1})}(u_{n,i} - hz_{n,i}). \tag{3.6}$$

This operation is allowed since the constants m and k satisfy relation (3.3), K satisfies (3.2) and the sets $K(t)$ are assumed to be r-prox-regulars. Indeed, we have $u_{n,0} \in K(t_{n,0})$ and then

$$\begin{aligned} d(u_{n,0} - hz_{n,0}, K(t_{n,1})) &\le d(u_{n,0} - hz_{n,0}, K(t_{n,0})) + \mathcal{H}(K(t_{n,1}), K(t_{n,0})) \\ &\le \|u_{n,0} - hz_{n,0} - u_{n,0}\| + k|t_{n,1} - t_{n,0}| \\ &= h(\|z_{n,0}\| + k) \le h(m+k) \le \frac{r}{2} < r. \end{aligned}$$

By Proposition 2.9, we have that $P_{K(t_{n,1})}(u_{n,0} - hz_{n,0})$ is a nonempty single-valued set. Then we define the point $u_{n,1} \in K(t_{n,1})$, by

$$u_{n,1} = P_{K(t_{n,1})}(u_{n,0} - hz_{n,0}).$$

Similarly, we can define, by induction, all the points $(u_{n,i})$, $0 \le i \le n$.

Observe that relation (3.6) and Proposition 2.9 give

$$u_{n,i+1} = P_{K(t_{n,i+1})}\left(u_{n,i+1} + r\frac{u_{n,i} - hz_{n,i} - u_{n,i+1}}{\|u_{n,i} - hz_{n,i} - u_{n,i+1}\|}\right), \tag{3.7}$$

that is,

$$\frac{u_{n,i} - hz_{n,i} - u_{n,i+1}}{\|u_{n,i} - hz_{n,i} - u_{n,i+1}\|} \in \Gamma^r(K(t_{n,i+1}), u_{n,i+1}).$$

On the other hand, we have by (3.4), for all $t \in I_{n,i} = (t_{n,i}, t_{n,i+1}]$

$$u_n(t) = u_{n,i} + \left(\frac{t}{h} - i\right)(u_{n,i+1} - u_{n,i}),$$

and for all $t \in I_{n,i-1} = (t_{n,i-1}, t_{n,i}]$

$$u_n(t) = u_{n,i-1} + \left(\frac{t}{h} - (i-1)\right)(u_{n,i} - u_{n,i-1}).$$

Then

$$\begin{aligned} u_n(t_{n,i}) &= u_{n,i-1} + \left(\frac{t_{n,i}}{h} - (i-1)\right)(u_{n,i} - u_{n,i-1}) \\ &= u_{n,i-1} + (i - i + 1)(u_{n,i} - u_{n,i-1}) = u_{n,i}, \end{aligned}$$

and

$$\lim_{t \to^{>} t_{n,i}} u_n(t) = u_{n,i} + \left(\frac{t_{n,i}}{h} - i\right)(u_{n,i+1} - u_{n,i}) = u_{n,i}.$$

Consequently, for each $n \geq n_0$, the mapping u_n is continuous and $x_n \in \mathbf{C}^1_E(I)$.

Step 2. We look for a differential inclusion satisfied by the mapping u_n. For almost every $t \in I_{n,i}$, we have by (3.4)

$$\dot{u}_n(t) = \frac{1}{h}(u_{n,i+1} - u_{n,i} + hz_{n,i}) - z_{n,i}.$$

We set

$$\Delta_n(t) = \dot{u}_n(t) + z_n(t) = \frac{1}{h}(u_{n,i+1} - u_{n,i} + hz_{n,i}).$$

We claim that $-\Delta_n(t) \in \Gamma^{r/(m+k)}(K_n(t), u_{n,i+1}) \cap \overline{\mathbf{B}}(0, (m+k))$, which is equivalent to

$$\|\Delta_n(t)\| \leq (m+k) \quad \text{and} \quad u_{n,i+1} \in P_{K_n(t)}\left(u_{n,i+1} - \frac{r}{(m+k)}\Delta_n(t)\right).$$

First, we check that $\Delta_n(t)$ is a bounded vector. Using the construction of the points $u_{n,i+1}$ (see relation (3.6)) and the fact that $u_{n,i} \in K(t_{n,i})$, we have for almost every $t \in I_{n,i}$

$$\begin{aligned} \|\Delta_n(t)\| &= \left\| \frac{1}{h}(u_{n,i+1} - u_{n,i} + hz_{n,i}) \right\| \\ &= \frac{1}{h} \|P_{K(t_{n,i+1})}(u_{n,i} - hz_{n,i}) - (u_{n,i} - hz_{n,i})\| \\ &= \frac{1}{h} d(u_{n,i} - hz_{n,i}, K(t_{n,i+1})) \\ &\leq \frac{1}{h}(d(u_{n,i} - hz_{n,i}, K(t_{n,i})) + \mathcal{H}(K(t_{n,i+1}), K(t_{n,i}))) \\ &\leq \frac{1}{h}(\|u_{n,i} - hz_{n,i} - u_{n,i}\| + k|t_{n,i+1} - t_{n,i}|) \\ &\leq \frac{1}{h}(h(m+k)) = m + k. \end{aligned}$$

That is, for almost every $t \in I_{n,i}$,

$$\|\Delta_n(t)\| \leq (m+k). \tag{3.8}$$

Then, by considering the vector $v = u_{n,i} - hz_{n,i}$, since K has r-prox-regular values, the relation (3.7) gives

$$u_{n,i+1} = P_{K_n(t)}(P_{K_n(t)}(v) - r\frac{P_{K_n(t)}(v) - v}{\|P_{K_n(t)}(v) - v\|}). \tag{3.9}$$

Observe, that by the relation (3.8), we have $\|P_{K_n(t)}(v) - v\|/(h(m+k)) \leq 1$. Then, applying Lemma 2.7 to the relation (3.9), with $\lambda = \|P_{K_n(t)}(v) - v\|/(h(m+k))$, we get

$$\begin{aligned} u_{n,i+1} &= P_{K_n(t)}\bigg(P_{K_n(t)}(v) - \frac{r}{h}\frac{\|P_{K_n(t)}(v) - v\|}{(m+k)}\frac{P_{K_n(t)}(v) - v}{\|P_{K_n(t)}(v) - v\|}\bigg) \\ &= P_{K_n(t)}\bigg(P_{K_n(t)}(v) - \frac{r}{h(m+k)}(P_{K_n(t)}(v) - v)\bigg) \\ &= P_{K_n(t)}\bigg(P_{K_n(t)}(v) - \frac{r}{(m+k)}\Delta_n(t)\bigg), \end{aligned}$$

that is, $-\Delta_n(t) \in \Gamma^{r/(m+k)}(K_n(t), u_{n,i+1})$, or equivalently

$$u_{n,i+1} \in P_{K_n(t)}(u_{n,i+1} - \frac{r}{(m+k)}\Delta_n(t)) \quad \text{a.e. on } I_{n,i}. \tag{3.10}$$

Step 3. Existence of limit mappings. First, we will to prove the convergence of the sequence $(u_n(\,\cdot\,)) \in \mathbf{C}_E(I)$. By the relation (3.8), we have for almost every $t \in I$,

$$\|\dot{u}_n(t)\| \leq \|\dot{u}_n(t) + z_n(t)\| + \|z_n(t)\| = \|\Delta_n(t)\| + \|z_n(t)\| \leq 2m + k,$$

i.e.

$$\|\dot{u}_n(t)\| \leq 2m + k. \tag{3.11}$$

This shows that $(\dot{u}_n(\,\cdot\,))$ is uniformly bounded by $(2m+k)$. So $(u_n(\,\cdot\,))$ is a bounded sequence of $\mathbf{C}_E(I)$ since for every $t \in I$

$$\|u_n(t)\| \leq \|u_0\| + \int_0^t \|\dot{u}_n(s)\|\,ds \leq \|u_0\| + T(2m+k) := M. \tag{3.12}$$

Now, we will show that $(u_n(\,\cdot\,))$ is relatively compact. Obviously, $(u_n(\,\cdot\,))$ is equicontinuous. Let us prove that for every fixed t, the sequence $(u_n(t))_{n\geq n_0}$is relatively compact. For each i and for $t \in I_{n,i}$, using the relation (3.8) and the fact that $u_{n,i} \in K(t_{n,i})$, we have

$$\begin{aligned} d(u_n(t), K(t)) &\leq d(u_n(t), K(t_{n,i})) + \mathcal{H}(K(t_{n,i}), K(t)) \\ &\leq \|u_n(t) - u_{n,i}\| + k|t - t_{n,i}| \\ &= \bigg\|u_{n,i} + \bigg(\frac{t}{h} - i\bigg)(u_{n,i+1} - u_{n,i}) - u_{n,i}\bigg\| + k|t - t_{n,i}| \end{aligned}$$

$$\begin{aligned}
&\le \left\| \left(\frac{t_{n,i+1}}{h} - i \right) (u_{n,i+1} - u_{n,i}) \right\| + k|t_{n,i+1} - t_{n,i}| \\
&= \|u_{n,i+1} - u_{n,i}\| + kh \\
&= \|u_{n,i+1} - u_{n,i} + hz_{n,i} - hz_{n,i}\| + kh \\
&= \|h\Delta_n(t) - hz_{n,i}\| + kh \le h\|\Delta_n(t)\| + hm + kh \\
&\le h(m+k) + hm + kh = 2h(m+k) = 2\frac{T}{n}(m+k).
\end{aligned}$$

Then, for each $n \ge n_0$ there is $y_n(t) \in K(t)$ such that

$$d(u_n(t), K(t)) \le \|y_n(t) - u_n(t)\| < 2\frac{T}{n}(m+k) + \frac{T}{n}(m+k) = 3\frac{T}{n}(m+k).$$

Set $e_n(t) = u_n(t) - y_n(t)$, and observe that

$$\|e_n(t)\| < 3\,\frac{T}{n}(m+k) \quad \text{and} \quad u_n(t) - e_n(t) \in K(t).$$

Using this last inequality and the relation (3.12) we get for each $n \ge n_0$,

$$\|u_n(t) - e_n(t)\| \le M + 3\,\frac{T}{n}(m+k) \le M + 3\,\frac{T}{n_0}(m+k) := M',$$

that is, $(u_n(t) - e_n(t)) \in K(t) \cap \overline{\mathbf{B}}(0, M')$, or equivalently

$$u_n(t) \in K(t) \cap \overline{\mathbf{B}}(0, M') + (\{0\} \cup \{e_k(t)/k \ge n_0\}) := \widetilde{K}(t). \tag{3.13}$$

Remark that the set $\widetilde{K}(t)$ is compact since $K(t)$ is ball-compact and $\lim_{n\to\infty} e_n(t) = 0$. Consequently $(u_n(t))_{n\ge n_0}$ is relatively compact. By Ascoli–Arzelà's Theorem, the sequence $(u_n(\,\cdot\,))_{n\ge n_0}$ is relatively compact in $\mathbf{C}_E(I)$, by extracting a subsequence still denoted $(u_n(\,\cdot\,))$ we may suppose the uniform convergence of $(u_n(\,\cdot\,))$ to some mapping $u(\,\cdot\,) \in \mathbf{C}_E(I)$. Obviously $u(0) = u_0$, $u(\,\cdot\,)$ is a Lipschitz mapping and for all $t \in I$

$$u(t) \in K(t) \tag{3.14}$$

since $K(t)$ is closed.

Now, we will to prove the convergence of $(x_n(\,\cdot\,))$ in $\mathbf{C}_E(I)$. For all $t, s \in I$

$$\begin{aligned}
\|x_n(t) - x_n(s)\| &\le \left\| x_0 + \int_0^t u_n(\tau) d\tau - x_0 - \int_0^t u_n(\tau)\, d\tau \right\| \\
&\le \int_0^t \|u_n(\tau)\|\, d\tau \le M|t-s|.
\end{aligned}$$

That is, $(x_n(\,\cdot\,))$ is equicontinuous. Furthermore, for all $t \in I$,

$$\|x_n(t)\| = \left\| x_0 + \int_0^t u_n(s)\, ds \right\| \le \|x_0\| + \int_0^t \|u_n(s)\|\, ds \le \|x_0\| + MT.$$

On the other hand, as

$$x_n(t) = x_0 + \int_0^t u_n(s)\, ds,$$

by the relation (3.13), we get

$$x_n(t) \in x_0 + \int_0^t \overline{\mathrm{co}}(\widetilde{K}(s))\,ds := \Gamma(t),$$

which is a compact set since for all $t \in I, \overline{\mathrm{co}}(\widetilde{K}(t))$ is a convex compact set (see [12] for more details). Therefore, $(x_n(\,\cdot\,))_{n\geq n_0}$ is relatively compact. Then we can apply Ascoli–Arzelà's Theorem to conclude the existence of a subsequence, still denoted $(x_n(\,\cdot\,))$, which converges uniformly on I to some mapping $x(\,\cdot\,) \in \mathbf{C}_E(I)$. Obviously $x(0) = x_0$ and $x(\,\cdot\,)$ is a Lipschitz mapping with ratio M.

Observe that for all $t \in I$,

$$x(t) = \lim_{n\to\infty} x_n(t) = x_0 + \int_0^t \lim_{n\to\infty} u_n(s)\,ds = x_0 + \int_0^t u(s)\,ds \tag{3.15}$$

using Lebesgue's theorem since $(u_n(\,\cdot\,))$ is equibounded (relation (3.12)), hence, $\dot{x}(\,\cdot\,) = u(\,\cdot\,)$ almost everywhere.

Finally, let us prove the convergence of the sequences $(z_n(\,\cdot\,))$ and $(\dot{u}_n(\,\cdot\,))$.

First, set for every $t \in I_{n,i}$, $\theta_n(t) = t_{n,i+1}$, $\delta_n(t) = t_{n,i}$, and observe that

$$\lim_{n\to\infty} |\delta_n(t) - t| = \lim_{n\to\infty} (t - t_{n,i}) \leq \lim_{n\to\infty} (t_{n,i+1} - t_{n,i}) = \lim_{n\to\infty} \frac{T}{n} = 0,$$

that is, $\lim_{n\to\infty} \delta_n(t) = t$. By the same calculus we have $\lim_{n\to\infty} \theta_n(t) = t$. Then, for all $t \in I$,

$$\begin{aligned}\lim_{n\to\infty} \|x_n(\delta_n(t)) - x(t)\| &\leq \lim_{n\to\infty} (\|x_n(\delta_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\|)\\ &\leq \lim_{n\to\infty} (M|\delta_n(t) - t| + \|x_n(t) - x(t)\|) = 0.\end{aligned}$$

This shows that $\lim_{n\to\infty} \|x_n(\delta_n(t)) - x(t)\| = 0$.

Similarly, we have, for all $t \in I$,

$$\begin{aligned}\lim_{n\to\infty} \|u_n(\delta_n(t)) - u(t)\| &\leq \lim_{n\to\infty} (\|u_n(\delta_n(t)) - u_n(t)\| + \|u_n(t) - u(t)\|)\\ &\leq \lim_{n\to\infty} ((2m+k)|\delta_n(t) - t| + \|u_n(t) - u(t)\|),\end{aligned}$$

that is, $\lim_{n\to\infty} \|u_n(\delta_n(t)) - u(t)\| = 0$.

The convergence of the sequences $(x_n(\theta_n(\,\cdot\,)))_n$ and $(u_n(\theta_n(\,\cdot\,)))_n$ to $x(\,\cdot\,)$ and $u(\,\cdot\,)$ respectively, is also obtained.

Now, by relation (3.5) and the construction of u_n, x_n and z_n, we have for all $t \in I$,

$$z_n(t) \in F(\delta_n(t), x_n(\delta_n(t)), u_n(\delta_n(t))). \tag{3.16}$$

Since F satisfies the relation (3.1), we deduce that $(z_n(\,\cdot\,))_n$ is a bounded sequence in $\mathbf{L}^\infty_E(I)$, then we can extract a subsequence, still denoted $(z_n(\,\cdot\,))$ converging $\sigma(\mathbf{L}^\infty_E, \mathbf{L}^1_{E'})$ to some mapping $z(\,\cdot\,)$ in $\mathbf{L}^\infty_E(I) = (\mathbf{L}^1_{E'}(I))'$ since E is reflexive, i.e. for all $\zeta(\cdot) \in \mathbf{L}^1_{E'}(I)$, we have

$$\lim_{n\to\infty} \langle z_n(\cdot), \zeta(\cdot)\rangle = \langle z(\cdot), \zeta(\cdot)\rangle. \tag{3.17}$$

As $\mathbf{L}^\infty_{E'}(I) \subset \mathbf{L}^1_{E'}(I)$, by the relation (3.17) we deduce that for all $\zeta(\cdot) \in \mathbf{L}^\infty_{E'}(I)$,

$$\lim_{n\to\infty} \langle z_n(\cdot), \zeta(\cdot)\rangle = \langle z(\cdot), \zeta(\cdot)\rangle,$$

that is, $(z_n(\cdot))$ converges $\sigma(\mathbf{L}^1_E, \mathbf{L}^\infty_{E'})$ to $z(\cdot)$ in $\mathbf{L}^1_E(I)$, so Mazur's Lemma ensures that for almost every $t \in I$, there exists a sequence $(\xi_n(\cdot))$ (where $\xi_n(\cdot)$ is a convex combination of $\{z_k(\cdot),\ k \geq n\}$) which converges to $z(\cdot)$ in $\mathbf{L}^1_E(I)$. We can extract from the sequence $(\xi_n(\cdot))$ a subsequence which converges almost everywhere to $z(\cdot)$. Then,

$$z(t) \in \overline{\{\xi_n(t), \quad n \in \mathbb{N}\}} = \bigcap_{n\in\mathbb{N}} \overline{\{\xi_n(t)\}}, \quad \text{a.e. } t \in I,$$

and so,

$$z(t) \in \bigcap_{n\in\mathbb{N}} \overline{\text{co}}\{z_k(t),\ k \geq n\}, \quad \text{a.e. } t \in I.$$

Set $A_n = \{z_k(t),\ k \geq n\}$. Then, by Theorem 2.1, we obtain for all $x' \in E'$,

$$\begin{aligned} \langle x', z(t)\rangle &\leq \delta^*(x', A_n) && \text{for all } n \in \mathbb{N} \\ &= \sup_{k\geq n} \langle x', z_k(t)\rangle && \text{for all } n \in \mathbb{N}, \end{aligned}$$

that is,

$$\langle x', z(t)\rangle \leq \inf_{n\in\mathbb{N}} \sup_{k\geq n} \langle x', z_k(t)\rangle = \limsup_{n\to\infty} \langle x', z_n(t)\rangle.$$

From the relation (3.16), we get

$$\langle x', z(t)\rangle \leq \limsup_{n\to\infty} \delta^*(x', F(\delta_n(t), x_n(\delta_n(t)), u_n(\delta_n(t)))).$$

We define for all $x' \in E'$ the mapping

$$h_{x'}\colon [0,T] \times E \times E \to \mathbb{R}, \qquad (t,x,u) \mapsto h_{x'}(t,x,u) = \delta^*(x', F(t,x,u))$$

which is upper semicontinuous since F is upper semicontinuous, and hence

$$\limsup_{(t,x,u)\to(t_0,x_0,u_0)} h_{x'}(t,x,u) \leq h_{x'}(t_0,x_0,u_0).$$

Hence, since $\lim_{n\to\infty} \delta_n(t) = t$, $\lim_{n\to\infty} x_n(\delta_n(t)) = x(t)$ and $\lim_{n\to\infty} u_n(\delta_n(t)) = u(t)$, we conclude that,

$$\limsup_{n\to\infty} \delta^*(x', F(\delta_n(t), x_n(\delta_n(t)), u_n(\delta_n(t)))) \leq \delta^*(x', F(t, x(t), u(t)))$$

so,

$$\langle x', z(t)\rangle \leq \delta^*(x', F(t, x(t), u(t))), \quad \text{for all } x' \in E',$$

and then,

$$\sup_{x'\in E'} (\langle x', z(t)\rangle - \delta^*(x', F(t, x(t), u(t)))) \leq 0.$$

Since F has closed convex values, by the relation (2.1), we get

$$d(z(t), F(t, x(t), u(t))) \leq 0.$$

This shows that

$$z(t) \in F(t, x(t), u(t)), \quad \text{a.e. } t \in I. \tag{3.18}$$

On the other hand, we see by the relation (3.11), that $(\dot{u}_n(\,\cdot\,))_n$ is bounded in $\mathbf{L}^\infty_E(I)$, up to a subsequence, we may suppose that $(\dot{u}_n(\,\cdot\,))_n$ weakly* converges in $\mathbf{L}^\infty_E(I)$ to some mapping $w(\,\cdot\,)$ and that $w(\,\cdot\,) = \dot{u}(\,\cdot\,)$. Indeed, for all $y \in \mathbf{L}^1_{E'}(I)$,

$$\lim_{n\to\infty} \langle \dot{u}_n(\,\cdot\,), y(\,\cdot\,)\rangle = \langle w(\,\cdot\,), y(\,\cdot\,)\rangle,$$

i.e.

$$\lim_{n\to\infty} \int_0^t \langle \dot{u}_n(s), y(s)\rangle\, ds = \int_0^t \langle w(s), y(s)\rangle\, ds,$$

in particular for $y(\,\cdot\,) = \mathbf{1}_{[0,t]}(\,\cdot\,)e_j$, with $t \in I$, $\mathbf{1}_{[0,t]}$ the characteristic function of the interval $[0, t]$, and (e_j) a sequence of the space E' which separates the points of E (such a sequence exists since E is separable), then we obtain

$$\Big\langle \lim_{n\to\infty} \int_0^t \dot{u}_n(s)\, ds, e_j \Big\rangle = \Big\langle \int_0^t w(s) ds, e_j \Big\rangle, \quad \text{for all } j,$$

which ensures,

$$\lim_{n\to\infty} \int_0^t \dot{u}_n(s)\, ds = \int_0^t w(s)\, ds.$$

As $(u_n(\,\cdot\,))$ is a sequence of absolutely continuous mappings, we have the following equality

$$\lim_{n\to\infty} (u_n(t) - u_n(0)) = \lim_{n\to\infty} \int_0^t \dot{u}_n(s)\, ds = \int_0^t w(s)\, ds,$$

then

$$u(t) = u(0) + \int_0^t w(s)\, ds,$$

so $u(\,\cdot\,)$ is absolutely continuous, and hence $w(\,\cdot\,) = \dot{u}(\,\cdot\,)$.

Observe again, that for all $t \in I$

$$\mathcal{H}(K_n(t), K(t)) = \mathcal{H}(K(\theta_n(t)), K(t)) \leq k|\theta_n(t) - t| \to 0. \tag{3.19}$$

Let us prove now that for almost every $t \in I$

$$-\dot{u}(t) - z(t) \in \Gamma^{r/(m+k)}(K(t), u(t)),$$

or equivalently

$$u(t) \in P_{K(t)}(u(t) - \frac{r}{(m+k)}(\dot{u}(t) + z(t))).$$

Set $r' = r/(m + k)$. We have $\Delta_n(t) = \dot{u}_n(t) + z_n(t)$ and by the arguments given above we know that $(\Delta_n(\,\cdot\,))_n$ weakly*-converges in $\mathbf{L}^\infty_E(I)$ to $\dot{u}(\,\cdot\,) + z(\,\cdot\,) := \Delta(\,\cdot\,)$. Supplying the property "I-smoothly weakly compact" supposed on the space E to the sequence $(r'\Delta_n(\,\cdot\,))_n$ we obtain for all $y \in \mathbf{L}^\infty_E(I)$ and all $\phi \in \mathbf{L}^1_{\mathbb{R}}(I)$,

$$\lim_{n\to\infty}\int_I \langle J_p(y(t)-r'\Delta_n(t)) - J_p(-r'\Delta_n(t)), \Delta_n(t)\rangle \phi(t)\,dt \\ = \int_I \langle J_p(y(t)-r'\Delta(t)) - J_p(-r'\Delta(t)), \Delta(t)\rangle \phi(t)\,dt.$$

By (3.10) we know that for almost every $t \in I$

$$u_n(\theta_n(t)) \in P_{K_n(t)}(u_n(\theta_n(t)) - r'\Delta_n(t)),$$

and since the sequence $(u_n(\theta_n(\,\cdot\,)))_n$ strongly converges in $\mathbf{L}^\infty_E(I)$ to $u(\,\cdot\,)$, using the relation (3.19), we conclude by Proposition 2.16, that for almost every $t \in I$

$$u(t) \in P_{K(t)}(u(t) - r'\Delta(t)),$$

that is, $-\Delta(t) \in N_{K(t)}(u(t))$ (see Definition 2.5 and Remark 2.6), or equivalently,

$$-\dot{u}(t) - z(t) \in N_{K(t)}(u(t)), \quad \text{a.e. } t \in I,$$

and, by (3.18), we get

$$-\dot{u}(t) \in N_{K(t)}(u(t)) + F(t, x(t), u(t)), \quad \text{a.e. } t \in I.$$

Finally, by the relation (3.14) and (3.15) we conclude that

$$\begin{cases} -\ddot{x}(t) \in N_{K(t)}(\dot{x}(t)) + F(t, x(t), \dot{x}(t)), & \text{a.e. } t \in I, \\ \dot{x}(t) \in K(t), & \text{for all } t \in I, \\ x(0) = x_0; \quad \dot{x}(0) = u_0, \end{cases}$$

that is, our problem $(\mathcal{P}_F)$ has at least a Lipschitz solution $x \in \mathbf{C}^1_E(I)$. Furthermore,

$$\|\ddot{x}(t)\| \le 2m + k, \quad \text{a.e. } t \in I.$$

The proof of our theorem is then complete. □

Remark that in Theorem 3.1, the set-valued mapping K depends on the time. In the following, we extend this result to the case where K depends on the state variable x, that is, our aim is to give an existence result for the following differential inclusion

$$(\mathcal{P}'_F) \qquad \begin{cases} -\ddot{x}(t) \in N_{K(x(t))}(\dot{x}(t)) + F(t, x(t), \dot{x}(t)) & \text{a.e. } t \in I, \\ \dot{x}(t) \in K(x(t)) & \text{for all } t \in I, \\ x(0) = x_0, \quad \dot{x}(0) = u_0. \end{cases}$$

Theorem 3.2. *Let $I = [0,T]$ $(T > 0)$ and E be a separable, reflexive, uniformly smooth Banach space, which is I-smoothly weakly compact for an exponent*

$p \in [2, \infty)$. *Let* $F\colon I \times E \times E \rightrightarrows E$ *be an upper semicontinuous set-valued mapping with nonempty closed convex values. We assume that there exists a constant* $m > 0$ *such that*

$$F(t,x,u) \subset m\overline{\mathbf{B}}_E, \quad \text{for all } (t,x,u) \in I \times E \times E. \tag{3.20}$$

Let $r > 0$ *and* $K\colon E \rightrightarrows E$ *be a set-valued mapping taking nonempty and* r*-prox-regular values. We assume that* $K(\,\cdot\,)$ *satisfies the following assumptions*:

(a) $K(\,\cdot\,)$ *moves in a Lipschitz way, that is, there exists a constant* $k > 0$ *such that for all* $y, z \in E$,

$$\mathcal{H}(K(y), K(z)) \le k\|y - z\|;$$

(b) *there is a constant* $l > 0$ *and a ball-compact set* $L \subset E$ *such that*

$$K(y) \subset L \subset l\overline{\mathbf{B}}_E, \quad \text{for all } y \in E.$$

Then for all $x_0 \in E$ *and* $u_0 \in K(x_0)$, *the differential inclusion* $(\mathcal{P}'_F)$ *has at least a Lipschitz solution* $x(\,\cdot\,) \in \mathbf{C}^1_E(I)$.

PROOF (sketch). The proof is essentially the same as for Theorem 3.1. Fix $n_0 \in \mathbb{N}^*$ such that

$$\frac{T}{n_0}(m + 3kl) \le \frac{r}{2}. \tag{3.21}$$

The first step consists in defining the approximating mappings as in (3.4) and (3.5) by setting for every $n \ge n_0$ and for each $t \in I_{n,i}$

$$\begin{cases} u_n(t) = u_{n,i} + \left(\dfrac{t}{h} - i\right)(u_{n,i+1} - u_{n,i}), \\ x_n(t) = x_0 + \displaystyle\int_0^t u_n(s)\,ds, \\ x_{n,i} = x_n(t_{n,i}), \\ z_{n,i} \in F(t_{n,i}, x_{n,i}, u_{n,i}), \\ K_n(t) = K(x_{n,i}), \end{cases} \tag{3.22}$$

where $u_{n,0} = u_0$; $x_{n,0} = x_0$ and for all $0 \le i \le n-1$ the point $u_{n,i+1}$ is given by

$$u_{n,i+1} = P_{K(x_{n,i})}(u_{n,i} - hz_{n,i}). \tag{3.23}$$

First, observe that this last relation implies that $u_{n,i+1} \in K(x_{n,i})$ and then by using the hypothesis (b), we obtain for all $0 \le i \le n$

$$\begin{aligned} \|x_{n,i} - x_{n,i-1}\| &= \left\| \int_0^{t_{n,i}} u_n(s)\,ds - \int_0^{t_{n,i-1}} u_n(s)\,ds \right\| \le \int_{t_{n,i-1}}^{t_{n,i}} \|u_n(s)\|\,ds \\ &= \int_{t_{n,i-1}}^{t_{n,i}} \left\|u_{n,i-1} + \left(\frac{s}{h} - (i-1)\right)(u_{n,i} - u_{n,i-1})\right\|\,ds \end{aligned}$$

$$\leq \int_{t_{n,i-1}}^{t_{n,i}} \bigg(\|u_{n,i-1}\| + \bigg|\frac{s}{h} - (i-1)\bigg|(\|u_{n,i}\| + \|u_{n,i-1}\|)\bigg)\,ds$$
$$\leq \int_{t_{n,i-1}}^{t_{n,i}} \bigg(l + \bigg(\frac{t_{n,i}}{h} - (i-1)\bigg)(2l)\bigg)\,ds = 3l|t_{n,i} - t_{n,i-1}| = 3lh.$$

Let us prove now that the relation (3.23) is well defined. Indeed, since $u_0 \in K(x_0)$, we have

$$d(u_{n,0} - hz_{n,0}, K(x_{n,0})) \leq \|u_{n,0} - hz_{n,0} - u_{n,0}\| \leq hm \leq \frac{r}{2} < r$$

using the relation (3.21). Then we set $u_{n,1} = P_{K(x_{n,0})}(u_{n,0} - hz_{n,0})$ which implies that $u_{n,1} \in K(x_{n,0})$. Consequently, using (3.21) a second time, we can write

$$\begin{aligned} d(u_{n,1} - hz_{n,1}, K(x_{n,1})) &\leq d(u_{n,1} - hz_{n,1}, K(x_{n,0})) + \mathcal{H}(K(x_{n,1}), K(x_{n,0})) \\ &\leq \|u_{n,1} - hz_{n,1} - u_{n,1}\| + k\|x_{n,1} - x_{n,0}\| \\ &\leq hm + k3lh \leq \frac{T}{n_0}(m + 3kl) \leq \frac{r}{2} < r. \end{aligned}$$

Then, we set $u_{n,2} = P_{K(x_{n,1})}(u_{n,1} - hz_{n,1})$. Similarly, we can define, by induction, all the points $(u_{n,i})$, $0 \leq i \leq n$.

The second step of the proof still holds with the following estimate

$$\|\Delta_n(t)\| \leq m + 3kl, \quad \text{a.e. } t \in I_{n,i}$$

and

$$-\Delta_n(t) \in \Gamma^{r/(m+3kl)}(K_n(t), u_{n,i+1})$$

or equivalently,

$$u_{n,i+1} \in P_{K_n(t)}\bigg(u_{n,i+1} - \frac{r}{(m+3kl)}\Delta_n(t)\bigg).$$

In step 3, for proving that for every fixed t the sequence $(u_n(t))_{n \geq n_0}$ is relatively compact, we consider for each i and all $t \in I_{n,i}$

$$\begin{aligned} d(u_n(t), K_n(t)) &= d(u_n(t), K(x_{n,i})) \\ &\leq d(u_n(t), K(x_{n,i-1})) + \mathcal{H}(K(x_{n,i}), K(x_{n,i-1})) \\ &\leq \|u_n(t) - u_{n,i}\| + k\|x_{n,i} - x_{n,i-1}\| \\ &= \bigg\|u_{n,i} + \bigg(\frac{t}{h} - i\bigg)(u_{n,i+1} - u_{n,i}) - u_{n,i}\bigg\| + k\|x_{n,i} - x_{n,i-1}\| \\ &\leq \|u_{n,i+1} - u_{n,i}\| + 3khl = h\|\Delta_n(t) - z_{n,i}\| + 3khl \\ &\leq h\|\Delta_n(t)\| + hm + 3khl \leq 2\frac{T}{n}(m + 3kl) \leq 2\frac{T}{n_0}(m + 3kl), \end{aligned}$$

and since $K_n(t) \subset L$, we conclude that, for each $n \geq n_0$,

$$d(u_n(t), L) \leq 2\,\frac{T}{n_0}(m + 3kl).$$

Then $(u_n(t))_{n\geq n_0}$ is relatively compact since L is ball-compact. We complete the proof of our theorem as for Theorem 3.1. □

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