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Rudolf Taschner
The Continuum
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REVIEW

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Rudolf Taschner's idiosyncratic book is simultaneously charming and annoying.

Entitled *The Continuum*, it is an introduction to constructive analysis. The topics are the real numbers (a definition thereof and their basic properties), metric spaces (with such properties as completeness, compactness, locatedness, and separability), and continuous functions (including pointwise and uniform continuity, and sequences of such functions).

The next fact a potential reader would likely want to know is who the intended audience is. The preface indicates that these are students, advanced undergraduate and graduate. There are two problems with this, though. For one, there are no exercises. Even worse, there is virtually no discussion of constructivism. That is, in the introductory chapter there are just over two pages on this topic (p. 13-15). There the essence of the matter is described as follows: "The salient point of "Brouwer's new intuitionism" can be concentrated into one sentence: He replaced Dedekind cuts, i.e. vertical *lines* in the Farey table [rationals], by vertical *stripes*" (p. 13). And that, we are told, is Luitzen Egbertus Jan Brouwer's intuitionism. Excluded middle is not mentioned even once, nor are its consequences that constructivists routinely consider, such as Markov's Principle (If a decidable property of the natural numbers is not always false then it's somewhere true.). There is no discussion about the meaning of existence proofs. The limited principle of omniscience (LPO: Every binary sequence is all 0s or has a 1.) is the single exception, being mentioned once (p. 70), in the only Brouwerian counter-example in the text (If all subsets of

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a metric space are located, then LPO holds.), and even then LPO is not referred to by name. A student could learn constructivism from this book neither by explicit instruction nor by seeing it in action. (Of course, if the constructivism is supposed to be secondary, then many of the proofs and notions here will seem unnecessarily complicated.)

Moving on to the substance of the book, that is where the charm begins. The first topic is the Farey fractions, which is a way of generating the rationals by how a third grader would like to add fractions: the Farey mediant of $p\backslash n$ and $q\backslash m$ is $p+q\backslash n+m$. This construction is then found in the pentagram, which is connected with the Fibaonacci numbers, continued fractions, and certain square roots, culminating in Dedekind cuts. This mathematics is quite interesting, rich in content, easily accessible, and not so well known, and so, while not any of the mainstream approaches to the foundations of analysis, it is well worth one's time.

A similar *fin de siecle* elegance is also provided by an air of old-fashioned, Old World stateliness. This starts with the title, which is a self-conscious reference to Hermann Weyl's "Das Kontinuum" from 1918, almost a century ago. The photographs of Weyl and Brouwer at the beginning (p. $vi - vii$), in which even Hermann Hesse makes an appearance, put one in a historical frame of mind. The brief history of the subject seems to end with Weyl in 1927. Two sentences are dedicated to the later work of Errett Bishop, Douglas Bridges, and Bishop's "American school of constructivism;" this more recent tradition is then dismissed with an approving citation of an imaginary mathematician's rejection of "hard labor and fewer results," so that the author can return to Brouwer's intuitionism (p. 15). And so we remain in interregnum Europe.

A different kind of charm is afforded by the introduction to Dedekind cuts. While referring to Dedekind's published work on the subject, mention is made of the publisher of one of his booklets in the body of the text (p. 9), in contrast to normal modes of attribution and unique even in this book. The reason for this unusual decision is, presumably, that the publisher so referred to just so happens to be the publisher of the current work. This is of course completely unprofessional, but rather than being unpleasant, comes across as charming in an amateurish kind of way. Also contributing to this sense of amateurness are the diagrams, two of which have appeared already by this point (p. 4 & 7). The quality of their printing is really lousy. Better output can be produced with TeX, for instance. Other inexplicable layout decisions similarly distract when reading. For instance, proofs are introduced with the boldface word **Proof.** The corresponding action is not taken, though, when introducing the statements of theorems. Sure, named theorems do have their names in bold beforehand, and the statements of all theorems are italicized. But what happens when coming across an italicized assertion unflagged as a theorem or a lemma or something is that one immediately wonders why this particular statement is being emphasized, and then does a double-take when one realizes what's going on. Ignoring this nearly universal and quite sensible standard is jarringly amateurish.

The same sense that the author doesn't quite know how it's done comes across in the use of language. By way of praise, most of the time the language is idiomatic and does not distract from the content; that the author and both of his language assistants seem to be non-native English speakers, that's quite an accomplishment. Every once in a while, though, there is something strange that you just can't help stumbling over. The identity function is called "the identical function" (p. 96) (and more than once – see p. 100 and the index too – so it's not a typo). In the section entitled "topological concepts" (p. 78), the cover of a set (of points of a metric space) is defined to be what's usually called the closure of a set, and has nothing to do with open covers. He sometimes talks about detaching a real number (p. 81), and I still don't know what that means.

Often mathematicians abuse language and notation to make it easier on the reader, and not uncommonly one finds a phrase or symbol introduced with the apology "by abuse of notation." In this case, the author chose to give this apology in French – *"par abus de langage"* – even though it can be perfectly well expressed in English, as is commonly done. I assume this was done that way by reference, even if unintended, to Bourbaki, who first introduced the phrase, naturally enough in the language in which they were writing. Doing so fits in well with the historical atmosphere of the book, even if it anachronistically puts us in the postwar period. But the intended reader, a beginning English-speaking student, will know neither French nor the phrase in English to enable them to figure the translation out, leaving them to guess it has something to do with languishing on a bus. Furthermore, instead of indicating that the phrase is foreign with italics, as is typical, it is enclosed in quotes, and sometimes set off with dashes too for good measure, as in: "The assignment that appoints to each argument x one and the same value y_0 is of course a function that can be defined on every set. It is called a *constant* function which – "par abus de langage" – can be identified with its only value y_0" (p. 96).

Occasionally the phrasing is very cumbersome. Consider for instance the statement of the following theorem (p. 99):

Theorem 1. *Suppose that the function $f : U \to T$ and the point $\xi \in U$ fulfill the following condition: For each real number $\epsilon > 0$ it is possible to construct a real number $\delta > 0$ such that for all $x \in U$ the inequality $\|x - \xi\| < \delta$ implies the inequality $\|f(x) - f(\xi)\| < \epsilon$. Then f is continuous at the point ξ.*

While this looks to be the exact definition of continuity at a point, and so seems to be a definition and not a theorem, the content of this statement as a theorem is actually in order, as the definition of continuity in the text is non-standard. (Continuity was defined on the cover – that is, closure – of a set, so that a function f can be continuous at a point where it's not even defined, apparently so that f can be extended to these points and – "par abus de langage" – the same name used for the extended function. Why this benefit outweighs the disadvantage of using a notion of continuity unique in the literature was not explained.) My purpose in repeating this statement is to contrast it to the much lighter:

Theorem 2. *For $f : U \to T$ and $\xi \in U$, suppose that for all $\epsilon > 0$ there is a $\delta > 0$ such that $\|x - \xi\| < \delta$ implies $\|f(x) - f(\xi)\| < \epsilon$. Then f is continuous at ξ.*

The only thing the extra verbiage adds to the double-length original version is weight.

Returning finally to the substance of the book, after the introduction the real numbers are defined. They are defined, essentially, as numbers that are given by our standard, base ten, decimal representation. No explanation for this choice is given. As a mathematician, I find binary, or signed binary (whereby -1 is also allowed as a digit – much more sensible constructively), or arbitrary Cauchy sequences of rationals, natural, and base ten not. A reason for this unnatural choice should have been given. As for the rest of the chapter, it is quite in order that the mathematics is not advanced, given the intended audience. The downside is that the reader is presented with page after page of tedious verification of statements that look like trivialities, and little else. The theorems are the basic properties of $>$ and $\geq$ (transitivity, irreflexivity in the former case, and such like), absolute value, the triangle inequality (in three versions no less), equality and apartness, convergence, and so on. Perhaps the most frustrating part of this is that the differences with the classical theory, just the thing that might intrigue a reader, are nowhere brought out.

I could go into comparable detail about the remaining two chapters, but the upshot will be the same. With the essence of the constructivism hidden, the text reads like unnecessarily difficult proofs of things you would find in any standard, classical text, often just left to the reader there, except for those notions, inherently constructive, that, in this context, just make no sense at all.

If one wanted an introduction to constructive analysis, there are any of a number of other texts that stand up well against the current one (some of which are listed in the bibliograhy below). Perhaps the most apt comparison can be made with *Techniques of Constructive Analysis* by Bridges and Luminita Vita (sans diacritical marks), being dedicated solely to the same subject and having appeared just about a year after the text under review. Authored by arguably the current leading constructive analyst and a student of his, it does in 47 pages what *The Continuum* does in 128, plus a lot more: exercises, a section on constructive logic, a fuller history, more notions defined and principles identified. To say nothing of the remaining 153 pages. Admittedly parts of this book would be rough going for the audience in question of introductory students. But I'd rather have my students struggle with advanced, inherently difficult material than with tedium.

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