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REVIEW

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This collection of essays, as its title indicates, aims to increase our understanding of modern mathematics and, more generally, "to advance contemporary work in creating stronger links between the history and philosophy of mathematics" (p. 1). Modestly construed this is a commendable aim - the evolution of modern mathematics post-1850 is a complicated business, and all too often historians operate with tacit philosophical presuppositions and philosophers operate with questionable (or no) histories - and dialogue should have the desirable effect of removing such impediments to understanding. However, not all the essays in the collection equally advance our understanding of either modern mathematics or the history or philosophy of mathematics; some clearly do; others do not. Moreover, the editors seem to have a more ambitious understanding of their aims, though it is not entirely clear either what that understanding amounts to or the extent to which the collection accomplishes what it sets out to do. The book contains twelve essays (divided into three topical groups of four) flanked by an introduction by the editors and a coda by one of them (Gray). In what follows I will briefly describe the essays and present some commentary that will clarify these opening remarks.

Most—all but six pages summarizing the essays—of the forty-five page introductory essay is concerned primarily to situate the collection and motivate the importance of combining historical with philosophical studies of mathematics. Already in the introduction we find signs of tension in aims. On the one hand, when discussing the state of philosophy of mathematics, the editors rightly complain that twentieth century philosophy of mathematics in the Anglo-American tradition (with a few notable exceptions like Lakatos and Kitcher) has been pursued from an ahistorical systematic perspective, that historico-philosophical approaches in the German tradition expired during the Nazi period, and that philosophical approaches in the Latin tradition have groped toward historical perspective without achieving a satisfactory overall picture. This suggests that the editors favor a philosophical perspective that is both historically informed and systematic. On the other hand, however, while they praise the more recent "maverick" tradition exemplified in Kitcher's historically informed account of mathematical progress and rationality, the editors also criticize it for being too much guided by the quest for a systematic, all-inclusive answer to a grand problem. Similarly, when discussing the state of history of mathematics, the editors complain that history of mathematics *circa* 1970 was dominated by simplistic, overly general textbook treatments that tended to focus on famous names and results, to rely uncritically on earlier historical work, and to pay little attention to post-1800 mathematics. Nevertheless, while they praise history of mathematics written since the 1970s for its appreciation of the complexity of mathematical practice and the way in which mathematical results are generated in specific historical contexts "immersed in a human world of aims, purposes, and decisions" (p. 29), they also think it "unfortunate that there has never been, in the history of mathematics, the equivalent of a Thomas Kuhn" (p. 25). But Kuhn's approach to history of science is surely a systematic, all-inclusive answer to a grand problem!

Despite this evidence of mixed aims, however, we can discern some thematic unities (induced by commonalities within modern mathematics itself) as well as some clear purposes in various dichotomies the editors discuss. First, they distinguish mathematical philosophy from philosophy of mathematics: the former focuses on resolving philosophical questions about mathematics by using logical and mathematical methods and is exemplified in traditional foundational disputes between logicists, intuitionists, and formalists; the latter focuses on broader questions about "mathematical knowledge, its underpinnings, and its development ... including questions about mathematical practices" (p. 6). Clearly, the editors favor the latter approach. Second, they distinguish between static and dynamic conceptions of mathematics: static conceptions take mathematics either as a finished building with an eternal subject matter of fixed truths or, if the building is under construction, as a linear accumulation of floors each piling new truths atop previous levels; dynamic conceptions focus on how the building takes a complicated shape as mathematical practices unfold over time subject to constraints that themselves historically evolve. The editors again favor the latter kind of approach. They argue that the static conception informs traditional, mainstream philosophy of mathematics (especially the quest for philosophical foundations that would provide *a priori* certainty); and this is problematic because the received tradition takes mathematics to be a static product to which it applies the metaphysical and epistemological categories of general philosophy and thus subordinates the agenda of philosophy of mathematics to the main problems of general philosophy. Third, they distinguish between two conceptions of philosophy: systematic philosophy and philosophical reflection. Systematic philosophy uses a static, ahistorical snapshot of "the" building of mathematics as a testing ground for key issues in philosophy, issues that are more properly deconstructed as merely those that are conceived to be centrally important to the historically evolving practice of philosophy at the given time. The "philosophical reflection" conception, in contrast, recognizes that the history of mathematics presents many instances where mathematical practitioners wrestled with interesting problems of a conceptual and methodological nature, problems that arise within mathematical practice and that are driven by efforts to extend mathematics in novel ways and in response to specific mathematical problems. Moreover, this is especially true of modern mathematics (post-1850) where mathematicians increasingly relied on considerations of a broadly philosophical nature to justify their freedom to extend mathematics from its "intuitive" underpinnings in geometry or arithmetic into increasingly abstract structures axiomatically described. Important among these considerations are conceptual narratives that organize the mathematical subject matter in terms of what is deep and fruitful in consequences and that guide extensions and new applications of a theory in terms of how well they fit the conceptual setting.

Although the introductory essay beats about the bush quite a bit, this appears to be bird one finally rouses. The editorial guiding principle is the investigation of philosophical positions that developed largely out of considerations internal to modern mathematics itself; this is an investigation that can attract the interest of philosophers, historians, and mathematicians. One might add, though the editors don't go so far, that such an investigation promises to restore its lost moorings to philosophy of mathematics. This is a laudable guiding principle, and the best essays in the collection (those in Part I, Avigad's essay in Part II, and Sieg's in Part III) contribute significantly to philosophy of mathematics thus conceived. The remaining essays fall short of the guiding principle, and their inclusion can only be explained in terms of the lack of clarity about aims - they seem to be there (especially those in Part III) because of a yearning for a philosophical program that would tell us in all-inclusive terms what mathematics is.

The first group of essays, "Reinterpretations in the History and Philosophy of Foundations", includes papers by Beaney, Ferreirós, Tappenden, and Corry. As a group, these essays, especially the first three, stand out: they significantly promote the goal of better understanding foundational approaches to mathematics by uncovering some of the lost philosophical moorings that anchored mathematical developments between 1850 and 1920. Beaney's essay explores the role played by "elucidation" in Frege's thought. His logicist departure from Kant on the epistemological status of arithmetic and analysis required Frege to show that the truths of arithmetic are analytic (not synthetic). But Frege's understanding of this claim was itself a departure from Kant: for Frege it meant that the truth's proof depends only on general logical laws and definitions, whereas for Kant it meant that the predicate B is contained in the subject A if the truth is of form 'A *is* B'; moreover, for Frege the general laws of logic are the laws of quantificational logic that he himself had formulated, whereas for Kant they are the laws of Aristotelian logic. In order to motivate his project and not have it seem that he was merely changing the subject, Frege was therefore obliged to show that his understanding of analyticity, logic, and number drew upon, made explicit, and refined an understanding that was already part of the tradition: he was obliged to employ historical elucidation in his philosophical preambles. Beaney further uses this notion of historical elucidation to chart connections between Frege and several of his contemporaries (Dedekind, Hilbert, Russell) and argues that it indicates that philosophy of mathematics needs to be deeply intertwined with historical investigation.

Ferreirós' article explores Riemann's philosophical approach to mathematics and physics. Riemann is well known as a pioneer of the abstract conceptual orientation that modern mathematics took, freed from essential dependence on concrete and intuitive content, and exhibited in his contributions to (representation-free) function theory and intrinsic (metric-free) geometry (topology). What is less well known is the philosophical context in which Riemann worked; and this is what Ferreirs explores in his useful and interesting analysis of fragments not easily accessible in English and of Riemann's 1854 Inaugural Lecture. Under the influence of Herbart, Riemann held that all knowledge begins with experience, but we don't just "take in" objects in experience, we also take in concepts, relations, and primitive systems of relations. Moreover, we don't just take in material; active reflection enables us to form and reform precise concepts and organize and reorganize experience in theoretically fruitful ways. No concepts are immune from revision; thus apriorism is rejected. Natural science—and mathematics as an essential part of it—is just our attempt to conceive and reconceive Nature by means of precise concepts. Along the way Ferreirós lays out Riemann's research interests in experimental and theoretical physics (a unified continuum theory) and their connections with his research interests in mathematics. His idea that truly elementary laws can only occur in the infinitely small, for example, seems closely bound up with a view of foundations that differs significantly from what philosophers in the twentieth century came to think of as foundations, a view of foundations that render intelligible by means of a coherent system of precise, fruitful concepts both the very small and its relationship to global phenomena rather than a view of foundations driven by misplaced worries about evidential security and certainty.

The novel style of mathematics inspired by Riemann and centered in Göttingen was one of two styles practiced in Germany in the late nineteenth century. The other conservative style followed Weierstrass in Berlin. Tappenden's essay contrasts the two, especially in their approaches to definitional practice and complex function theory, and situates Frege, contrary to traditional wisdom, in the Riemannian camp. In analysis Weierstrass and his followers held that "there can be no dispute about the kind of thing that counts as a basic operation or concept: the basic operations are the familiar arithmetic ones like plus and times" (p. 111), whereas Riemann and his followers held that "[w]hat is to count as fundamental ... has to be *discovered*" (p. 112). Moreover, fruitfulness in applications (both mathematical and physical) was taken by Riemann and his followers as a criterion for the importance and centrality of a fundamental concept—as a criterion of discovery. Applied to complex analysis, the Weierstrassians favored the study of analytic functions defined in terms of the power series representations that render them computationally tractable; they favored a pure (= untainted by geometrical or intuitive considerations) arithmetization of both real and complex analysis. By contrast, the Riemannians favored representation-independent definitions of analytic functions (those satisfying the Cauchy-Riemann equations) that enabled existence proofs and the investigation of a function's properties (like discontinuities and boundary conditions) without attending to their analytic representations; by its very nature their methodology was "impure" and connected with geometrical representations (Riemann surfaces) and physics (potential theory). The standard interpretation of Frege's work places him in the Weierstrassian tradition: his definition of number, for example, is seen as extending Weierstrassian rigor all the way down to the natural numbers. Here and elsewhere Tappenden argues that Frege should be seen instead as a Riemannian: his work as a teacher and mathematical researcher, his criticisms of Weierstrass, his approach to definitional practice and the importance of fruitfulness, his emphasis on functions as fundamental, and his separation of objects from their representations—all point to a Riemannian influence and program. We better understand Frege's project if we understand that his logicism involves the reduction of arithmetic and Riemannian *complex* analysis to logic and his demands for rigor have a Riemannian, not a Weierstrassian, character.

Hilbert is perhaps best known in philosophical circles for his axiomatization of geometry (1899) and his formalist program to prove the consistency of infinitary theories by finitist methods (from about 1920 on). Corry's essay argues that it would be a mistake to see Hilbert as espousing a systematic formalism with respect to mathematics. In particular, although Hilbert's views about geometry changed between the 1890s and the 1920s, they were never formalist in the way his post-1920 views about arithmetic became. Throughout most of this period, Corry argues, Hilbert was attracted to, and vacillated between, images of mathematics that emphasized perceptual experience or *a priori* Kantian intuition as the source of our mathematical knowledge. His axiomatization of geometry provided a unified "network of concepts preserving meaningful connections with intuition and experience, rather than a formal game with empty symbols" (p. 142). Nevertheless, his emphasis on the logical analysis of axiomatic theories (the investigation of simplicity, completeness, independence, and consistency) detached from their anchoring in intuition or experience led to a tension in his thought: if all possible, "good", axiomatic systems have equal mathematical validity and value, how do we explain our inclination to grant a preferred status to Euclidean geometry (which Hilbert in 1905 was inclined to grant)? By 1916 his work on General Theory of Relativity convinced him that Euclidean geometry should not have preferred status and to adopt what later came to be called 'an empiricist view of geometry': pure geometries are correct if free from contradiction; the question of which geometry is physically correct is to be decided on empirical grounds; Euclidean geometry is physically correct only as a very good local approximation. According to Corry, formalism was no part of this picture. (Sieg's essay in the volume provides a useful complement.)

The second group of essays, "Explorations into the Emergence of Modern Mathematics", deals with the evolution of several features that are peculiar to modern mathematics—the investigation of structures and morphisms, the reliance on topological and model-theoretic methods, *e.g.*,—and contains papers by Avigad, McLarty, Mancosu, and Marquis. Avigad's essay traces Dedekind's progressively more abstract and set-theoretic attempts to develop the theory of ideal divisors between 1871 and 1895. Both Dedekind and Kronecker, using very different methodological approaches, aimed to extend Kummer's theory of ideal divisors for cyclotomic integers to arbitrary algebraic fields. Kronecker's general approach set a high value on algorithmically tractable representations: domain extensions consist in introducing new expressions for the new objects and rules for operating and determining equivalences on the expressions. This constructivist methodology is inevitably piecemeal (we expand our representations and calculi as the need arises) and has ontological presuppositions (the new objects are just what our notation says they are). Dedekind's general approach was Riemannian and set a high value on representation-independent definitions: domain extensions consist in the axiomatic characterization of essential structural features that both completely capture all the novel objects and their properties and preserve and smoothly carry over the central properties (like unique factorization in the case of ideal divisors) of the older domain. His methodology embraces non-constructive proof techniques (admitting pure existence proofs on infinite domains), completely general proofs (not subject to special conditions), and uniform definitions (without multiple case distinctions) that are underpinned by the intrinsic properties of the structure characterized without detours through extraneous features of particular representations of the structured elements. For Dedekind, questions about algorithms and explicit representation come after the proper development of the structure that determines conditions for the equivalence of representations. This type of approach also has ontological presuppositions: it strongly distinguishes the objects ("objects of thought" for Dedekind) from any syntactic representation of them. Avigad is careful to distinguish the respective costs and benefits of Kronecker's and Dedekind's approaches (in terms of algorithmic tractability versus theoretical generality). Because of the Riemannian influence on Dedekind, Avigad's essay provides a particularly useful accompaniment to those of Ferreirós and Tappenden in Part I. More generally, like the essays in Part I, Avigad's essay on these intertwined aspects of philosophy and mathematical methodology in Dedekind's thought uncovers some of the internal motivations for mathematical (especially set-theoretic) foundations, independently of the crisis to which the set-theoretic paradoxes were soon to lead.

Marquis's essay explores the history of homotopy theory (1930s-1960s), using it to argue (following suggestions by Hirsch and Polanyi) that there is an important distinction to be drawn between knowing-how (*technè*) and knowing-that (*episteme*) aspects of mathematical knowledge. Homotopy theory, he argues, is best understood as a *systematic mathematical technology*: its groups should be viewed as measuring instruments that provide information (classification by homotopy type) about the topological spaces studied by their means; its mapping operations (like the "lifting" operations provided by fibrations) should be viewed as tools that enable us to deploy the measuring instruments. This is a technically demanding read. While the metaphor is suggestive and interesting, its details are sketchy and programmatic (as Marquis acknowledges). Moreover, it is not clear how promising the program is, since there is some reason to question both the extent to which the metaphor will easily transfer from science to mathematics and the extent to which what Marquis calls 'the epistemology of scientific instrumentation' provides deep insights into science.

The essays by McLarty and Mancosu are straightforwardly historical investigations without pretensions to the drawing of philosophical lessons and show that good history doesn't require too much meta-philosophical baggage. McLarty traces Noether's development and extension of the axiomatic, structural approach to algebra pioneered in Dedekind's theory of ideals (and partly described, as sketched above, in Avigad's essay). Motivated by Dedekind's Riemannian search for uniform, general theories, Noether went beyond Dedekind by replacing his "arithmetic" conception of algebra with a "purely set-theoretic conception" that is "independent of any operation": "[t]hese [set-theoretic] methods do not look at addition or multiplication of the elements of a ring ... They look at selected subsets and the corresponding homomorphisms" (p. 193). McLarty describes how Noether's approach generalized Dedekind's, the central role played by homomorphisms in characterizing structures, her persuasion of Alexandroff and Vietoris to extend this structural conception from algebra to topology leading to the birth of modern algebraic topology, and how the correlations in algebraic topology between maps (between topological spaces) and morphisms (between groups) soon became the functors of category theory in the work of Eilenberg, Mac Lane, and others.

Tarski's analysis of logical consequence in 1936 appears to agree with the standard model-theoretic definition widely accepted today: sentence X is a logical consequence of class of sentences K if every model of K is a model of X. On the standard model-theoretic definition, a model is an assignment of appropriate objects from a universe of discourse to the non-logical constants of $K \cup X$, and logical consequence requires that every such assignment *from every non-empty universe* that satisfies K must also satisfy X; call this 'the variable-domain conception of model'. On Tarski's 1936 conception, however, a model is an assignment of appropriate objects to the variables of $K^* \cup X^*$ (where $K^* \cup X^*$ are the sentential functions that result from replacing all non-logical constants of K and X by variables of appropriate type), and logical consequence requires that every such assignment that satisfies K^* must also satisfy X^*. Unlike the standard model-theoretic definition of *model*, Tarski's conception doesn't explicitly specify the domain of a model (the set over which the individual variables range) which determines the evaluation of the quantifiers. Over the past two decades there is a debate among historians and philosophers of mathematical logic about what Tarski had in mind when giving his 1936 analysis. Did he intend the variable-domain conception of model? Or did he intend a fixed-domain conception, one that distinguishes a universe of discourse (the set that would be assigned to a primitive predicate, S, characterizing the intended universe of the theory) and a possibly wider fixed domain (the fixed range of the variables in all models of theory)? The difference can be illustrated as follows: on the variable domain conception, assuming that the language has a universe-characterizing primitive predicate, S, to be interpreted as satisfied by all members of (any) domain, $(\forall x)Sx$ would be true in all models; but on a fixed-domain conception (where the variables range over a domain that is wider than the universe of the theory), $(\forall x)Sx$ would be false on some models. Mancosu's essay clarifies what the fixed-domain conception is and argues that Tarski in 1936, and as late as 1940, intended a fixed domain (usually $= V$) conception of model: it was common practice at the time to distinguish the range of the quantifiers from the universe of discourse, and one can make sense of an unpublished archived 1940 lecture Tarski gave on semantic completeness and categoricity only on a fixed-domain conception.

The third group of essays (by Epple, Scholz, Benis Sinaceur, and Sieg), "Alternative Views and Programs in the Philosophy of Mathematics", is seemingly offered as a possible heterodox corrective to the logicism-intuitionism-formalism orthodoxy that dominated philosophy of mathematics in the first half of the twentieth century. While the lack of historicity in philosophy of mathematics of that period is unfortunate, with the exception of Sieg's essay the heterodoxy on offer here seems too idiosyncratic to provide much new insight into modern mathematics. Moreover, it seems counterproductive to the goal of providing an internalist perspective that the philosophical and historical excavations other essays in the volume so well promote. (I should note that this criticism bears on editorial aims and does not detract from the value of Epple's and Benis Sinaceur's articles as studies in the history of philosophy.)

Epple's essay explores the epistemological connections between the work of Hausdorff (the mathematician) and Hausdorff (the Nietzschean perspectivalist philosopher writing under the *nom de plume* 'Paul Mongré'). Epple explains how Hausdorff advocated a "considered empiricism" (a precursor of logical empiricism), whose key theses are: (i) mathematics is an autonomous creation of disciplined thinking; (ii) no other scientific knowledge can claim to be more than a plausible, economic organization of empirical phenomena. His arguments supporting (ii) anticipated the underdetermination arguments of Poincaré and Schlick for the conclusion that physics + geometry is underdetermined by evidence (since empirical phenomena can be regimented in alternative mathematical guises). Hausdorff also advocated the use of mathematics to self-critique science, or, more properly—though Epple seems not always to clearly distinguish the two—extravagant metaphysical interpretations of, and conservative epistemological attitudes toward, science. The development and investigation of new mathematical structures encourages us to see in precise terms new underdetermination possibilities and to combat two tendencies—to mistake (as some of his contemporaries did) the pragmatic success of a theory for its truth, and to mistake (as Poincaré did in the case of Euclidean geometry) the seeming simplicity and naturalness (due to our biological makeup) of a theory for its inevitability. Epple's essay shines an interesting light on the little-known philosophical reflections of Hausdorff, who was clearly both a perceptive and prescient critic of science, and the connection between Hausdorff's considered empiricism and Nietzsche is interesting (though one doubts that perspectivalism, unless clarified and qualified, is a good perspective for the investigation of science and mathematics). In this sense the essay is a nice piece of history of philosophy, but, given that so much of considered empiricism was retained in logical empiricism, it hardly provides a very novel alternative to mainstream philosophical thought about science and mathematics in the first half of the twentieth century.

Benis Sinaceur's essay provides an extensive description and analysis of French philosophy of mathematics centered on Cavaillès. Cavaillès offered what appears to be (though Benis Sinaceur doesn't explicitly say so) essentially a Hegelian critique of Kant. He rejected Kant's *a* *priori* forms (shown to be mistaken by non-Euclidean developments) in favor of an account whereby mathematical concepts progressively and autonomously develop according to their own internal dialectical logic (as a living organism grows, to the extent that it does, according to its own internal principle of development). At any given stage of historical development, the structure of concepts is determined by laws (axiom systems and transformations that define the structures) and limits (impossibility theorems), and it determines the mathematical objects (as formal nodes in the structure). That structure is determined by the inability of the preceding structure to resolve inherent problems that it could express, and due to its own inherent limitations it develops into a successor structure. This progress is governed by rational processes of idealization (domain extensions) and thematization (where operations are transformed into higher-level objects) so that form becomes content for the construction of new form. A grand synthesis that combines mathematical progress (considered *in abstracto*), historicity, rationality, and anti-foundationalism (justification of mathematical concepts is realized in the *internal* development of mathematics) is suggested: mathematics unfolds according to its own internal rationality and necessity (a necessity which is "there" even though we see it only with hindsight). As Benis Sinaceur acknowledges, the view leads to a puzzling coexistence of two distinct orders of reality—the contingent sequence ¡Archimedes, Leibniz, Riemann¿ and the "necessary" sequence ¡quadratures, definite integrals, Riemann sums¿. This is a fascinating study in history of philosophy of mathematics, but those of us who favor a more piecemeal, "underlabourer" approach to philosophy are unlikely to find in it more than that.

Weyl's philosophical influences and post-1927 reflections are the subject of Scholz's essay. Scholz sketches how Weyl's views were shaped by the transcendental idealist tradition (from Kant, through Fichte, to Husserl), by the crisis in mathematical foundations, and by his work in relativity and quantum mechanics. Rejecting metaphysical absolutes and transcendental ideals as vestiges of classical metaphysics, Weyl was led to think of mathematics as constructive (created by the symbol-producing activity of human beings) and as inextricably intertwined with physics (providing a historically relativized *a priori* framework that structures experience). Although he believed that classical views about space and time could be recovered by limiting processes from relativistic views, he thought that we are stuck with a two-world view in quantum mechanics and in constructivist mathematics (the macroscopic world and its formal description in QM, the world of constructions and marks and its symbolic description in mathematics), and he was dissatisfied with extant attempts to gloss over these distinctions. Instead he turned to existentialism (Heidegger, Jaspers) and thinkers who had dealt with the representational uses of symbols (von Humboldt, Helmholtz, Wittgenstein) in his efforts to formulate an account of meaning that would glue the two worlds back together. The result was a "practice-based symbolic realism"—symbolic representations are creatively produced like tools; the goodness of a symbolic representation like the goodness of a tool is constrained by the raw materials available for construction, by the abilities of the maker, and by the goals and values of the practice in which it is used; what we do with symbols in our cultural and material practices is what gives them meaning and connects them with the world. Scholz recommends that we use Weyl's views as a model for a humanistic, culturally informed study of mathematics in our times. I cannot make the same recommendation. Scholz's study is too sketchy to know what to make of it. It seems to confuse questions about the nature of mathematics with questions about human existence that are better kept separate—here I let Scholz's astonishing closing sentence speak for itself: "In ... our context, it may be more than useful to take up Hermann Weyl's thoughts on a symbolical realism for the mathematical sciences and to fuse them with Ivan Illich's challenge to reorient *all* our practices in accordance with conviviality" (pp. 308-9).

It is now universally accepted that Hilbert's goal of establishing the consistency of analysis and set theory within finitist mathematics was shown to be unattainable by Gödel's second incompleteness theorem. Influenced by the writings of Bernays, Sieg's essay (previously published in *Synthese* in 1990) makes a case for the possibility of salvaging something akin to Hilbert's program. Recent work in proof theory shows that a considerable portion of classical mathematics, including all of analysis, can be carried out in a small part of V that is consistent relative to the quasi-constructive principles formalized in intuitionistic number theory. Sieg argues that such quasi-constructive principles should be considered to preserve the spirit of Hilbert's program and proposes that their investigation should be a project for future philosophical reflection. The argument has philosophical and historical moves. The philosophical move consists in a characterization of "accessible" domains that are "quasi-constructive". The paradigm of an accessible domain is the natural numbers considered as ordinals: their representation by concrete numerals has the special characteristic that the relations between the numbers represented can be "read off" the relations between the representing numerals in such a way that we can directly apprehend the structure of the numbers by considering the structure of the numerals: the symbols wear their meaning on their sleeves so that we can grasp the build-up of the structure; understanding the generating procedure amounts to grasping the intrinsic build-up of the objects. The trick then is to extend the paradigm by structural analogy to higher levels (constructive ordinals and the elements of other accessible inductively defined classes, uniformly iterated accessible i.d. classes, *etc.*) and see how far it will go. According to Sieg (p. 363), it goes quite far: "[s]egments of $[V]$—that contain some ordinals (0, ω, or large cardinals) and are closed under the powerset, union, and replacement operations—are accessible in this extended sense: the uniquely determined transitive closures of their elements are construction trees" (analogous to those used by Brouwer). Moreover, if we grasp the build-up of the objects in such a segment of V by understanding its generating procedure, then the axioms of ZF^* (ZF with a suitable axiom of infinity) will be evident to us, since they just formulate the principles generating the construction. The historical move is to motivate this kind of project as being in the spirit of Hilbert's program by tracing Hilbert's concerns to the debates between Dedekind and Kronecker concerning the proper understanding of the arithmetization of analysis—and here Sieg's essay can be read as a companion to those of Avigad and Corry. Dedekind's emphasis on wholesale complete definitions of systems of numbers raised the question of consistency; the point of consistency proofs was to guarantee the existence of sets (systems of numbers) with an axiomatically characterized structure. Sieg argues that Hilbert's program is better seen as uniform structural reduction (projection of the intended structure of the reduced theory to the number theoretic domain where its consistency can be recognized from a standpoint that is more elementary than the assumption of the intended structure) than as a justification of an instrument that yields correct contentual finitist output when given similar input. If this is correct—though it seems more clearly Bernays than Hilbert—then we have a motivation for Sieg's program.

Gray's coda asks whether we should view the emergence of modern mathematics as continuous with prior mathematics or as signaling a disruptive break with the past. Clearly, modern mathematics brought about changes that "were unexpected, massive, and permanent" (p. 373) and that were related in complex ways to logic, philosophy, and physics. More generally, what kind of framework should a historian use to answer such questions given the scale of the changes and their complex relationships to other practices? Gray proposes that we employ the trope of Modernism as a framework, one that is already tried and tested in diverse cultural areas - music, painting, narrative literature, poetry, architecture. Modernism he characterizes as "an autonomous body of ideas, pursued with little outward reference, maintaining a complicated ... relationship with the day-to-day world and drawn to the formal aspects of the discipline ...introspective to the point of anxiety ...the *de facto* view of a coherent group of people ..." (p. 374). He argues that modern geometry and analysis fit this characterization well: "mathematics was no longer based on the primitive acts of counting and measuring and ...was no longer any kind of idealized, abstracted, simplified science" (p. 382). Its objects were axiomatically defined independently of science, freely created; its relationship with the world was complicated; its discourse became more turned in on itself (concerning the correct way to reason, develop new concepts, etc.) and increasingly formal. Gray further considers the relationship between philosophy and logic and this radically new mathematics (in the Leibnizean/neo-Kantian and the psychologistic/antipsychologistic debates). He concludes that modernism provides a framework that allows us to recognize both continuity and radical change and welcomes its pluralistic possibilities. I am not convinced. The answer to the continuity question could be arrived at more simply by saying that, as is usual with such questions, there was more continuity than proponents of disruptive fractures claim and less than proponents of Whiggish history claim. It is not at all clear that the modernist trope will transfer from the arts and culture in general to the far more constrained practice of mathematics. Moreover, it is not at all clear how well it works for the areas to which it has been extensively applied—saying that Joyce's *Ulysses* is a modernist novel is hardly edifying; the hard work involves excavating Joyce's relationship to Ibsen, his views about Ireland, his ideas about the classics and music, *etc.* As for pluralism, "Let a thousand flowers bloom" is highly commendable, but only if we also acknowledge the need for weeding, and modernism (either with or without the "post" prefix) hasn't forged sharp hoes.

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