John Burgess,
Fixing Frege,
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REVIEW

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This volume is divided into three main chapters: 1) *Frege, Russell, and after*; 2) *Predicative theories*; 3) *Impredicative theories.* There follows an appendix with ten *Tables* (abridged overviews of various complex topics, such as "Fregean Categories in Adjukiewicz-Bar Hillel notation", "Axiomatizations of Frege's set-theory", etc.). More than ten pages of *Notes* and a substantial bibliography complete the volume[1]
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The one fundamental issue on which the entire book hinges is the contradiction discovered by Russell (1902) in the logical basis of Frege's system for the foundations of arithmetic, and the various ways of removing the inconsistency, or "fixing Frege", that have been proposed in the past one hundred years. (I would like to point out, as something not mentioned in Burgess's volume, that not everybody has been scared by the antinomies. Asenjo, in *Logic of antinomies*, says that there are three ways of looking at antinomies: as undesirable anomalies, harmless abnormalities, useful logical entities; the first section of Asenjo's *Antinomicity*, is titled "A positive view of antinomies").

After some preliminary observations (sections 1-4) my main discussion is devoted to the role allegedly played by "abstracts" in the task of fixing Frege (section 5). My conclusion is briefly summarized in section (6).

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[1]A few misprints have been detected. On p. 20, line 6 from the bottom: "principle" should be replaced by "principal". On p. 81, line 4 from the bottom: a "the" should be deleted. On p. 231, note 34, "Crisipin" should be replaced by "Crispin". On pp. 21, 43, and Index the name "Weiner" should be replaced by "Wiener".

1. *Historical development of "fixing Frege".*

The history of "fixing Frege" has not been a continuous process- quite on the contrary. After two early attempts, by Frege himself and by Russell, Frege's system, along with its inconsistency and desired repair, were "increasingly forgotten", until a recently developed "revival" (p. 46). The two early attempts are examined by Burgess in chapter 1, while the recent work is the topic of his chapters 2 and 3.

Frege's own immediate response was published in a *Nachwort* to the second volume of the *Grundgesetze der Arithmetik* (1903). Frege's proposal to meet Russell's paradox, as observed by the author (pp. 32-34) "involved a minimal modification" of his logic and turned out to be ineffective.

Russell was "the second philosopher-logician who attempted to repair the *Grundgesetze*" (p. 32). Burgess' approximately fifteen pages devoted to Russell's solution—the sophisticated and famous "theory of types"—concludes with the negative remark that the offered summary "should make it unsurprising that mathematicians ultimately preferred", rather than Russell's creation, the axiomatic set theory stemming from Zermelo (p. 46).

2. *Two criteria for the evaluation of the attempts to "fix Frege".*

Two types of guidelines are considered by Burgess for the evaluation of the various attempts to fix Frege: *mathematical* and *philosophical.* The former are, for the author, the most important, as clearly stated at the very beginning: "The thought underlying the present monograph is that however wonderful the philosophical benefits of Frege-inspired reconstructions of mathematics, the assessment of the ultimate significance of any such approach must await a determination of just how *much* [author's emphasis] of classical mathematics can be reconstructed, without resort to *ad hoc* hypotheses, on that approach" (p. 2). That this is the author's "main goal" is subsequently confirmed (for example, p. 49). Naturally, in order to measure the amount of mathematics that can be reconstructed in any given attempt to fix Frege a good weighing apparatus must be available. In fact, a special section (1.5: *Mathematical targets*) is devoted to describe "the scale by which one measures the scope and limits of a given approach". Burgess explains that "the discussion of this scale will take the form of a survey of stronger and stronger theories of arithmetic, analysis, and set theory that have been developed by mathematical logicians in the period from Russell to the present" (p. 49-50). Philosophical criteria also

matter in the evaluation of the attempts to fix Frege, but are far less pressing within the author's plan: "The present work aims merely to characterize, not to resolve, philosophical issues" (p. 49).

In my view, Frege would not like Burgess' ranking of the two types of criteria. Frege nicely compared arithmetic "with a tree that unfolds upward into a multitude of techniques and theorems while its root drives into the depths", and distinguished, accordingly, two "impetus", which he called, in German, *Wurzeltrieb* (impetus of the roots) and *Wipfeltrieb* (impetus of the top of the tree, *Grundgezetze I*, p. XIII). What Frege calls *Wipfeltrieb* Burgess calls "Mathematical targets" (chapter 1, 1.5), what Frege calls *Wurzeltrieb* our author calls "Philosophical targets" (chapter 1, 1.6). While Frege favors the *Wurzeltrieb* Burgess gives priority to the *Wipfeltrieb*.

I think that fixing Frege should be first of all a philosophical enterprise, not guided by how much of mathematics can be salvaged. The quantity of leaves, branches, or even flowers engendered by the *Wipfeltrieb* is philosophically worthless if the *Wurzeltrieb* turns out to be defective.

3. *On how to name the attempts to fix Frege.*

Burgess distinguishes a broad sense and a narrow sense of "neo-Fregeanism": "in a broad sense [neo-Fregeanism] is the logical project of developing consistent modifications of Frege's inconsistent system. Neo-Fregeanism in a narrower sense is the logico-philosophical project of developing such systems with an aim resembling Frege's, which is to say, with the aim of establishing that a substantial amount of mathematics has some special epistemological status" (p. 75).

As I understand the author, the "resembling" characterizing the narrow sense consists in sharing the plan presented by Frege in the Preface of *Begriffsschrift*, which aims at showing that arithmetic does not depend on any sort of intuitive knowledge, not even on the "pure" Kantian intuition.

I find it difficult to agree with this terminology, for at least two reasons. First, the "resembling" condition for being a narrow neo-Fregean is too weak. Let us recall Frege's definition of number in *Grundlagen*. It includes, first, the so-called Hume principle, and secondly the assignment of the corresponding equivalence class as denotation of the singular terms of the form "the number of the concept...". Now, consider those who in their efforts to fix Frege want to keep all of him except the second part of the definition of number. They would qualify as neo-Fregeans in the narrow sense, assuming that they accept the

above mentioned epistemological plan with regard to arithmetic. In my view, however, the rejection of the second part is such an un-Fregean move that it becomes questionable whether one can still talk, meaningfully, of neo-Fregeanism, even in the broadest sense. Secondly, the phrase "neo-Fregean" has been so tied up, recently, with very particular ways of trying to remove the inconsistency from Frege's creation, that it sounds a bit misleading to apply it to any other possible attempt to achieve the same.

4. *The two original Fregean characters.*

The original Fregean story boasts two principal characters: concepts and extensions of concepts, which have in Burgess' presentation the prominence they deserve.

Burgess points out that Frege's system appears "to have *two* features, each of which contributes to the paradox": Frege's being "very free in assuming the existence of concepts" and Frege's being "very free in assuming the existence of extensions of concepts" (p. 47). The latter includes the assumption that "to every concept there may be associated an object, called its extension" (p. 18). The first type of freedom is best exemplified, in my view, by Frege's analysis of the notion of *ancestor* (in the *Begriffsschrift*). Charlemagne is my ancestor iff I have all the hereditary properties that Charlemagne's children have (had). While this analysis of the ancestral appears to this reviewer as irritatingly circular, Burgess seems to view it as a successful refutation of Kant's philosophy of arithmetic as based on intuition (p. 17).

In chapter 2, Burgess studies the attempts involving some restriction on concepts, an approach called "predicativity" since the early Russell. The predicative attempts examined in chapter 2 impose no restrictions at all on the existence of extensions "for whatever concepts" (cf. for example p. 146). The alternative approach, studied in chapter 3, looks at theories that freely admit concepts (in particular, the dangerous "quantification over all concepts", p. 146) but "restrict the assumption of the existence of extensions" (p. 146). The reader should not be confused by the fact that chapter 3—*Impredicative Theories*—is not designated in a way that refers to the freedom in assuming extensions but in terms of "taking the opposite tack" with respect to chapter 2.

The examination of predicative attempts to fix Frege ends in a rather pessimistic note, relative to the author's criterion of how much of mathematics has been saved. The last sentence of chapter 2 reveals this lack of optimism: "It is time, therefore, to have a look at the impredicative

option". Although the impredicative fixing Frege (chapter 3) fares better, it still falls short of what is desired. In sum, the author's global evaluation is the following: "In the work of all the authors named [this refers to a list of the most important scholars involved in the project of "fixing Frege"], one encounters limits to how much of classical mathematics one can develop (or at any rate, to how much one can develop without resort to ad hoc hypotheses)" (p. 49).

As said, philosophy is another (albeit, for the author, secondary) criterion to evaluate the "fixing Frege". At the end of his overview of philosophical issues to be taken into account in the evaluation (section 1.6: *Philosophical targets*) Burgess says: "I will for the most part leave the ultimate philosophical evaluation [...] to the reader" (p. 85). Being myself a reader, I will take advantage of this permission and I will straightforwardly move to the issue that is, in my view, central.

5. *The third man (abstraction).*

The reader soon discovers that Frege's original notions of concept and extension of concept are not the only principal characters in the author's horizon. There is, as in the classical movie, a "third man" involved, that overshadows both concepts and their extensions. This third man appears throughout the book under a number of words: "abstract", "abstraction", "abstractionism", "abstractionist". These terms in turn occur often as part of larger technical phrases, with a specific meaning of their own: "abstraction theory", "principle of abstraction", "general abstraction", "abstractionist definition of number".

What do these words, essentially "abstract" and "abstraction", mean in Burgess' monograph? There seem to be three sources of information. 1) After reminding his readers that equivalence relations are reflexive, symmetric, and transitive, the author writes: "We may then want to introduce *abstracts* with respect to the equivalence, these being, intuitively speaking, objects associated with the original objects in such a way that the abstracts associated to two objects will be the same if and only the two objects are equivalent" (p. 22).
2) The author says that "the abstract with respect to an equivalence may be identified with the set of equivalents" (p. 23). If we ask: Why?, the reason given is that the set of equivalents "can be used to serve any purpose that would be served by abstracts" (p. 23).
3) Burgess hints at what the abstracts are *not*: such entities should not be conceived as being the product of any mental activity (p. 166, cf. also p. 80). That is, "abstraction" should not indicate any sort of

mental "leaving out" and "retaining", nor abstracts should be regarded as products thereof.

Readers who share the traditional understanding of abstraction as essentially involving a "not paying attention to" will hardly know what to do with the third, negative comment, and will turn to the positive explanations (1) and (2), hoping to be enlightened by them. Now, what the first comment gives us is extremely thin. To appreciate its thinness let us consider biconditionals of the following form: Rab iff $f(a) = f(b)$, with R an equivalence relation relating the objects a and b, and $f(x)$ some functional expression that , by insertion of the singular terms "a" or "b". becomes another singular term of the form "the f of a", "the f of b", *etc.* Suppose we are well acquainted with the objects a and b, as well as with what to be related by R means, but for some reason we have forgotten the meaning of "$f(x)$". In this strange half-amnesia situation we can hardly say that we know what the objects $f(a)$ and $f(b)$ are...all we know about them is what the biconditional says: those objects are the same iff a and b stand in the relation R. Such is, exactly, the information on the nature of the abstracts provided by comment (1). This lack of knowledge is not improved by turning to the second comment. On the contrary, the second comment is irritatingly circular: with the infinitesimal amount of information provided by the first comment, that is, having practically no idea of what is the nature of the alleged abstracts, it is hardly understandable how can we discover "the purposes that might be served by them".

In sum, the answer to the above stated question: "What do the words abstract and abstraction mean in Burgess' monograph?" is extremely unsatisfactory—or perhaps better to say non-existent.

This painful situation with regard to the meaning of the term "abstraction" and cognates is not an isolated phenomenon, restricted to some of the current Frege scholarship. It is, unfortunately, just an instance of a generalized, curiously well- established fact in the past century history of logical and foundational studies. In view of the lack of information on the nature of the objects $f(a)$, $f(b)$..., or in order to make it (allegedly) more precise, scholars "look around" and try to find a denotation for the singular terms "$f(a)$", "$f(b)$"... guided by the principle that *any* entity whatsoever can be assigned to them, provided the assignment is compatible with the biconditional. The equivalence classes of a, b are the easiest candidates, but there are indefinitely many others that are compatible with the biconditional. The assignment of any particular entity among those that are compatible is of course frivolous and unjustified—another *harakiri of reason*, to use the

phrase Hermann Weyl applied to the Russellian axiom of reducibility (*Philosophie*, p. 41). Unfortunately, and sadly enough for those of us who are fond of Frege, it was Frege who probably started the practice of this harakiri, when, after stating the so-called Hume principle (two concepts F and G are equinumerous iff the number of F = the number of G), he moved to the second part of the definition of number (the number of the concept F = the extension of the concept being equinumerous with F, *Grundlagen*, §68). Such a two-stages procedure was used by Frege also in his introduction of sets (Axiom V of *Grundgesetze*), and it can be argued that it underlies Frege's theory of *Bedeutung* as well. Thus, it seems correct to speak of "Frege's method" (I have used alternative designations, such as "looking-around" or "circumspection" method; *cf.* my "Abstraction, Looking-around, and Semantics").

How could a method of so little philosophical value[2] manage to acquire a highly respectable status throughout the 20th c. mainstream of logical and foundational studies? One important cause of this strange development was that very early, in connection with Peano and his school, the method began to be mistakenly referred to as "abstraction" (*cf.* my "The Troubled History of Abstraction"). The awesome terminology of "abstraction" and "abstracts" suggests to the unprepared reader that there is, behind the scene, something very sophisticated going on, whereas in reality there is either nothing (when the singular terms "f(a)" are left uninterpreted, as it happens among those who want to keep, in the case of number, the Hume principle without the second part of the definition) or just some object arbitrarily chosen as denotation (extensions or what not).

When the frivolous choice is performed, i.e. when, as instructed by comment (2), abstracts are "identified" with extensions, my former remark that abstracts overshadow extensions has to be revised, or rather reversed: abstracts disappear, or perhaps keep a ghostly existence only as linked to extensions in such a way that *to assume extensions is tantamount to assuming abstracts for all equivalences* (p. 23, author's emphasis, repeated on p. 48). This ghostly, linked existence appears to be qualified when the author, after announcing that the plan of chapter 3 is "to restrict the assumption of the existence of extensions", adds "or replace it by the assumption of the existence of abstracts for some equivalence other than coextensiveness" (p. 146). The exception "other than coextensiveness" is understandable: to assume the existence of

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[2]Husserl writes: *Ich kann nicht einsehen, dass diese Methode eine Bereicherung der Logik bedeute*: "I cannot see that this method represents an enrichment of logic", *Philosophie der Arithmetik*, p. 134, within the section titled *Freges Versuch*: "Frege's attempt".

abstracts with respect to coextensiveness of concepts would amount to assert the famous, or infamous Axiom V of Frege's *Grundgesetze der Arithmetik*.

Moreover, the identification of abstracts with extensions collides, in Burgess' presentation, with his claim that "extensions are a special case of abstracts" (p. 23). Thus, extensions seem to be explained in terms of abstracts and abstracts in terms of extensions. Obviously, this two-way explanation is untenable. One has to choose: either triumphantly claim, with the mainstream, that classes explain abstraction (as done by Reichenbach, *Elements*, §37 *The principle of abstraction*), or take the opposite course, but not both. Interestingly, in Lorenzen, who takes the opposite course and emphasizes that extensions (sets, classes) are to be explained in terms of abstraction[3], one finds the same phrase "special case" as in Burgess: "die Klassen nichts anderes als ein *spezieller Fall* von abstrakten Objekten sind" (extensions are nothing else but a *special case* [of abstracts], emphasis mine, *Einführung*, p. 101). Lorenzen, however, does not at the same time claim that abstracts are classes.

6. *Concluding remarks.*

The *philosophical* (*Wurzeltrieb* oriented) evaluation of the various attempts to fix Frege reported by Burgess, *to the extent* that they are based on the just seen notion of abstracts and abstraction, cannot be but negative, regardless of how much mathematics can be reconstructed (of how intensive and exuberant the *Wipfeltrieb* is).

Burgess' work includes a great amount of helpful, interesting, sharp observations regarding Frege's logic, mathematics, and philosophy, that have not been presented or discussed in this essay. My review has been limited to the criticism of the *philosophical* quality of the recent attempts to fix Frege, as reported by the author. *Fixing Frege* fills a serious gap in the Frege literature (always increasing but perhaps with an excessive attention paid to semantics and the philosophy of language) and should remain for a long time a necessary reference for scholars in the field.

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                   tics," *Studia Leibnitiana*, Sonderheft 8, 108-123, 1979.

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[3]When Lorenzen reduces, so to speak, classes to abstraction, what he means by "abstraction" continues to be the classical or traditional, "not paying attention to" but subject to a logico-linguistically precise analysis and formulation.

[Angelelli 2005] ———, "The Troubled History of Abstraction," in *Logical Analysis and History of Philosophy*, Uwe Meixner and Albert Newen (eds.), Paderborn: Mentis Verlag, **8**, 2005, pp. 157-175.

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