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The Mathematical Works of Bernard Bolzano
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REVIEW

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Bernard Bolzano (1781-1848) devoted the better part of his life to the cause of reform in his homeland Bohemia, then part of the Hapsburg Empire. Beginning in 1805, he held a chair as Professor of Religious Science at the Charles University in Prague, where he soon became known as a fearless critic of the abuses of the regime and a leader of the "Bohemian Enlightenment," a movement which combined a rationally clarified Catholicism with an ambitious programme for social and political reform. In his spare hours, Bolzano took up his other great passions: philosophy, mathematics and logic. He was, he tells us in his autobiography, especially drawn towards the "part of mathematics that is at the same time philosophy," ([8], p. 19) that is, in what we would call foundations in the broad sense of the term. A large part of his research accordingly focused on the search for proofs of propositions most mathematicians thought too obvious to require proof (such as the intermediate value theorem), and for definitions of concepts generally thought to be so clear in themselves that they neither admit of nor require definition (*e.g.*, "line", "surface" or "solid"). More or less alone in taking this approach in the early part of the nineteenth century, he had the field almost to himself, and accomplished far more than one might have thought possible for an amateur, arriving at a number of important results usually thought to stem from a later period. Among his achievements, we mention the following: in 1816, he gave one of the first rigorous treatments of power series; in 1817, he gave the first rigorous proof of the intermediate value theorem, which involved a precise definition of continuity, a statement of the sufficiency of the Cauchy criterion for the convergence of a sequence, and a statement and proof of the least upper bound theorem; in his later work (dating from the 1830s), he developed an arithmetical theory of real numbers, used the

Bolzano-Weierstrass theorem[1] to prove central results about continuous functions, distinguished pointwise from uniform continuity, stated Heine's theorem[2], and constructed a variety of "monster functions", including one which is continuous at every point in an interval, but not monotone on any subinterval, and also nowhere differentiable. These mathematical discoveries are interesting enough, but all the more so in view of the fact that Bolzano was engaged at the same time in systematic reflections on logic and scientific methodology. In his logical writings, especially the monumental *Theory of Science* of 1837 [9], we see important parts of the modern conception of axiomatic theories take shape, among them a sophisticated understanding of definition, a general theory of collections (including sets), and the first viable definition of consequence. Bolzano is a major thinker, one of the truly great philosophers of the nineteenth century, and deserves to be more widely known.

With the publication of *The Mathematical Works of Bernard Bolzano*, Steve Russ has done great service for this cause. Here we find a substantial collection of translations from Bolzano's mathematical writings, the beginnings of which date back some thirty years [23]. There are complete translations of all five of the mathematical works Bolzano published between 1804 and 1817 ([3], [4], [5], [6], [7]), significant parts of the *Theory of Quantities* [12], (including Bolzano's theory of "measurable numbers", which is often taken to be an arithmetical theory of real numbers, and his *Theory of Functions*) and a complete translation of the posthumously published *Paradoxes of the Infinite* [13].[3] This volume will go a long way towards making Bolzano's mathematical work better known, and not only on account of the translation. Most of these writings have been extremely difficult to find even in the German originals, the notable exceptions being the often reprinted *Purely Analytic Proof* and the *Paradoxes.*[4]

[1]Any infinite set of real numbers contained in a closed, bounded interval has at least one limit point in the interval.

[2]A real-valued function continuous on a closed, bounded interval is uniformly continuous there.

[3]Readers should not be misled by the title of Russ's collection, which by no means contains all of Bolzano's mathematical writings. The introductory parts of the *Theory of Quantities* are not included, for example, nor are any of Bolzano's later geometrical writings.

[4]Apart from the original publications, The Institute of Czechoslovak and General History published, in 1981, facsimile reprints of the five early mathematical works (*Acta historiae rerum naturalium nec non technicarum* Special issue 12). Not surprisingly, copies are hard to come by. The *Theory of Functions*, for its part, only appeared in the critical edition of Bolzano's works in 2004 (ed. B. van Rootselaar;

A few words on Bolzano's writings on mathematics and logic are in order here. At first, inspired by the example of German-language authors such as Ch. Wolff and A. G. Kästner, Bolzano seems to have aimed at producing nothing less than a complete, systematic presentation of the entirety of mathematics. The 1810 treatise, *Contributions to a Better-grounded Presentation of Mathematics* (pp. 83–137), was intended to be the first installment of such a work. In it, Bolzano presents the programme, including his views on mathematical method, (which he claims to be nothing other than logic [p. 103]), as well as a proposed definition of mathematics as a "science of forms" (pp. 91-95) and a classification of its sub-disciplines (pp. 95-102). Not surprisingly, the response to this poorly circulated publication by an obscure Bohemian priest was less than enthusiastic. Bolzano was nevertheless disappointed, and decided, as he tells us,

> to attempt [...] to make myself better known to the academic world by publishing some works whose titles would be better suited to arouse attention (p. 262).

The fruits of this attempt were three papers of 1816-1817: the *Binomial Theorem* (pp. 155–248), the *Purely Analytic Proof* (pp. 253–277), and the *Three Problems of Rectification, Complanation and Cubature* (pp. 279–344). Though the first and third of these are tough sledding for the reader, they do contain original and significant contributions. The *Purely Analytic Proof*, however, is a gem: concise and focused, it presents in a few short pages a compelling justification for the "arithmetization of analysis" and a paradigm of how to go about it.

These works would eventually find the right readers. The *Purely Analytic Proof*, especially, was greatly admired by Weierstrass and his students (see [22], p. 85 ff.). But this recognition came too late for Bolzano. Discouraged again by the poor reception of these shorter works by his contemporaries, he returned to his original plan of writing comprehensive, systematic treatises. Logic came first: writing the *Theory of Science* occupied most of the twenties. Next came the turn of mathematics, as Bolzano began work on another massive treatise which was to be called the *Theory of Quantities* [*Größenlehre*].[5] This decision was regrettable in many ways. First, due to advancing age

[16], Series 2A Vol. 10/1); beforehand it was only available in a rare edition by K. Rychlík published in 1930 by the Royal Bohemian Society of Sciences [14]. As a result, even quite knowledgeable historians of mathematics are sometimes unaware of Bolzano's accomplishments.

[5]Part of the *Theory of Quantities* is a brief account of Bolzano's logic ("On the Mathematical Method"), a mature counterpart to the *Contributions* of 1810. This has been translated into English in Bolzano [2004].

and illness, Bolzano never finished his project, and never published his discoveries. Small fragments of his later writings appeared in the *Paradoxes of the Infinite*, but the bulk of his manuscripts were passed on to a promising student (Robert Zimmermann, who would become professor of Philosophy at Vienna), who did nothing with them. It was only in the 1920s, after many of Bolzano's results had been independently rediscovered, that scholars began to pay attention to the *Theory of Quantities*. Second, perhaps precisely on account of the scope of the treatise, Bolzano's work is very uneven. On the one hand, we find astonishing precision, great insight and creativity, on the other, some rather elementary blunders. Had he chosen to publish even a handful of research papers containing his more important discoveries, his work might have been of more consistent quality and had considerably more influence on the development of mathematics.

The first piece translated here, *Considerations on some Objects of Elementary Geometry* [3], is derived from Bolzano's doctoral thesis. Many of the themes of his mature mathematical work are already clearly present. He is interested to begin with in the nature of rigorous, or scientific proof, and sets out two rules (p. 31 f.): first, simply because we find a proposition to be obviously true does not mean that it does not need to be proved. Rather, we should follow the example of the ancients in seeking proofs even for obvious truths. This is done not for the sake of certainty, but rather for the sake of understanding. For if we know the objective connections between the truths of mathematics, we will not only find it easier to grasp them, we will also be better able to discover new truths. Second, in a correct proof, no "alien" intermediate concepts should be introduced—a rule which Bolzano assimilates to Aristotle's ban on crossing from one genus to another ([1], I, 7). In the case of geometry, for example, no appeal should be made to the concept of motion. For if we do appeal to motion, for instance in proving the congruence of two figures, we must establish that such a motion is possible—*e.g.*, that one and the same object can occupy a certain collection of spaces, *etc.* But in order to do this, we will have to presuppose the very geometrical proposition we are attempting to prove. This situation is by no means untypical: often, crossing from one genus to another simply produces a vicious circle, where a special case is invoked to support the general result which grounds the special case (cf. p. 126). Somewhat more controversial are Bolzano's claims that in the theory of triangles and parallel lines no use should be made of the concept of a plane, and that no use of the concept of an angle or a triangle should be made in the theory of the straight line.

The *Considerations* then attempts to reformulate geometry in line with these methodological strictures. The key notions are those of *determination*, *similarity*, and *equality*, the key principles the following:

(1) If the determining elements ("pieces") of the spatial objects A and B are equal, then A and B are equal (p. 36).
(2) If the determining elements of the spatial objects A and B are similar, then A and B are similar (p. 40).

Similarity is defined as by Leibniz and Wolff:

> Two spatial objects are called similar if all the characteristics which arise from the comparison of the parts of each one among themselves are equal in both ... (p. 40)

Unfortunately, Bolzano does not give a precise definition of determination in the 1804 treatise, perhaps taking it for granted as sufficiently understood (for Bolzano's later views on the subject, see [7], §§13-14; [9], §180; cf. [2], p. 15 ff.). He adopts as an axiom (§19, p. 40) the proposition that there is no absolute measure of length, an assumption which later permits him to prove the parallel postulate (§59, p. 60).

The second part of the *Considerations* is given over to a sketch of a theory of the straight line. Bolzano takes as primitive notions the concepts of distance, direction, and opposition, and defines betweenness as follows:

> A point m may be said to be [...] between a and b if the directions ma, mb are opposite (p. 25).

The straight line (segment) $\overline{ab}$ can then be defined as "an object which contains all and only those points which lie between the two points a and b" (p. 76). We notice here that Bolzano is already clearly inclined to conceive of geometrical objects as structured point-sets.

The second work in Russ's collection is the 1810 essay *Contributions to a Better-grounded Presentation of Mathematics*. As remarked above, this work contains Bolzano's first sketch of a logic, as well as his definition of mathematics and a classification of its branches. Once again Bolzano's boldness is in evidence, as he not only disagrees completely with Kant's claims about mathematics (pp. 132-137) but also repeats the charge, implicit in the *Considerations*, that Euclid's presentation of geometry is thoroughly defective and indeed past saving (pp. 87-88). Analysis and arithmetic, too, are in rough shape:

> Have not the greatest mathematicians of modern times recognised that in arithmetic the theory of negative numbers, together with all that depends on it, is still not

> clear? Is there not a different presentation of this theory in almost every textbook? The chapter on irrational and imaginary numbers is still more ambiguous, and in places full of self-contradictions—not to mention the defects of higher algebra and the differential and integral calculus (p. 87).

Bolzano's prescription is a reconstruction of mathematics from the ground up, in line with an improved methodology, which is sketched in part II of the work. Again he stresses that the function of scientific proof is not to engender conviction, but rather to display the "objective dependence" of one truth upon others. Where Descartes had written:

> None of the conclusions one deduces from a principle which is not evident can themselves be evident, even if the deduction is evident... ([18], Vol. IX-2, p. 8)

Bolzano counters that, since proofs concern the objective dependence of truths, "which is independent of our subjective recognition of it," (p. 103) it is entirely conceivable that an axiom may be less evident than the propositions it supports.

> Indeed, it could even be that an axiom may appear questionable and dubious ... because we do not immediately see that the things we recognise at once as true can be derived from it (p. 119).

The things we are most certain of, he maintains, are usually provable propositions. Axioms are propositions which are by their very nature unprovable, but this does not mean they must be self-evident. They too may need to be justified in the Cartesian sense, that is, we may need to be convinced that they are true, often by showing that they support exactly the (provable) results which we take to be certain (*ibid.*).

With definitions, similarly, the Cartesian view that indefinables are those notions which are so clear in and of themselves that no clearer terms can be found to define them is replaced by a corresponding objective notion: indefinable concepts are those which, in and of themselves, have no parts. This property is by no means the same as the Cartesian one. How then are we supposed to convey the primitive concepts of mathematical theories to others? Here Bolzano appeals to a notion quite like implicit definition:

> But how does he begin to reach an understanding with his readers about such simple concepts and the words he chooses for their designation? This is not a great difficulty. For either his readers already use certain words or

> expressions to denote the concept and then he need only indicate these to them, *e.g.*, 'I call *possible* that of which you say that it *could be*', or else they have no particular symbol for the concept he is introducing, in which case he assists them by stating several propositions in which the concept to be introduced occurs in *different combinations* and is designated by its own word. From the comparison of these propositions the reader himself then abstracts which particular concept the unknown word designates (p. 107).

A final point of interest in the *Contributions* are two criteria for the correctness of proofs set out in §28 and §29 of Part II (p. 122 ff). The second of these (pp. 123-126) is a slightly more general version of the ban on crossing from one genus to another in the course of a proof. The first reads as follows:

> If the subject (or the hypothesis) of a proposition is as wide as it can be so that the predicate (or the thesis) can be applied to it, then in any correct proof of this proposition, all characteristics of the subject must be used, *i.e.*, they must be applied in the derivation of the predicate, and if this does not happen, the proof is incorrect (pp. 122-123).

To take a simple example: if, in proving a proposition which is true of the real numbers but not of the rational numbers (*e.g.*, every polynomial of odd degree with integral coefficients has a root), we make no mention of any property which the reals have and the rationals lack, then our proof must be incorrect.

It might seem as if these principles were so obvious that no mathematician could possibly violate them. Yet the next two works in Russ's collection show the opposite to have been the case. The *Binomial Theorem* [5] takes up the problem of power series, attempting to produce a rigorous proof of Newton's binomial theorem:

$$(1+x)^n = 1 + nx + n \cdot \frac{(n-1)}{2}x^2 + \cdots + \frac{n!}{r!(n-r)!}x^r + \cdots$$

for all real values of n.

Now when n is not a positive integer, Newton's expansion results in an infinite series. Bolzano, who had looked through dozens of attempted proofs, noted that everyone knew that in these cases the series only converged for $|x| < 1$, but—amazing as it may sound—no one had made any use of this condition. (p. 159) Thus all of these proofs break

Bolzano's first rule: if they were correct, they would show that the series converges for every value of x. They prove too much, and hence prove nothing (*ibid.*).

In order to prove Newton's theorem, Bolzano claims, we must first come to a precise understanding of what it says (p. 157 ff.). Conceptual analysis—which Kant had claimed to be worse than pointless in mathematics ([20], A712/B740 ff.)—is thus very much the order of the day. Bolzano begins by carefully setting out a definition of what we would call the convergence of an infinite series[6] in terms of the convergence of the associated sequence of partial sums. With this definition in hand, he is able to give a clear meaning to the binomial equation even in the cases where the right hand side is an infinite series. He then proves the theorem, first for positive integral values of n, then for rational values of n (when $|x| < 1$), and finally for irrational n and $|x| < 1$.

Bolzano's second criterion shows its worth in the *Purely Analytic Proof* [6]. Again, his aim is to prove the obvious: a real-valued function which is continuous on an interval $[a, b]$ with $f(a) < 0 < f(b)$ must have a zero on (a, b). Interestingly, some previous mathematicians had tried to prove this theorem. Yet they all violated Bolzano's second rule, by appealing to geometry, the theory of motion, or the like. Consider, for example, the "purely analytic proof" offered by Lagrange ([21], p. 133.). Suppose $g(x)$ is a polynomial with $g(\alpha) < 0 < g(\beta)$, where $0 < \alpha < \beta$ (which we may suppose without loss of generality). Let $f(x)$ represent the sum of the terms of $g(x)$ preceded by a plus sign, and $\phi(x)$ the sum of terms of $g(x)$ preceded by a minus sign (taken in absolute value), so that $g(x) = f(x) - \phi(x)$. Then for $x = \alpha$, we have $f(x) < \phi(x)$, while for $x = \beta$, $f(x) > \phi(x)$. Lagrange continues:

> Now from the form of the quantities f and ϕ, which contain only positive terms and positive, integral exponents, it is obvious that these quantities will necessarily increase as x does, and that, in making x increase through all the insensible degrees, they will also increase by insensible degrees, but in such a way that f will increase more than ϕ, since from being smaller it becomes the greater of the two quantities. Therefore there will necessarily be a term between the two values α and β where f will equal ϕ, just as two bodies which one assumes to be

[6] At that time, the term "convergence" was usually taken to mean that the terms of a series $\sum a_n$ went to zero with increasing n—a condition clearly compatible with what we would call the divergence of the series.

> moving in the same direction along the same trajectory and which, leaving at the same time from two different points, arrive at the same time at two other points, but in such a way that the one which was behind at first is afterwards ahead of the other, must necessarily meet on their path (*ibid.*[7]).

Bolzano's criticisms of Lagrange's proof are simple but devastating. If, as Lagrange himself maintains, the concepts of motion and time are foreign to analysis, they can play no role in a purely analytic proof. At best, we can consider their use here to be merely figurative, and hence eliminable in favour of terms belonging to analysis. But at this point, the emptiness of the proof becomes obvious;

> The deceptive nature of the whole proof really rests on the fact that the concept of *time* has been involved in it. For if this were omitted, it would soon be seen that the proof was nothing but a re-statement in different words of the proposition to be proved. For to say that a function fx, before it passes from the state of being smaller than ϕx to that of being greater, must first go through the state of being equal to ϕx is to say, without the concept of time, that among the values that fx takes if x is given every arbitrary value between α and β, there is one that makes $fx = \phi x$, which is exactly the proposition to be proved (p. 257).

Having shown the nullity of previous proofs of the intermediate value theorem, Bolzano proceeds to develop his own. Again, a clear understanding of the proposition is a necessary preliminary to a correct proof, and he begins by formulating a definition of (pointwise) continuity:

> According to a correct definition, a function fx varies according to the law of continuity for all values of x inside or outside certain limits means only that, if x is any such value, the difference $f(x + \omega) - fx$ can be made smaller than any given quantity, provided ω can be taken as small as we please (p. 256).

Again, the result concerns real-valued functions, so a correct proof will have to appeal to some characteristic properties of the reals. Bolzano formulates two:

[7] I have changed the names of the functions and the points in order to make Lagrange's text uniform with Bolzano's, quoted below.

(1) Any infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers with the "Cauchy property" (in modern language, $\forall \epsilon > 0\ \exists N\ (m, n > N \rightarrow |x_n - x_m| < \epsilon)$ has a real limit (p. 266 ff).
(2) Any set of real numbers which is bounded above has a least upper bound (p. 269 ff).

The proof of the first theorem is flawed, but the proof of the second is a thing of beauty, as Bolzano shows how to determine the least upper bound through successive approximation by means of a (possibly infinite) convergent sequence of the form:

$$u, u + \frac{D}{2^m}, u + \frac{D}{2^m} + \frac{D}{2^{m+n}}, \cdots, u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \frac{D}{2^{m+n+\cdots+r}}, \cdots$$

Where u is not an upper bound of the set but $u + D$ is.[8]

The main result is then proved as follows. Given that $f(a) < 0$ and f is continuous on $[a, b]$, it follows that $f(x) < 0$ for all x in a one-sided neighborhood of a, $[a, a + \omega)$ for some $\omega > 0$. On the other hand, since $0 < f(b)$, the set of ω such that $f(a + \omega) < 0$ is bounded above. By the second theorem stated above, this set has a least upper bound, say h, and it is easy to show (appealing to continuity, trichotomy, and the fact that h is the least upper bound) that $f(a + h) = 0$.

The *Purely Analytic Proof* is Bolzano's greatest single mathematical work. It is here that anyone interested in his mathematical thought should begin.

The last of Bolzano's early publications, *The Three Problems of Rectification, Complanation, and Cubature* combines two rather different inquiries. On the one hand, Bolzano is concerned to demonstrate, with the help of his concept of determination, the correctness of the well-known formulae for the length of a line, the area of a surface, and the volume of a solid:

(1) $L = \int \sqrt{dx^2 + dy^2 + dz^2}$
(2) $A = \int \int dxdy \sqrt{1 + (\frac{dz}{dx})^2 + (\frac{dz}{dy})^2}$

[8]This proof was much admired in Weierstrass's circle, and on account of the similarities between Bolzano's proof method and the technique of repeated bisection of an interval, Bolzano's name was attached along with that of Weierstrass to the theorem that every infinite set of points contained in a closed, bounded interval has a limit point on the interval (which Weierstrass proved using the technique of iterated bisection). It was not known at that time that Bolzano had stated and used the "Bolzano-Weierstrass theorem" in the (unpublished) *Theory of Functions* (see pp. 459-460; also p. 582). Bolzano did show the *Theory of Functions* to some other mathematicians and he presented some of his discoveries at meetings of the Bohemian Royal Society, but I am aware of no evidence that would suggest that Weierstrass did not simply re-discover the Bolzano-Weierstrass theorem.

(3) $V = \int\int\int dxdydz$

His second concern is to advance his project of reconstructing geometry from the ground up, here by formulating adequate definitions of a number of geometrical concepts, among them "line", "surface" and "solid". (p. 301 ff; p. 322 ff; p. 334 ff). Geometrical objects are explicitly conceived of as structured point sets (p. 301, note), and his proposed definitions are clearly topological in character. A line, for example, is defined as "a spatial object, at every point of which, beginning at a certain distance and for all smaller distances, there is at least *one* and at most a *finite* set of points as neighbours" (p. 301), a surface as "a spatial object at each point of which, beginning from a certain distance and for all smaller distances, there is at least *one* and at most only a *finite* set of *separate lines* full of points" (p. 322)[9].

The second half of Russ's collection begins with Bolzano's theory of *infinite quantity concepts*. These are number concepts which involve an infinite number of the elementary operations $+, -, \times, \div$ performed on natural numbers, *e.g.*,

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots.$$

A number concept A is *measurable*, for Bolzano, if for every positive integer q, there is an integer p and number concepts P_1, P_2 (where P_1 is either zero or "purely positive" and P_2 is purely positive), such that:

$$A = \frac{p}{q} + P_1 = \frac{p+1}{q} - P_2$$

($\frac{p}{q}$ and $\frac{p+1}{q}$ are called the *measuring fractions* of A.) A number concept is said to be infinitely small and positive if it is not simply zero yet for every q, the corresponding p is 0, *e.g.*, $\frac{1}{1+1+1+1+\cdots}$. A number concept A is infinitely small and negative iff $-A$ is infinitely small and positive. Infinitely large, positive number concepts are those for which for all q, only the equations $A = \frac{p}{q} + P_1$ can be satisfied, while A is infinitely large and negative if only the equations $A = \frac{p+1}{q} - P_2$ can be satisfied. Thus all infinitely large number concepts are unmeasurable, but the converse doesn't hold, witness:

$$1-2+2-2+2-2+\cdots$$

[9]For more on Bolzano's work on these and related questions, see [19] and [24], p. 56 ff.

Finite as well as infinite numbers concepts can be measurable, and Bolzano proves among other things that all rational numbers are measurable. In §54 Bolzano suggests that the notion of equality (or equivalence) might be extended in the case of measurable number concepts from simple identity to identity of measuring fractions (equality "in the process of measuring"); similar extensions are suggested for the relations of order. Bolzano sees the desirability of having $A \pm J = A$ in this extended sense whenever A is measurable and J infinitely small. Unfortunately, with his original definitions, this claim is an antitheorem, for if A is rational (say $= \frac{3}{4}$ and $J = \frac{1}{1+1+1+1+\cdots}$, then for $q = 4$, the corresponding p for A is 3 while for $A - J$ it is 2. At this point, Bolzano suggests changing the definition of equality (in the extended sense) to the following: $A = B$ iff $|A - B|$ behaves like zero in the process of measuring. He remarks: "this section is to be rewritten" (p. 391), but to judge from the manuscripts that survive, he never found the time for the required revisions.

If the foundations of Bolzano's theory were somewhat unsettled, it is clear from the later parts that he had an exceptionally clear idea of just what a theory of real numbers should accomplish. Among other things he states and proves that the measurable numbers are closed under the elementary operations, that they have the Archimedean property (§74, p. 399), that they form a dense set (§79, p. 400), that every sequence of measurable numbers with Cauchy's property converges (§107, p. 412 ff), that every set of measurable numbers which is bounded above has a least upper bound (§109, p. 416 ff), and that every measurable number is representable as a Cantorian series (§48, p. 385 ff.):

$$\frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma}{abc} + \frac{\delta}{abcd} + \cdots$$

where $a \geq 1, b, c, d, \ldots > 1$ are natural numbers, α is an integer and $\beta, \gamma, \delta, \ldots$ are natural numbers with $\beta < b, \gamma < c, \ldots$

On the whole, Bolzano's theory compares unfavorably with the later, more polished theories of H. Méray, G. Cantor, R. Dedekind, *et al.* Nonetheless, his recognition that such a theory was called for, and his mastery of so many important details, make his efforts worth a closer look.

The next selection is Bolzano's *Theory of Functions*, probably the most polished part of the *Theory of Quantities*. Among the highlights of this work are Bolzano's precise definitions of left-, right- and two-sided continuity (pp. 448-449), his distinction between pointwise and uniform continuity (§49, p. 456; cf. p. 575 ff., where Bolzano states

Heine's theorem and attempts a proof), his proofs of central theorems about functions continuous on closed intervals (namely, that such functions are bounded [§§56-57, pp. 459-460], assume extreme values on the interval [§§58-60, pp. 460-461], and take on all values between any two values they assume on the interval [§65, pp. 463-464]), and the construction of the "Bolzano-function", which is continuous on an interval $[a, b]$, but not monotone on any subinterval of $[a, b]$, and nowhere differentiable (§111, pp. 487-489; Bolzano only proves [§135, pp. 507-508] that his function is not differentiable at a set of points which is dense in $[a, b]$).

The *Theory of Functions* is as remarkable for its general approach as it is for such individual results. It is clear from the start that Bolzano has rejected the traditional conception of functions as analytical expressions (better, whatever is designated by these expressions) in favour of the modern notion of an arbitrary, many-one correlation between objects. This break with the older concept of a function forces a host of others; generally speaking, functions may not be continuous, may not have derivatives, still less (as Lagrange had maintained) be representable by their Taylor series (except perhaps at certain isolated values). New definitions are needed, conditions need to be pinned down, before the usual theorems of the calculus can be properly stated and proved. Here we see Bolzano at his best, producing a treatise that is, for all its mistakes, strikingly modern, living proof of the basic soundness and fruitfulness of his methodology.

By way of illustration, consider Bolzano's proof that a function continuous on a closed interval $[a, b]$ is bounded there. (pp. 459-460) Suppose that a function $f(x)$ is unbounded on an interval $[a, b]$. Then for $n = 1, 2, 3, \ldots$, there must be $x_1, x_2, x_3, \ldots \in [a, b]$ such that $f(x_n) > n$. By the Bolzano-Weierstrass theorem, the set of x_n must have a limit point in $[a, b]$:

> ...the infinitely many numbers $x_1, x_2, x_3, \ldots$, either all of them, or a part of them which is so large that its multitude is itself infinite, can be enclosed in a pair of limits p and q, which can approach one another as close as we please, and it follows from § that one of these limits could be represented by c, the other by $c \pm \omega$, if we denote by c a constant number lying not outside of a and b, but by ω a number which can decrease indefinitely (p. 459)[10].

[10]Though Bolzano claims that a proof of this theorem is to be found in the theory of measurable numbers (marginal note to the *Functionenlehre*, reproduced in [16],

The function f is unbounded in every neighborhood of c, and hence cannot be continuous there, nor *a fortiori* on the interval. Hence a function which is continuous on $[a, b]$ must also be bounded there.

Like the theory of measurable numbers, the *Theory of Functions* is quite uneven. There are moments of true brilliance. At times, though, Bolzano is just downright sloppy, carelessly extending results from the finite to the infinite case (*e.g.*, §155, p. 527), and overlooking complications caused by multiple quantification or multiple variables. The manuscript also raises some puzzles. As Bob van Rootselaar has pointed out ([16], 2A 10/1, p. 155 note), Bolzano became aware in 1832 of Cauchy's example of a function which has derivatives of all orders at a given value but is not represented there by its Taylor series, yet Bolzano's manuscript betrays no awareness of this: in §206, for instance, he states a proposition for which Cauchy's function is a counterexample (p. 561).

The final work in the collection is the *Paradoxes of the Infinite*. Readers will once again notice the unevenness of Bolzano's work. On the one hand, we find a number of very acute observations: the usual objections to the existence of actual infinites are shown to be unfounded, useful distinctions are introduced (*e.g.*, in spatial terms, unbounded is not at all the same thing as infinite), and Bolzano recognises as a characteristic, non-contradictory property of infinite sets that they can be mapped one to one onto proper subsets of themselves. (§20, p. 615 ff) On the other hand, Bolzano seems to have been reluctant to abandon the principle that the whole is greater than the part, when this is interpreted to mean that the parts of the whole must always be more numerous than the parts of its parts. Thus he never took the first step towards Cantor's transfinite arithmetic, in that he never recognised a conception of number according to which the existence of a bijection between two sets establishes equinumerousness. Even after establishing that the set of points of any line segment can be mapped 1:1 onto those of any other, he continues to maintain that the multitudes of points in the two cases are not equal, but rather proportional to the lengths of the segments (p. 626).

Though the shortcomings of Bolzano's technique are evident on almost every page of his mathematical writings, the overwhelming impression is one of solid achievement. In his introduction, Russ sums up the situation quite well:

2A 10/1, p. 47, but not in Russ's translation), none has so far been found. Bolzano does sketch part of a proof in the manuscript "Improvements and Additions to the Theory of Functions," (Russ, p. 582).

> Some of Bolzano's mathematics is very good by any standards: some is rather amateurish and long-winded, some is plain eccentric. But in each of the works included here, whether it be with notions of proof, concepts of number, function, geometry or infinite collections, his thinking is fundamental, pioneering, original, far-reaching, and fruitful. In each case his key contributions were taken up later by others, usually independently and of course improved upon, but they all entered into the mainstream of mathematics. I know of no other mathematician, working in isolation, with such a consistent record of independent, far-sighted, and eventually successful initiatives (p. xiii).

For the most part, Russ's translations are accurate, and the mathematics has been faithfully transcribed.[11] Equations have been carefully typeset so that they resemble the originals as closely as possible, and well-executed diagrams have been inserted into the text at appropriate places (in the originals, diagrams were generally included in the end-matter). The translation of the *Paradoxes*, however, could have benefitted from further revision. There are more translation problems here than with the other works, among them some which are serious enough that Bolzano's point is lost entirely. This is unfortunate, especially given that the previous English translation by D. A. Steele [15] sometimes has a more just or fluid rendering.[12]

[11]Russ writes (p. xix) that he will maintain a web-site with a list of errata (http://www.dcs.warwick.ac.uk/bolzano/).

[12]Here is one example: in §12, Bolzano writes:

> Ebensowenig kennt der Mathematiker an der Kreislinie und an so vielen anderen in sich zurückkehrenden Linien und Flächen eine Grenze, und betrachtet sie doch nur als endliche Dinge (es müßte denn sein, daß er auf die unendliche Menge der in ihnen enthaltenen Punkte zu sprechen käme, in welchem Betrachte er aber auch an jeder begrenzten Linie etwas Unendliches anerkennen muß).

Russ has (p. 605):

> Just as little does the mathematician regard the circumference of a circle and many other lines and surfaces which turn back on themselves as a limit and consider them only as finite things (It would have to be that he may come to speak of the infinite multitude of points contained in them, and in that respect he must also recognise in every bounded line something infinite.)

While Steele's translation runs (p. 83):

> Or for another example: mathematicians know of no bounding point in the circular periphery, of no bounding point or line in the

The translations are accompanied by a general introduction, including some reflections on the process of translation in general, and the challenges of translating Bolzano in particular. Each subdivision of the work is also furnished with an introduction, setting out the historical context of Bolzano's writings, sketching their contents, and providing helpful commentary along with references to the secondary literature. There are a few rough spots, but on the whole, Russ's collection is an admirable achievement, one which will do much to advance the study of Bolzano in the English-speaking world.

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[7] Bernard Bolzano, *Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst; zugleich als Probe einer gänzlichen Umstaltung der Raumwissenschaft, allen Mathematikern zur Prüfung vorgelegt*, Leipzig, 1817.

[8] Bernard Bolzano, *Lebensbeschreibung des Dr. B. Bolzano*, Sulzbach: Seidel, 1836.

[9] Bernard Bolzano, *Wissenschaftslehre*, Sulzbach, 1837. (Partial English translations in [10] and [11]).

[10] Bernard Bolzano, *Theory of Science*, ed. and tr. R. George, Berkeley and Los Angeles: University of California Press, 1972.

[11] Bernard Bolzano, *Theory of Science*, tr. B. Terrell, Dordrecht: Reidel, 1973.

[12] Bernard Bolzano, *Größenlehre*, [16], Series 2A, Vols. 7-10 (several parts have yet to appear).

[13] Bernard Bolzano, *Paradoxien des Unendlichen*, ed. F. Přihonský, Leipzig, 1851.

numerous closed curves and surfaces, and consider them for all that to be finite objects—unless indeed they come to speak of the infinite set of points contained in them, from which point of view, however, the like infinitude must be recognised in every bounded rectilinear segment.

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