PROBLEMS OF RELEVANT LOGIC IN V. A. SMIRNOV'S BOOK *FORMAL DEDUCTION AND LOGICAL CALCULI*

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Relevant logic is, undoubtadly, one of the most interesting trends of modern logic at least in its theoretical aspect. At present it attracts the attention of many logicians, mathematicians, and philosophers, for it deals with serious problems of operations with information and the laws of reasoning.

The purpose of this paper is to provide an exposition of some results obtained by Professor Vladimir Alexandrovich Smirnov in the field of relevant logic. The exposition concerns his book *Formal Deduction and Logical Calculi* [Smirnov *1972*] without any consideration of any of other of his works in which relevant logic was investigated. In the first part of the paper the definitions of formal iferences and the formulations of deduction theorems for implicative fragments of relevant systems are reproduced according to [Smirnov *1972*]; in the second part, one of the systems of relevant logics, Smirnov's absolute system, is described.

1. Deduction theorems for the implicative fragment of relevant system R in V. A. Smirnov's book *Formal Deduction and Logical Calculi*

In his *Formal Deduction and Logical Calculi* [Smirnov *1972*] Smirnov formulated different notions of inference and several kinds of deduction theorems. Considering the question about the necessary and sufficient conditions of validity of one or another kind of deduction theorem for logical sysnems with a single rule of inference, *modus ponens*, Smirnov obtained a number of interesting results concerning properies of relevant logic. Particularly, he proved the deduction theorem in different forms for the implicative fragment of system **R**, one of the more widely known and important relevant logical systems. To explain these we require the definitions of inference given in [Smirnov *1972*].

The inferences in question have a form of tree. "A system of formulae which is called a tree is introduced by the following inductive definition which, at the same time, explains what is the last formula of a system of formulae.

1. $\langle\ \rangle$ is a system of formulae without the last formula.

2. If A is formula then $\langle A\rangle$ is a system of formulae and A is its last formula.

3. If $\alpha_1, \ldots, \alpha_k$ are systems of formulae and E is formula then $\langle\alpha_1, \ldots, \alpha_k, E\rangle$ is a system of formulae and E is its last formula.

4. Nothing different from that mentioned in items 1–3 is a system of formulae.

The systems of formulae above can be represented in two dimensions: system $\langle\ \rangle$ is associated with the empty word; system $\langle A\rangle$ is associated with the object A; and system $\langle\alpha_1, \ldots, \alpha_k, E\rangle$ — with the object

$$\frac{\alpha_1, \ldots, \alpha_k}{E}$$

The system which fits the definition given above will be called a *tree*. The *height* of the tree h is defined as follows:

$$h(\langle\ \rangle) = 0$$

$$h(\langle A\rangle) = 0$$

$$h(\langle\alpha_1, \ldots, \alpha_k, E\rangle) = max\ (h\ (\alpha_1), \ldots,\ h(\alpha_k)) + 1$$

Now we define different notions of inference for the cases when the minimal height of the inference is equals 0 and 1. The definitions will vary depending upon whether deduction is from a set, a list, or a sequence of premises, is defined, and on whether induction on one or two variables is used. Each of the definitions of inference well be denoted according to the scheme **DI nTm**[1] , where **n** = 1, 2 is the maximum height of the inference, **T** = =**S, L, Seq**[2] and **m** = 1, 2 shows whether the notion is introduced by induction on one or two variables." (see [Smirnov *1972*, 52–53].)

Let us start with the definition **DI 0L2**.

"Definition **DI 0L2**:

1. If A is an axiom, then $\langle\langle\ \rangle A\rangle$ is an inference from the empty list of premises.

2. $\langle A\rangle$ is an inference from the list of premises $[A]$.

3. If α_1 is an inference of the formula A_1 from the list of premises Γ_1, $\ldots$, α_k is an inference of formula A_k from the list of premises Γ_k and E is directly deducible from formulae $A_1, \ldots, A_k$, then $\langle\alpha_1, \ldots, \alpha_k, E\rangle$ is an inference from the list of premises $\Gamma_1, \ldots, \Gamma_k$." (see [Smirnov *1972*, 53–54].)

"Definition **DI1L2**:

1. If A is axiom, then $\langle\langle\ \ \rangle A\rangle$ is an inference from empty list of premises.

2. If the formula E is directly deducible from $A_1, \ldots, A_k$, then $\langle\langle A_1\rangle, \ldots, \langle A_k\rangle E\rangle$ is an inference from the list of premises $[A_1, \ldots, A_k]$.

3. If α_1 is an inference of the formula A_1 from the list of premises Γ_1, $\ldots$, α_k is an inference of formula A_k from the list of premises Γ_k and E

[1] The letters "**DI**" are used here for the first two letters of Smirnov's abbreviation mean simply: "*Vyvod Deduktivnyj*" (deductive inference). — *Translator's note*

[2] Here "**S**" means "set (of premises)", "**L**" — "list (of premises)" and "**Seq**" — "sequence (of premises)". Naturally, Smirnov uses different letters according to Russian transcriptions of the words. — *Translator's note.*

is directly deducible from formulae $A_1, \ldots, A_k$, then $\langle \alpha_1, \ldots, \alpha_k, E \rangle$ is an inference from the list of premises $\Gamma_1, \ldots, \Gamma_k$." (see [Smirnov *1972*, 53–54].)

The definitions **DI0Seq2** and **DI1Seq2** can be obtained by replacement of each word "list" by the word "sequence" and a combination of lists by a combination of sequences in the definitions **DI 0L2** and **DI1L2** accordingly.[3]

According to the table presented in [Smirnov *1972*, 76] and comments given at [Smirnov *1972*, 75], eight theorems which are directly concerned with relevant logic hold. Some of these theorems could better be called "meta-theorems".

"Theorem 5. "Deduction theorem of the form

$$\frac{A, \Gamma \mid B}{\Gamma_A \vdash A \supset B^{\,4}}$$

holds with respect to **DI 0L2** and the system **H** with the single rule of *modus ponens* if and only if the following formulae are provable in **H**:

1. $A \supset A$

2. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

3. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

4. $(A \supset (A \supset B)) \supset (A \supset B)$"

Replacement of "**DI 0L2**" by "**DI 0L2**" and the reference to the deduction theorem of the form

[3] For the definitions of a sequence of formulae and a list of formulae; see [Smirnov *1972*, 42–44].

[4] Γ_A is a result of elimination of some occurences of A (probably, all or none of them) in the list of formulae Γ; the same notation is used for consequences of formulae.

$$\frac{A, \Gamma \mid B}{\Gamma_A \mid A \supset B} \qquad \begin{matrix} - \\ - \end{matrix}$$

by the reference to the deduction theorem of the form

$$\frac{\Pi, A, \psi \mid B}{\Pi_A, \Psi_A \mid A \supset B} \qquad \begin{matrix} - \\ - \end{matrix}$$

for consequences of premises in the formulation of this theorem gives another theorem from [Smirnov *1972*].

It is known that implicative the fragment of the system **R** can be formalized by means of the implicative syllogistic system $\mathbf{R}_{\supset}$ with the single rule of inference *modus ponens* and axioms 1–4. Therefore, due to the theorems quoted above, deduction theorems of the forms

$$\frac{A, \Gamma \mid B}{\Gamma_A \mid A \supset B} \qquad \begin{matrix} - \\ - \end{matrix}$$

and

$$\frac{\Pi, A, \psi \mid B}{\Pi_A, \Psi_A \mid A \supset B} \qquad \begin{matrix} - \\ - \end{matrix}$$

hold for $\mathbf{R}_{\supset}$ with the notions of inference **DI0L2** and **DI0Sec2** respectively.

Another interesting theorem is

"Theorem 7. "Deduction theorem of the form

$$\frac{A, \Gamma \mid B}{\Gamma_A \vdash A \supset B} \qquad \begin{matrix} - \\ . \end{matrix}$$

holds with respect to **DI 0L2** and the system **H** with the single rule *modus ponens* if and only if the following formulae are provable in **H**:

1. $A \supset A$

2. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

3. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$" (see [Smirnov *1972*, 69]).

Replacement of "**DI 0L2**" by "**DI 0Sec2**" and the scheme

$$\begin{array}{c} A, \Gamma \mid B \qquad - \\ \hline \Gamma_A \mid A \supset B \qquad - \end{array}$$

by the the scheme

$$\begin{array}{c} \Pi, A, \psi \mid B \qquad - \\ \hline \Pi, \Psi_A \mid A \supset B \qquad - \end{array}$$

in the formulation of this theorem gives another theorem proved in [Smirnov *1972*]. Therefore, deduction theorems of the forms

$$\begin{array}{c} A, \Gamma \mid B \qquad - \\ \hline \Gamma_A \mid A \supset B \qquad - \end{array}$$

(when **DI0L2** is used) and

$$\begin{array}{c} \Pi, A, \psi \mid B \qquad - \\ \hline \Pi, \Psi \mid A \supset B \qquad - \end{array}$$

(when **DI0Sec2** is used) hold for $\mathbf{R}_{\supset}$ with the single rule of inference *modus ponens* in which formulae 1–3 are provable.

"Theorem 6. "Deduction theorem of the form

$$\begin{array}{c} A, \Gamma \mid B \qquad - \\ \hline \Gamma_A \vdash A \supset B \qquad . \end{array}$$

holds with respect to **DI1L2** and the system **H** with single rule *modus ponens* if and only if the following formulae are provable in **H**:

1. $(A \supset B) \supset (A \supset B)$

2. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

3. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

4. $(A \supset (A \supset B)) \supset (A \supset B)$" (see [Smirnov *1972*, 69]).

Replacement of "**DI1L2**" by "**DI Sec2**" and

$$\frac{A, \Gamma \vdash B}{\Gamma_A \vdash A \supset B}$$

by

$$\frac{\Pi, A, \psi \vdash B}{\Pi_A, \Psi_A \vdash A \supset B}$$

in the quotation gives another theorem from [Smirnov *1972*].

It follows from the last two theorems that deduction theorems

$$\frac{A, \Gamma \vdash B}{\Gamma_A \vdash A \supset B}$$

(when **DI1L2** is used) and

$$\frac{\Pi, A, \psi \vdash B}{\Pi_A, \Psi_A \vdash A \supset B}$$

(when **DI1Sec2** is used) hold for $\mathbf{R}_{\supset}$ and each of its subsystems with single rule of inference *modus ponens* in which formulae 1–4 are provable.

"Theorem 8. "Deduction theorem of the form

$$\frac{A, \Gamma \mid B \qquad -}{\Gamma_A \vdash A \supset B}$$

holds with respect to **DI1L2** and the system **H** with single rule *modus ponens* if and only if the following formulae are provable in **H**:

1. $A \supset A$

2. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

3. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$" (see [Smirnov *1972*, 71]).

Replacement of "**DI1L2**" by "**DI1Sec2**" and the scheme

$$\frac{A, \Gamma \mid B \qquad -}{\Gamma_A \mid A \supset B \qquad -}$$

by the the scheme

$$\frac{\Pi, A, \psi \mid B \qquad -}{\Pi, \Psi \mid A \supset B \qquad -}$$

in the formulation of this theorem gives another theorem proved in [Smirnov *1972*].

It follows from the last two theorems that deduction theorems

$$\frac{A, \Gamma \mid B \qquad -}{\Gamma_A \mid A \supset B \qquad -}$$

(when **DI1L2** is used) and

$$\frac{\Pi, A, \psi \mid B \qquad -}{\Pi, \Psi \mid A \supset B \qquad -}$$

(when **DI1Sec2** is used) hold for $\mathbf{R}_{\supset}$ and each of its subsystems with the single rule of inference *modus ponens* in which formulae 1–3 are provable.

2. Absolute predicate calculus

V. A. Smirnov constructed and investigated a logical system which he called the "absolute calculus".

We reproduce two Hilbert-type versions of the calculus which, for convenence, will be referred to as **HfA** and **HA**. The language of **HfA** is standard first-order language which contains two sorts of individual variables (free and bounded), predicate symbols, propositional constants $\wedge$, $\vee$, $\supset$, $\neg$ and **f**. The language of **HA** is the language of **HfA** without the propositional constant **f**.

Axioms of **HfA** are the following schemes of formulae:

1. $(A \supset B) \supset (A \supset B)$

2. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

3. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

4. $(A \supset (A \supset B)) \supset (A \supset B)$

5. $A \wedge B \supset A$

6. $A \wedge B \supset B$

7. $(C \supset A) \wedge (C \supset B) \supset (C \supset A \wedge B)$

8. $A \supset A \vee B$

9. $B \supset A \vee B$

10. $(A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C)$

11. $\neg A \supset (A \supset \mathbf{f})$

12. $(A \supset \mathbf{f}) \supset \neg A$

13. $\forall x\, F_x^{\,w}\, Aw \supset F_t^{\,w}\, Aw$

14. $\forall x\, F_x{}^w\, (C \supset Aw) \supset (C \supset \forall x\, F_x{}^w\, Aw)$, where C does not contain w.

15. $F_t{}^w\, Aw \supset \exists\, x\, F_x{}^w\, Aw$

16. $\forall x\, F_x{}^w (Aw \supset C) \supset (\exists\, x\, F_x{}^w\, Aw \supset C)$, where C does not contain w.

Here and everywhere below $F_x{}^w\, Aw$ and $F_t{}^w\, Aw$ are the results of correct substitution of w by x and t respectively in Aw.

There are tree rules of inference in **HfA**:

$$\frac{A \quad A \supset B}{B} \qquad (modus\ \ ponens)$$

$$\frac{Aw}{\forall x\, F_x{}^w\, Aw} \text{ (rule of generalization)}$$

(see [Smirnov *1972*. 87–88]).

$$\frac{A \quad B}{\mathrm{A} \wedge \mathrm{B}} \text{ (rule of introduction of conjunction)}$$

Calculus **HA** contains 15 axioms. Fourteen of them are axioms 1–10 and 13–6 of **HfA**, and the fifteenth is

$$(A \supset \neg B) \supset (B \supset \neg A)$$

(see [Smirnov 1972, 205–206]).

Rules of inference of **HA** are those of **HfA** and the definition of an inference is as usual. It can be shown that for every formula A of the language of **HA**, A is provable in **HA** if and only if A is provable in **HfA**.

But what is the connection between **HA** and relevant logic? It can be proved that:

1) the calculus **HA** is a subsystem of the first-order version **RQ** of Anderson-Belnap's system **R**,

and

2) implicative fragment of **HA** coincides with implicative fragment of **R**.

We need the following version of the definition of strong inference for the systems **HA** and **HfA**. It is a slight modification of the definition given at [Smirnov *1972*, 205-206].

1. If E is an axiom, then $\langle\langle\ \rangle\ E\rangle$ is an inference from the empty sequence of premises.

2. $\langle E\rangle$ is an inference from the sequence of premises E.

3. If α is a n inference from the sequence of premises Γ, A is the last formula of α, β is an inference from the sequence of premises Δ and $A \supset B$ is the last formula of β, then $\langle \alpha\, \beta\, B\rangle$ is an inference from the sequence of premises Γ, Δ.

4. If α is a n inference from the sequence of premises Γ, Aw is the last formula of α and w does not occur in any premise, then $\langle \alpha\ \forall x\, F_x{}^w Aw\rangle$ is an inference from the sequence of premises Γ.

5. If α and β are inferences from empty sequences of premises, A and B respectively are their last formulae, then $\langle \alpha\, \beta\ A{\wedge}B\rangle$ is an inference from the empty sequence of premises.

The standard notation $\Gamma \vdash_{\mathbf{HA}} A$ will mean that there is an inference of A from the sequence of premises Γ definrd above. Similarly for the **HfA**.

It can be shown that the formula A is provable in **HA** (respectively, in **HfA**) if and only if there is an inference of A from the empty sequence of premises in **HA** (respectively, in **HfA**). Really, it is easy to prove by induction on the length of the inference of A in **HA** (respectively, in **HfA**) that $\Gamma \vdash_{\mathbf{HA}} A$ (respectively, $\Gamma \vdash_{\mathbf{HfA}} A$) holds. The proof in the opposite direction is carried out by means of the following lemma:

If $\Gamma \vdash_{\mathbf{HA}} A$ (respectively, $\Gamma \vdash_{\mathbf{HfA}} A$), then the formula $\Gamma^{\wedge} \supset A$ is provable in **HA** (respectively, in **HfA**).

Here $\Gamma^\wedge \supset A$ is A when Γ is empty , $B \supset A$ when Γ is formula B and $(\ldots (B_1,, \wedge B_2) \wedge \ldots \wedge B_n) \supset A$ when Γ is $B_1, , ,B_2. , \ldots , B_n$ and $n >$ 1. This lemma can be proved by induction on the length of the inference of A from the sequence of premises Γ in **HA** (respectively, in **HfA**).

V. A. Smirnov presented his absolute system in the forms of the sequent calculi and the system of natural deduction as well. We shall deal with one of them, the sequent calculi **SLA** constructed in terms of sequences (instead of lists) of formulae. This insignificant modification of Smirnov's version is used for an equivalence of **SLA**, and **HA** will be proved without consideation of any other types of these systems.

Sequent the expression $\Gamma \rightarrow \Theta$, where Γ is a sequence (probably, empty) of a formulae of **HA** and Θ is a formula of **HA** (or empty expression).

The calculus **SLA** contains:

Basic sequent (axiom) $A \rightarrow A$

Logical rules of inference:

$$\frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B}$$

$$\frac{\Gamma \rightarrow A \qquad B, \Delta \rightarrow \Theta}{A \supset B, \Delta, \Gamma \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow A \qquad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}$$

$$\frac{A, \Gamma \rightarrow \Theta \quad \text{or } B, \Gamma \rightarrow \Theta}{A \wedge B,\ \Gamma \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow A \quad \text{or} \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}$$

$$\frac{A, \Gamma \rightarrow \Theta \qquad B, \Gamma \rightarrow \Theta}{A \vee B, \Gamma \rightarrow \Theta}$$

$$\frac{A, \Gamma \rightarrow}{\Gamma \rightarrow \neg A}$$

$$\frac{\Gamma \rightarrow A}{\neg A, \Gamma \rightarrow}$$

$$\frac{\Gamma \rightarrow Aw}{\Gamma \rightarrow \forall x A x} \ (\mathrm{I}\forall\mathrm{R})$$

$$\frac{\Gamma \rightarrow At}{\Gamma \rightarrow \exists x A x}$$

$$\frac{Aw, \Gamma \rightarrow \Theta}{\exists x A x, \Gamma \rightarrow \Theta} \ (\mathrm{I}\forall\mathrm{L})$$

$$\frac{At, \Gamma \rightarrow \Theta}{\forall x A x, \Gamma \rightarrow \Theta}$$

Structural rules of inference:

$$\frac{\Delta, A, B, \Gamma \rightarrow \Theta}{\Delta, B, A, \Gamma \rightarrow \Theta} \text{ (permutation)}$$

$$\frac{A, A, \Gamma \rightarrow \Theta}{A, \Gamma \rightarrow \Theta} \text{ (contraction)}$$

$$\frac{\Gamma \rightarrow M \quad \Delta_1, M, \Delta_2 \rightarrow \Theta}{\Delta_1, \Delta_2, \Gamma \rightarrow \Theta} \text{ (cut)}$$

The rules (I$\forall$R) and (I$\forall$L) are regulated by the usual restrictions: w does not occur in their conclusions (see [Smirnov *1972*, 184]). A proof in **SLA** having the form of tree is defined in the standard way.

By induction on the height of the inference in **HA** the following lemma can be proved.

Lemma 1. If $\Gamma \vdash_{\mathbf{HA}} A$, then the sequent $\Gamma \rightarrow A$ is provable in **SLA**.

Next consider:

Lemma 2. If the sequent $\Gamma \rightarrow A$ is provable in **SLA**, then $\Gamma \vdash_{\mathbf{HA}} A$; if $\neg A, \Gamma \rightarrow$ is provable in **SLA**, then $\Gamma \vdash_{\mathbf{HA}} A$.

This lemma can be verified by induction on the height of the proof in **SLA**. Particularly, in the case when the end of proof in **SLA** is

$$\frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B},$$

a deduction theorem of the type

$$\frac{A, \Gamma \vdash_{\mathbf{HA}} B}{\Gamma \vdash_{\mathbf{HA}} A \supset B}$$

is to be used.

It follows directly from these two lemmas that **SLA** and **HA** are deductively eqivalent: $\Gamma \vdash_{\mathbf{HA}} A$ if and only if $\Gamma \to A$ is provable in **SLA**.

When sequent calculi are considered (especially in the aspect of proof search), the question about cut-elimination is extremaly significant. Gentzen's initial proof of the cut-elimination theorem for classical and intuitionistic sequent calculus is based on the following two propositions:

• any proof which contains applications of cut can be transformed in a proof of the same sequent only by replacement of each application of cut by application of rule called "mix"

$$\frac{\Gamma \to \Psi, M \qquad M, \Delta \to \Theta}{\Delta^*, \Gamma \to \Theta, \Psi^*} \text{ (mix)}$$

where Δ^* and Ψ^* are the result of deleting of every occurence of M from Ψ and Δ;

• any proof which contains applications of (mix) can be transformed into a proof of the same sequent which does not contain applications of (mix).

But the point is that these propositions do not hold for **SLA**, for their verification is based on application of structural rules of weakening

$$\frac{\Gamma \to \Theta}{\Gamma \to \Theta, A}$$

$$\frac{\Gamma \to \Theta}{A, \Gamma \to \Theta}$$

which are not included in **SLA**. Thus Gentzen's initial method is not directly applicable to **SLA**.

V. A. Smirnov [Smirnov *1972*, 185] proposed a new method of cut-elimination and used it to prove the cut-elimination theorem for his absolute calculus. The key idea of the method is to replace cut by generalized mix: "... each cut can be replacet by generalized mix

$$\frac{\Gamma \rightarrow M \quad \Phi,\ M, \Delta \rightarrow \Theta}{\Delta_m\,, \Gamma\,, \Phi_m \ \rightarrow \Theta} \text{ (GM)}$$

where Δ_M, and Φ_M are the result of deleting of some (probably, all or none) occurences of M from Δ and Φ; M will be called a formula of generalized mix.

It is easy to see that cut is a particular case of generalized mix (when $\Delta_M = \Delta$ and $\Phi_M = \Phi$) and vise versa a generalized mix can be replaced by a figure constructed of contractions, permutations and cut."

Thus to prove the cut-elimination theorem for **SLA**, it is sufficient to show that for any proof in **SLA** obtained only by replacement of cut by generalized mix , there is a proof with the same bottom sequent which does not contain applications of generalized mix. This can be verified by induction of a number of applications of generalized mix in a proof in the callculus which is obtined from **SLA** by replacement of cut by generalized mix. The following lemma proved in [Smirnov *1972*] holds.

"Principal lemma. A proof with the only generalized mix at the end can be transformed in a proof with the same bottom sequent wich does not contain generalized mix." (see [Smirnov *1972*, 185])[5]

This is sufficient to prove the cut elimination theorem for **SLA** which allowed to show dicidability of propositional part of **HA**.[6]"

The method of cut-elimination proposed by V. A. Smirnov can be applied to an analysis of a wide class of sequent calculi without rules of weakening. Various examples of such calculi can be found in the field of relevant and paraconsistent logics.

Finally, let us consider the sequent calculus **SLfA**. It can be obtained from **SLA** when the rules of inference for negation are replaced by the following two rules:

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[5] Certainly only proofs which do not contain cut are considered here.

[6] See [Popov *1977*], where, taking to Smirnov's advice, the author applied the method used in [Belnap and Wallace *1965*].

$$\frac{A, \Gamma \to \mathbf{f}}{\Gamma \to \neg A}$$

$$\frac{\Gamma \to A}{\neg A, \Gamma \to \mathbf{f}}$$

SLfA is deductively equivalent to **HfA**: $\Gamma \vdash_{\mathbf{HfA}} A$ if and only if $\Gamma \to A$ is provable in **SLfA**. The cut-elimination theorem holds for **SLfA** as well. It can be proved in the following way. Firstly, we show that the sequent $\Gamma \to A$ is provable in **SLA** if and only if it is provable in **SLfA**. To prove this it is sufficient to prove the following two propositions.

Proposition 1. If the sequent $\Gamma \to A$ is provable in **SLA**, then $\Gamma \to A$ is pdovable in **SLfA**.

This follows from

Lemma 1. If the sequent $\Gamma \to \Delta$ is provable in **SLA**, then $\Gamma \to \Delta$ is provable in **SLfA**, if Δ is non-empty, and $\Gamma \to \mathbf{f}$ is provable in **SfLA** if Δ is empty.

This lemma can be verified by induction on the height of the inference of $\Gamma \to \Delta$ in **SLA**.

Proposition 2. If the sequent $\Gamma \to A$ is provable in **SLfA**, then $\Gamma \to A$ is provable in **SLA**.

This proposition follows from the cut-elimination theorem for **SLfA** and Lemma 2.

Lemma 2. If **f**-free sequent[7] $\Gamma \to \Delta$ is provable in **SLfA** without cut, then $\Gamma \to \Delta$ is provable in **SLA**, if Δ differs from **f**, and $\Gamma' \to \neg A$ is provable in **SLA**, if $\Gamma \to \Delta$ is $A, \Gamma' \to \mathbf{f}$.

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[7] Sequent $\Gamma \to \Delta$ of the calculus **SLfA** is **f**-free if any formula in Γ does not contain a subformula **f** and Δ is **f** or a formula which does not contain a subformula **f**.

This lemma can be proved by induction on the height of the cut-free inference of the **f**-free sequent $\Gamma \to A$ in **SLfA**.

Now we are in position to prove that for any formula A in the language of **HA**, A is provable in **HA** if and only if A is provable in **HfA**. Really, this is a direct consequence of the following sequence of equivalences:

1) For any formula A: A is provable in **HA** if and only if $\Gamma \vdash_{\mathbf{HA}} A$;

2) For any formula A: $\Gamma \vdash_{\mathbf{HA}} A$ if and only if the sequent $\to A$ is provable in **SLA** (this follows from equivalence between **SLA** and **HA**);

3) For any formula A from the language of **HA**: sequent $\to A$ is provable in **SLA** if and only if $\to A$ is provable in **SLfA** (this follows from equivalence between **SLA** and **SLfA**);

4) For any formula A: sequent $\to A$ is provable in **SLfA** if and only if $\Gamma \vdash_{\mathbf{HfA}} A$ (this follows from equivalence between **SLfA** and **HfA**);

5) For any formula A: $\Gamma \vdash_{\mathbf{HfA}} A$ if and only if A is provable in **HfA**.

A few words in conclusion

Almost 25 years passed since V. A. Smirnov's book *Formal Deduction and Logical Calculi* was published in Russian. Now it is one of the most often quoted work in Russian logic. We believe that the problems, ideas and methods formulated in this book still remain actual and stimulating for current logical investigations. We tried to show their non-trivial character in this paper and one can judge this part of Smirnov's logical work himself.

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