John P. Cleave, *A Study of Logics*. Oxford Logic Guides 18, Clarendon Press, Oxford, 1991. xiii + 417 pp.

Reviewed by

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There is much to learn and to enjoy in Cleave's *A study of Logic*. It is a rich source of mathematical results useful in the study of systems of formal logic, and a good introduction to a particular view of formal logic. In this review article, I will give an overview of the material covered in the book, placing it in the context of other work in the area; then I shall comment more critically on some of it. None of these criticisms cast doubt on Cleave's general project, but rather, indicate directions in which it could be pursued, and made more general.

1. Scope.

In this book Cleave seeks to find unity in the diverse landscape of systems of logic. The unifying theme of the work is the claim that the consequence relation (between sets of formulæ and formulæ) is central in the study of systems of logic. So, the work stands firmly in the tradition of Tarski's original work in axiomatising consequence relations and developing logic from that standpoint [Tar56, Chapter 5].

Cleave starts with a discussion designed to motivate the plural nature of logic. He gives a clear exposition of the place of logical systems in science. He is critical of positivism, which takes scientific theories to be simple applied axiomatic theories. On the contrary, Cleave argues, scientific theories are much richer and more varied, involving an observation language (which he takes to be naturally associated with a three-valued logic, because not all observation is definite), correlative definitions which relate observation to theoretical entities, and an abstract calculus which gives structure to the theoretical entities. At this abstract stage many different logical calculi may be useful. Theories of necessity or many-valued logics (or formalisms such as quantum logic) may be important in manipulating theoretical entities.

The author then proceeds to cover classical logic in Chapter 2. Standard results are proved, covering truth tables, a Hilbert style proof theory together with soundness and completeness results, normal forms, an excursion into some topics from logic programming, compactness, a Gentzen formulation — each for both propositional and predicate logic — Herbrand's theorem and Skolem functions. It is a brisk introductory chapter containing standard
classical results.

In Chapter 3 Cleave takes up some of these considerations and generalises them to motivate abstract consequence relations. For these he begins by defining *prelogics* which are simply collections of formulæ together with a function C defined from sets of formulæ to sets of formulæ. From a set of formulæ X one gets $C(X)$, the set of consequences of X. As already noted, this approach is rooted in Tarski's pioneering work. However, instead of moving straight to conditions needed to make prelogics *logics*, Cleave spends time studying homomorphisms between prelogics, and other useful notions that apply to prelogics (and hence logics). These concepts are used when defining mappings from languages (which are prelogics) into algebras (which are logics).

We then see the conditions for a prelogic to be a logic: C must satisfy $X \subseteq C(X)$, $C(C(X)) \subseteq C(X)$, and if $X \subseteq Y$ then $C(\mathrm{X}) \subseteq C(Y)$. Cleave departs from Tarski by allowing logics to fail to be compact, and by not demanding that each logic have a contradictory formula f, where $C(f)$ is the set of all formulæ. Next, we survey compactness, finite logics, the relationships with Gentzen systems and ways to compare abstract logics with each other. This approach is quite general and the algebraic results shed light on abstract systems.

However, logics are not very useful if the formulæ have no structure of their own. In Chapter 4 the discussion moves to logical operations. Different conditions are given for the relationships between the connectives (implication, negation, conjunction and disjunction) and consequence. For example, a conditional '$\supset$' is *normal* when $B \in C(X \cup \{A\})$ if and only if $A \supset B \in C(X)$.

Chapter 5 is a general discussion of order and lattices. Instead of quickly moving the discussion to partially ordered sets, Cleave lingers on quasi-ordered sets. This is helpful because formulæ in a logic are quasi-ordered by entailment, but usually this ordering is not a partial order ($A \wedge B$ entails $B \wedge A$ and *vice versa*, but these are not identical formulæ). After this, the discussion moves to partially ordered sets and lattices, completeness, distribution, complementation (of various kinds) and finally, Boolean algebras. These algebras serve as the truth values for logics.

In Chapter 6 the hard work pays off. We can construct consequence relations (from Chapter 3) naturally from lattices (Chapter 5) by defining consequence in terms of the lattice ordering. Then, given a language, a consequence relation on it can be defined by a homomorphism from the language into the lattice, and lifting the lattice consequence relation into the language. (Commonly construed as an evaluation of the formulæ.) This section is particularly neat, and we see different threads coming together into a coherent whole. Then the discussion moves to particular applications, such as quantum logic and temporal logics. Although the discussion here is worthwhile for its presentation, none of the results are particularly novel.

Chapter 7 brings with it a discussion of lattices relevant to the three-valued logic Cleave finds important for the observational stage in science. Quasi-Boolean algebras (QBAs) are distributive lattices with an involuted negation $'$. A QBA is *normal* if $x \cap x' \leq y \cap y'$. So, each of Łukasiewicz's logics are normal QBAs. This chapter contains a clear discussion of ways to generate QBAs from topological spaces, their relation to Boolean algebras, and two structure theorems. One shows that every non-trivial QBA is a subdirect power of the four element QBA Q_4. (This algebra has four elements, $\{0, n, m, 1\}$ ordered with $0 < n, m < 1$, $n \nless m$, and $m \nless n$. Negation maps 0 to 1 and back, and n and m are fixed points.) The other shows that every non-trivial normal QBA is a subdirect power of the three element normal QBA N_3 (the $\{0, n, 1\}$ fragment of Q_4). This result is important, because it shows that N_3 and Q_4 are paradigmatic normal QBAs and QBAs respectively. All other normal QBAs and QBAs are made up of copies of N_3 and Q_3 respectively. The chapter ends with a discussion of Körner's work on inexact classes and empirical continuity showing how the three-valued logic N_3 naturally arises from the observational stages of scientific practice.

Chapter 8 focuses on N_3 and Q_4. Cleave proves normal form theorems, Gentzenisations, and other interesting results. This chapter contains a rich mine of results about these logics.

Chapter 9, on relevance, is more controversial because Cleave rejects most of the work done on relevance in the tradition of Anderson and Belnap, choosing instead to analyse relevance syntactically by taking irrelevant subformulæ of a formula to be those formulæ which result in no change to the truth value of a proposition when replaced by their negation (or omitted altogether). This produces a number of different kinds of relevant implication and entailment. Then in the final section, Cleave gives a semantic account of relevant implication, using considerations of propositions having different subject matters. Cleave proves that the logic arising from these semantic considerations is simply the logic of Q_4.

In the penultimate chapter, Cleave discusses intuitionistic logic under the guise of *effective logic*. Effective logic encodes reasoning about consequence in our logics. Cleave motivates Gentzen rules for intuitionistic logic in the context of provability. He then goes on to discuss other interpretations of intuitionistic logics. In the final chapter, Cleave motivates modal logics, not by way of looking at possible worlds or Lewis' systems, but by mapping provability in a logic into the syntax of the logic itself. Modality is a matter of logic, not primarily of possible worlds. This chapter ends with a sketch of a deontic logic defined in these terms.

2. Evaluation.

Cleave's work is an excellent chart of a great deal of the logical landscape. The unifying vision is important, and it helps tie together a number of important themes in recent work in logic. However, the work is not *quite* as comprehensive as it could be. Cleave misses out on a number of important lines of current research. Most of my evaluation will consist of showing how this blind spot biases his approach, and offering suggestions to how further work in the area could proceed to fill out Cleave's approach in order to give a truer map of the logical landscape.

Is There But One Truth? Cleave gives us a natural way of extracting a consequence relation from a lattice of truth values. Consequence is naturtally modelled by the ordering in the lattice. It is also natural (or at least current dogma) to assume that the tautologies in a system are the consequences of the empty set. Cleave agrees with this. In the context of a lattice this means that the only tautologous truth value is the maximal one, or to use common parlance, only one truth value is designated. This result is an immediate consequence of the definition of tautologies as the consequnces of the empty set, and the identification of consequence with lattice ordering. However, as Cleave himself remarks on page 194, it is perfectly permissible to have more than one designated value in a lattice.

Even though this restriction to one designated value does eliminate many systems of logic from his discussion (such as linear logic, relevant logics and paraconsistent logics) this is not very significant because Cleave does not place much emphasis on designated values among truth values or tautologies among formulæ. The consequence relation is more important. Each of these logics has a perfectly acceptable consequence relation, modelled by the ordering on its lattice of truth values. If we take tautologies to be consequences of the empty set, then these logics have no tautologies: but as Cleave rightly points out, a logic is determined by its consequence relation, not by its tautologies.

However, this tacit dogma of 'only one true truth value' seems to blind Cleave to other possibilities for interpreting logics. For example, he discusses Q_4 in some detail. Those who work in relevant logic will recognice this as the extensional fragment of the logic **BN4** (Bel77, MGB84, Sla91) though Cleave never mentions any of these references. The difficulty Cleave has with interpreting Q_4 comes with the notion of *refinement*. Logics like N_3 can be equipped with an ordering of refinement on truth values which can be interpreted as the gaining of information. In N_3, n refines to either 0 or 1. Something that is neuter can become either false or true as more information comes in, but once there, it will never return to being neuter. Cleave takes it that the natural refinement on Q_4 has n and m both refining to 0 and 1. Both n and m are to be thought of as 'neuter' truth values which can be resolved to one of the classical values as information comes in (page 265). Then Cleave notes that this is not an acceptable definition because refinement is not preserved over the connectives. For example, $n \vee m = 1$; but we may have n and m both refining to 0, in which case the disjunction also changes to 0. However, a definite truth ought not be refined to a falsehood as information comes in. Cleave takes this to be a problem, and one to which he has no solution.

There *is* a readily available solution, and it has been in the literature for quite some time (Dun76, Bel77, Fit91, and many others). There is only one neuter value, n. The other value m, is the *maximum* of the refinement ordering. Both 0 and 1 can refine to m as yet more information comes in. Given this refinement ordering, all of the connectives behave as they ought, just as in the three-valued case. Unfortunately, Cleave does not consider this possibility. One reason may be that this approach only makes sense when m is another designated value. If refinement is the addition of information, m must be both true and false, so it is at least true, and so, designated.

The lattice Q_4 with this interpretation has an obvious model in the logic of inexact classes. Cleave has already shown us that inexact classes lead to N_3 when we allow extensions and anti-extensions of classes to be mutually exclusive but not necessarily inclusive, as often happens with predicates. However, this is asymmetric. It is quite possible for extensions and anti-extensions to be neither inclusive nor *exclusive*. Perhaps there are problem cases for predicates that manage to fall both into the extension and the anti-extension of a predicate. (If we have two independent tests, one for membership, and another for non-membership of some class, it is just as possible that something will manage to pass both tests as something failing both tests.) Given this situation, the natural logic is Q_4, with values interpreted as being true, false, neither and *both*. This is one way that Cleave's approach can be broadened.

Where are the Conditionals? Cleave reminds us that logic is about consequence. But it is one thing to be consequence oriented instead of being tautology oriented, and it is another to encode consequence into the language itself by way of conditionals. And surprisingly, for one who takes consequence to be central, conditionals do not feature greatly in Cleave's work. They appear in the section on connectives and consequence operators, but they disappear from scene in the section on QBAs. Admittedly, they are of little importance for the results Cleave intends to cover, but they are a source of much interest in themselves. If consequence is the heart of logic, then conditionals — the object language renderings of consequence — are vital. For example, each of Łukasiewicz's logics is a normal QBA. Their $\langle \wedge, \vee, \neg \rangle$ fragments are identical; Cleave has shown that they are all subdirect powers of the three valued logic N_3 (page 243), so no matter how large they are, they add no extra logical structure over and above the three-valued algebra. This result disappears once conditionals are added. The logics with conditionals are much richer and more varied than those without.

One way that this richness is exhibited is to observe that Łukasiewicz's conditionals don't qualify as conditionals in Cleave's account of Chapter 4. There it is required that for $\rightarrow$ to be a conditional, we must have $B \in Cn(\{A, A \rightarrow B)$. The conditionals in Łukasiewicz's logics do not satisfy this (for example, $0 \notin Cn(\{n, n \rightarrow 0\}) = \{n, 1\}$), but they nonetheless are genuine conditionals. More work has to be done to fit these into the general work on consequence.

Relevance. In the section on relevance, Cleave admits that he has little time for the work done by Anderson and Belnap. He argues that their systems 'certainly have no relevance to mathematics' (page 286). There is a kernel of truth behind this strong claim. Current mathematical practice pays little attention to matters of relevance. However, this doesn't mean that relevance is irrelevant to mathematics. Meyer has shown that there are arithmetics formulated in relevant logics that both prove as much as classical arithmetic, yet have finite non-triviality proofs (MM84). This seems enough to secure the claim that these logics are at least of some relevance to mathematics.

Cleave also states that relevant logics lack concrete applications (page 286). This depends on how far you look. A simple search shows that some applications are: reasoning under inconsistent information, characterising relevant properties of objects, and many others (Bel77, Dun87, Sla91). In addition, relevant logics are a part of the whole family of substructural *logics* which include linear logic (much used by computer scientists) the Lambek calculus (with applications to theoretical linguistics) and others. The area is full of applications.

Cleave argues that work in relevant logics suffers from being based on a proof-theoretic or axiomatic approach rather than semantics. Taking this to apply to the history of the formulation of relevant logics, this is true. But this is also true of the early history of most other systems, such as intuitionistic logic and modal logics. Adequate semantic structures of relevant logics have been found (Fin74, RM73) and these turn out to have understandable interpretations. Cleave's only references to the relevant literature are an early Anderson and Belnap paper, and *Entailment* volume 1; these contain little work on semantics.

Most ironic is that the little work Cleave does on the semantics of relevance (in terms of distinct subject matters for propositions) is yet another motivation for Anderson and Belnap's work. Cleave shows that the natural semantically motivated logic of relevance is modelled by the QBA Q_4. The resulting logic is Anderson and Belnap's first-degree entailment; not the logic that results from any of Cleave's syntactic considerations from earlier in that chapter. The only difference between Cleave's work here and Anderson and Belnap's is that the latter went on to consider what ought to happen when a relevant conditional is added to the language of the logic. This is an issue that Cleave doesn't consider. So, Cleave's approach is much closer to the work of Anderson and Belnap than he realises.

Is Effective logic really Constructive? The section on effective logic displays what seems to be an inconsistency in methodology. Cleave rightly takes issue at a purely 'postulational' approach to logical theorising. Presumably the problem is with simply starting with what we take to be plausible axioms and then simply working from there, with no heed to semantics. Cleave is right; working just like this is problematic. Even granting that we have a firm grip on particular truths or valid inferences, unless we have a semantic structure associated with the logic, we have no general way to show that an inference is invalid.

It seems that Cleave's section on effective logic fails on just this point. Cleave motivates each of the Gentzen rules for intuitionistic logic for his system of effective logic. He mentions that the proof theory ought to be a single-consequent Gentzen system because it models the consequence relation, which he takes to be naturally understood as a set to formula relation. But this is not enough. It is just as possible to construe consequence as a set to set relation, in which case the resulting effective logic will be classical (recall the difference between Gentzen formulations for intuitionistic and classical logic). Why choose one over another? Do they axiomatise different notions? This is not discussed, yet it is an important problem. It seems that there is no principled way of privileging set-formula consequence over set-set consequence, and if this is done, our effective logic will look decidedly classical. It seems that the way to deflect such an approach must involve explaining why intuitionistic non-theorems such as Peirce's law and excluded middle ought to fail on this interpretation. Ideally, this will be by way of appeal to a related semantic structure such as the Kripke semantics for intuitionistic logic. However, this is not done.

Matters of Style. The reader may get the impression that the production of the book was rushed. There are far too many typographical errors, and beyond this, there are a number of significant mistakes in content. For example the proof of Theorem 2.6 is incorrect as it stands (though the theorem is true nonetheless) and at times, new concepts are used before they are defined, if they are defined at all. Readers should not have to search through the earlier pages of a book to see where new notation is defined.

Finally, the prospective reader ought to be aware that Cleave's exposition is somewhat terse. For example, Theorem 6.19 tells us that if a lattice is finite, then the predicate logic based on that logic is compact. In this section, the theorem is proved, and we then go on to a new topic. We are not told whether this is the best result possible, or whether there are counterexamples to compactness in the infinite case. (There is a counterexample in the infinite case: Łukasiewicz's infinitely valued predicate logic is not compact.) Having such examples would increase the reader's understanding but at the cost of more pages. Cleave has consistently chosen the course of fewer pages. This is not a problem in itself, provided the reader is prepared to do the extra work.

This book is in the series of Oxford Logic Guides, and as expected from Oxford, the book is rich in content and well worth having for logicians of many stripes. Keeping in mind that some of the approaches are not quite as general as we might wish to have, this books is useful for anyone wanting to broaden their horizons of systems of logic, and to obtain useful formal tools for studying these logics.

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