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Brouwer And Griss On Intuitionistic Negation

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Abstract. During the 1940s, a debate about the intuitionistic definition of negation took place between G.F.C. Griss and L.E.J. Brouwer: namely, Griss criticized Brouwer's definition as inconsistent with the intuitionistic framework. In this paper I present firstly Brouwer's definition of negation; then Griss' criticism of it and his proposal of a "negationless" mathematics; finally Brouwer's reply to Griss.

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1. Introduction. In 1944, the first of a series of papers appeared in which the Dutch mathematician George François Cornelis Griss criticized the intuitionistic definition of negation proposed by Luitzen Egbertus Jan Brouwer and put forward his own definition, thus revising the basis of mathematics and logic. Although Griss was not able to develop his initial ideas due to his premature death in 1953 at the age of 55, his work pointed out nevertheless a weak and problematical aspect of Brouwer's theory. Therefore, it is interesting to consider the question with respect to its various components: the original Brouwerian definition, Griss' criticism of this definition and its consequences, Brouwer's reply to the criticism.

2. Brouwer's definition of negation. Brouwerian intuitionism defines mathematics as a mental activity that develops by passing from a (simpler) piece of mental evidence to another (more complex) piece of mental evidence. Therefore, mathematics proceeds autonomously without applying schemes from outside. As for logic, Brouwer accepts its traditional definition as the set of the laws of thought. But, within this framework, logical laws cannot have the role of giving mathematics heuristical schemes. The only meaning they can have is as schemes of linguistic expressions, that is in collecting the regularities present in mathematical expressions. Hence, mathematical reasonings have to correspond to logical assertions. What is needed is a redefinition of the classical meaning of negation: negative assertions can no longer mean the lack of a correspondence, because lacking is not a mental construction.

Brouwer realizes this between 1907 and 1908, probably by reflecting on the laws of logic (he never formulated a systematic approach to logical constants, but considered them individually as he came across them in the topics he was studying). Namely, he explicitly presents the new meaning of negation for the first time in his new interpretation of the law of excluded middle in his paper *De onbetrouwbaarheid der logische principes* [*1908*, 109–110], where he announces that this law no longer held, correcting the position he had assumed on the issue in 1907 in his doctoral thesis. There he first affirmed that logic expresses mathematical reasoning, then [*1907*, 75] stated that:

> While in the syllogism a mathematical element can be discerned, the proposition:
>
> A function is either differentiable or not differentiable says *nothing:* it expresses the same as the following proposition:
>
> If a function is not differentiable, then it is not differentiable.

and finally declared that the three classical laws of identity, non-contradiction and excluded middle hold. In this way an inconsistency arose, because he himself had said that both identity and excluded middle did not reflect mathematical reasoning. It can be supposed that Brouwer became aware of this inconsistency and understood that before judging the validity of logical laws, it was necessary to verify that they referred at least potentially to a mathematical reasoning. Hence, he began to redefine the three laws and negation within them. Since, after redefining their meaning, he realized that the validity of the law of excluded middle could no longer be maintained, Brouwer decided to set forth his new interpretation of only this law [*1908*, 109]:

> Now, the principle of *tertii exclusi:* it claims that every supposition is either true or false; in mathematics this means that for every supposed imbedding [sic] of one system into another, satisfying certain given conditions, the construction aiming to achieve it either concludes its task or comes to a stop because it is impossible to go on.

From the quotation above it can be concluded that negation expresses a construction that comes to a stop, that cannot continue any further. In 1907, at the end of his treatment of reasonings that seem to be hypothetical, Brouwer discussed this kind of construction and pointed out that it is what is usually considered a reasoning that ends in a contradiction [*1907*, 72–73]:

> Here it seems that the construction *is supposed* to be effected, and that a chain of hypothetical judgments is deduced from this hypothesis. But this is no more than apparent: what actually happens is the following: one starts by building a structure which fulfills part of the required relations, thereupon one tries to deduce from these relations, by means of tautologies, other relations, in such a way that finally these new relations, combined with those that had been kept in store, yield a system suitable as a starting point for the construction of the required structure. Only by this construction it will be proved that the original conditions can be fulfilled. 'But', the

> logician will retort, 'it may also happen that during these reasonings a contradiction has turned up between the new conditions and the original ones not yet used [...] Where you announce a contradiction, I simply perceive that the construction *no longer goes,* that the required structure can not be imbedded [sic] in the given basic structure'.

That is why Brouwer, from [*1923a-c*] on, expressed negation as reasoning that leads to an absurdity or, briefly, as an absurdity.

3. Griss' criticism of the definition of negation. Griss bases his criticism solely on the intuitionistic standpoint, which he shares with Brouwer, so as to make the definition of negation consistent with the intuitionistic standpoint. He says that at first he had simply doubted the acceptability of double negation, and only later, through reflection, came to criticize the definition of negation itself as "reasoning carried to an absurdity", that is, as an assertion corresponding to a reasoning that ends in a contradiction. Griss points out the incompatibility of such a meaning with an intuitionistic framework, by considering an example of reasoning carried to an absurdity [*1948a*, 71]:

> On suppose par exemple qu'il satisfasse une fraction à l'équation $x^2 = 2$ et on trouve une contradiction, car pour chaque fraction qu'on substitue à x le premier membre diffère de deux. Faire la supposition qu'une preuve soit donnée, tandis que cette preuve parait être impossible, est incompatible avec le point de départ constructif et évident, car l'existence d'une preuve est identique au fait qu'elle a été donnée.

and comparing it with a reasoning which he considers as intuitionistically acceptable [*1948a*, 71–72]:

> On peut dire p.e.: si un nombre est divisible par 4 il est divisible par 2, puisqu'il existe des nombres qui sont divisibles par 4.

He concludes [*1948a*, 71] that:

> La suppression du raisonnement par négation entraine donc le fait qu'il ne faut pas faire de suppositions à moins qu'il ne soit connu qu'il existe des systèmes mathématiques satisfaisant aux suppositions.

Griss' criticism is grounded in the following reflection: reasoning carried to an absurdity ends in a contradiction; therefore, the hypothesis from which it starts cannot have a mental counterpart, because it is impossible that something true later becomes false. Given that intuitionism considers as mathematical only those reasonings which start with a truth and reach other truths, reasoning carried to an absurdity cannot be included in mathematics.

Griss places this kind of reasoning in what he calls the "premathematical stage" of searching for proof, which he believes to be similar to Bouligand's "synthèse globale." In a first draft of his epistemology appearing in the *Comptes rendus de l'Academie des Sciences,* Bouligand had distinguished the "synthèse globale" from the truly definitive solution of problems, by defining them as Griss does for the premathematical and mathematical stages respectively. In fact, the "synthèse globale" is the "système en expansion et base commune à laquelle les mathématiciens qualifiés, a cet instant, font appel pour aborder les problèmes" and "s'accomode de la *seule absence de contradiction,* ce qui permet d'y raisonner par l'absurd une fois admis le tiers exclu," while "le praticien s'efforcera de *construire* la solution" [Bouligand *1947*, 1747–1748]. The reference to Bouligand may have been Griss' way of saying that he had borrowed from him the idea of the premathematical stage.

4. A new definition of negation. Griss' criticism of reasoning carried to an absurdity requires a redefinition of negation. However, in order to convince mathematicians, Griss [*1955*, 137] also offers a non-philosophical reason for redefining negation: "Le remplacement d'un résultat négatif par un résultat positif est toujours considéré comme une amélioration."

The new meaning that Griss gives to negation consists in a comparison between two already constructed entities and the realization that one of them has more properties than the other (and *vice versa*).

Griss also refers to Bergson's *L'évolution créatrice* in his proposed new definition of negation. In analyzing the meaning of "nothingness," Bergson [*1914*, 298–322] gave a parallel definition of "nothingness" (at the ontological level) and of negation (at the logical-linguistic level). Since thought is always thought of something, nothingness does not in reality exist, but refers to two existence's: that of the missing thing and that of the present thing, and it originates by comparing them and by noting that the characteristics of the present thing are distinct from those of the missing thing. As nothingness is in practice the substitution of one object by another (the presence of an object in place of another desired object), so negation — defined according to Aristotle as contradictory (and not contrary) to affirmation — is the substitution of an affirmative statement by another that is less definite (saying "non-red" gives little information about the precise color of the object). As is evident, all this led Griss to the idea, which he adopted, that "negative" consists in comparing two positive realities and noting a difference between their properties.

The intuitionistic admissibility of such operations is permitted by the original intuition of two-oneness: by grounding both unity and two-ity, it also grounds the possibility of distinguishing between them. As for the operation of comparison in practice, Griss specifies what it consists of in the case in which it occurs between numbers: from the notions of equality and apartness between real numbers it is possible to derive directly those for points (defined as couples of real numbers). On the basis of these latter notions both projective and Euclidean analytic geometries can be constructed. Griss [*1951b-c*] presents some results for the second kind.

The double operation of comparison and noting differences is simple in the case of natural numbers, whereas it is more complicated in other cases. For natural numbers, it consists merely in thinking of them as signs (for instance numerals or bars) and trying to superpose them or put their points into a bijective correspondence. This procedure is insufficient for rational numbers, because fractions like $^1/_1$ and $^4/_4$ — which are concretely different signs — represent the same number. Griss, therefore, introduces the notions of identity/difference and equality/distinguish ability between fractions: two fractions (a, b) and (c, d) are identical if $a = c$ and $b = d$; they are different if at least $a \neq c$ or $c \neq d$; they are equal if $ad = bc$ and distinct if $ad \neq bc$. As regards real numbers (defined as sequences of nested intervals) Griss takes over from Brouwer the notions which subsequently became known as of "equality" and "apartness": two real numbers are equal if for each step k the interval (a_k, b_k) of one of them "covers in the strict sense" the corresponding interval (c_b, d_k) that is, if $a_k < d_k$ and $c_k < b_k$; two real numbers are apart if there is a step k where the interval (a_b, b_k) of one of them "lies outside in the strict sense" the corresponding interval (c_b, d_k) of the other number, that is, if $d_k \leq a_k$ or $c_k \leq b_k$. These notions had appeared for the first time in Brouwer's 1919 paper [Brouwer *1919*, 191], where he refers to points on a plane that are defined as indefinitely proceeding sequences of "λ–squares" such that the inner domain of each is contained in the inner domain of the preceding square. Two points on a plane are said "to coincide," if it is possible to find for each square of the first a square of the second contained in it and *vice versa*; two points on a plane are said "to lie apart" (*örtlich verschieden*) if a square of the first lies outside a square of the second. The term "apartness," as opposed to the idea itself, was introduced by Griss and later taken up by Brouwer.

Griss points out that the change in meaning of negation requires that the theory of species also be revised: the empty species can no longer be accepted since the criticism of Brouwerian negation can be applied to it, that is, it concerns hypothetical properties without any corresponding mental event. Indeed, the empty species is defined by means of a contradictory property. He adds, however, that the empty species can no longer be retained, though it is defined as the intersection of two disjointed species, because it would require the existence of intersection in all cases, which is in contradiction to the requirements of constructivism. As far as concerns relations between species, Griss preserves those of Brouwer, that is equality and deviance: two species are equal if it is possible to find for each element of the first an identical element of the second and vice versa; a species deviates from another, if it is possible to find an element of it that is different from each element of the other, where difference is defined already in advance in a way acceptable from Griss' standpoint. Obviously, he eliminates the relation of difference understood as absurdity of the identity between species.

Griss also contests the definition of spread given by Brouwer. The latter had written (for instance in [*1918*, 150]) that at each step in the construction of a spread there are three possibilities: the sequence is sterilised, or it is continued by attributing to it a mathematical entity or by attributing nothing to it. With the universal tree as the constant starting-point, sterilisation is introduced to form the specific tree that represents each single spread, while the attribution of "nothingness" serves to take into consideration also finite sequences within the spread. Griss stresses the intuitionistic unacceptableness both of the attribution of nothing and of sterilization. Given a countable set A of signs, he proposes the following definition of spread [*1951a*, 193]:

> there is a rule that determines which signs from A (at least one) we may choose as first sign, then after having chosen a first sign, the rule determines which signs of A we have to choose as second signs, etc. Each *infinite* sequence created in this way is an element of the set. But we also admit *finite* sequences as elements of the set, by defining that at any stage there is a possibility to stop further choosing.

It is interesting to note that, after his Cambridge lectures (1946-51), Brouwer modified his description of the spread construction, by eliminating sterilization but retaining the attribution of nothing (see, for instance [Brouwer *1952*, 512]).

5. The revision of logic. From the intuitionistic standpoint, modifications within mathematics cause changes also within logic. In this field, Griss' reference point was not Brouwer, but Heyting, since the latter had elaborated an intuitionistic formal system.

Griss stresses, to begin with [Griss *1948b*, 99–100], that Heyting's interpretation of propositions as problems waiting for a solution no longer holds, because a logical system that reflects intuitionistic mathematics must contain only true formulae, solved problems and completed constructions. It must be precised that Heyting had firstly [*1930*, 958] defined a proposition as "un problème, ou mieux encore un certain attente," and later [*1931*, 113] distinguished between propositions and assertions by defining the first as expectation of a proof and the second as the affirmation that a proof has been concluded. Furthermore, in [*1930*, 960] he stated that in his logical calculus he included only assertions *in order to avoid confusion*; in [*1932*, 272] he said that propositions are not mathematical or logical theorems; finally, in [*1934*, 15] (to which Griss refers) he underlined that all the axioms of Heyting's calculus are assertions. However, only in [*1956*, 97] he explicitly interpreted logical constants in terms of conditions of assertibility.

On his side, Griss proposes that each assertion expresses a non-empty species. He then goes on to revise propositional logic. So Griss declares [Griss *l951d*, 41]. Yet Heyting notes [*1954*, 96] that Griss' interpretation of logic "makes a propositional calculus impossible and forces us to start at once with a predicate calculus," since Griss founds logic on the species. The first connective to be modified is negation, which is introduced through the notion of complementary species $\neg A$, understood as the proper subspecies of all elements of a given species U that are distinguishable from the elements of A. But negation is not the only connective that is redefined. Still attempting to avoid false assertions, Griss also changes the meaning of implication: $A \rightarrow B$ can be asserted if B follows from A and A is true. Griss affirms that it is the "natural meaning of negation." Lastly, disjunction undergoes revision. It must contain only true assertions, and so $A \vee B$ can no longer mean that A is true or B is true; nonetheless, it must have its own meaning different from that of conjunction. Therefore, Griss uses it to express two alternative properties holding for the elements of a species and such that for each property (that is for each disjoint) there is at least one element of the species that possesses it. For instance, if it can be proved for each element of a species that it is pair (P) or odd (O), and if there is at least one element of the species that is pair and one that is odd, $P \vee O$ may be asserted.

6. Brouwer's reply to Griss. By reading the passage of Brouwer's 1907 thesis that we quoted at the end of the second section of this paper, we can argue that Brouwer had anticipated, and shared, Griss' criticism of reasoning containing a hypothesis and that he thought he had proved that certain reasonings that seem hypothetical (including that containing negation) are not. Indeed, Brouwer had described the reasoning terminating in a contradiction as the reasoning which 1) starts by constructing a subject that fulfills some of the given relations; 2) obtains from these other relations; 3) compares the latter with the initial relations; and 4) ends by ascertaining that two relations are in conflict with one another and that, therefore, the construction of the object cannot be completed in such a way that it fulfills all the given relations. Brouwer's strong point is that in this reasoning an object is effectively given which fulfills part of the initial relations. Yet his argument does not hold completely, because it concludes with a comparison between two terms, one of which is the given object, while the other consists in properties for which no object is given that fulfills them.

There is, however, an explicit and direct reply from Brouwer to Griss. Brouwer had read Griss' writings and even presented some of them as contributed papers at meetings. He did not remain indifferent to the attack: his strategy consisted in showing the necessity of the concept of negation in mathematics, by pointing out that it is not always possible to find positive definitions of entities. There exist essentially negative properties, that is, not transformable into properties that are expressible through affirmative assertions. As instances he demonstrates [*1948*, 478–479] that:

1) We are not able to prove that $\rho \neq O$ is equivalent to "$\rho^\circ > O$ or $\rho < {}^\circ O$", that is, we are not able to prove that the negative property is translatable into the expression "measurably larger (°>) or measurably smaller (<°)" which is defined in merely affirmative terms, because $a \,^\circ\!> b$ means that $a - b > 2^{-n}$ for some suitable natural number n, and $a <^\circ b$ means that $b - a > 2^{-n}$ for some suitable natural number n. (Remark: "property" is the term used in 1948. However, in 1949, he talks of "relation," the term commonly used in the literature).

2) We are not able to prove that, if $\rho > O$ holds, then $\rho \,^\circ\!> O$ also holds ($a > O$ *is* a relation defined in negative terms since it is equivalent to "$a \neq O \wedge \neg a < 0$").

In a later paper of 1949, *De non aequivalentie van de constructieve en de negatieve orderelatie in het continuum,* Brouwer presents an even stronger result: instead of the mere "improbability" of the equivalence between ">" and "°>" on the continuum, he obtains the contradiction of such equivalence. Nevertheless, it is sufficient to analyse only the 1948 paper, in order to avoid getting bogged down in too much detail. The debate with Griss continued with reference to this paper, and the essential point is that it already contains Brouwer's way of treating the question.

Here is the proof for the first result:

ρ is a real number obtained as follows: let α be a mathematical assertion that cannot be tested, that is such that no method is yet known for proving either its absurdity or the absurdity of its absurdity. Let the "creative subject" build a sequence $a_1, a_2,\ldots$ by choosing:

$a_n = O$ as long as, in the course of choosing the a_n, the creating subject has experienced neither the truth, nor the absurdity of α;

$a_{r+n} = 2^{-r}$, if between the choice of a_{r-1} and that of a_r the creating subject has found a proof that α is true;

$a_{s+n} = -2^{-s}$, if between the choice of a_{s-1}, and that of a_s the creating subject has found a proof that α is absurd.

The sequence $a_1, a_2,\ldots$ is positively convergent; ρ is its limit. $\rho \quad O$, because its being equal to 0 would mean that it is ever impossible to prove either the truth or the absurdity of α: $\neg(\alpha \vee \neg\alpha)$, which is equivalent to $\neg\alpha \wedge \neg\neg\alpha$, that is, a contradiction. In order to prove that we are not able to affirm $\rho^\circ > O$ (that is $\rho > 2^{-n}$ for some suitable natural number n), Brouwer shows that we are not even able to affirm $\rho > O$: in such a case $\rho < O$ would be impossible; hence the absurdity of the absurdity of α should already be a certainty, while it is not. In the same way, in order to prove that we are not able to affirm $\rho <^\circ O$, Brouwer shows that we are not even able to affirm $\rho < O$: in such a case, $\rho > O$ would be impossible; hence the absurdity of α should be a certainty, while it is not.

Here is the proof for the second result:

let α again be a mathematical assertion such that we do not know of a method of proving

either its absurdity or the absurdity of its absurdity. Let the creative subject build the sequence $a_1, a_2, \dots$ by choosing:
$a_n = O$ as long as, in the course of choosing the a_n, the creating subject has experienced neither the truth, nor the absurdity of α;
$a_{r+n} = 2^{-r}$ if between the choice of a_{r-1} and that of a_r the creating subject has proved either that α is true or that it is absurd.

The sequence $a_1, a_2, \dots$ is positively convergent; ρ is its limit. $\rho \quad O$ holds for the same reason as in the preceding proof; and at the same time $\neg\rho < O$ on the basis of the construction of ρ: in no case can a negative term be chosen as a_n. Therefore $\rho > O$. However, we are not able to affirm $\rho^\circ > O$ (that is $\rho > 2^{-n}$ for some suitable natural number n) because the creative subject should already either have proved that α is true or have proved that α is absurd, but this is not the case, otherwise the creating subject would also possess a method of proving either the absurdity or the absurdity of the absurdity of α.

In these two proofs, the "creative subject" is explicitly mentioned for the first time in Brouwer's writings. The significance of this term, which has given rise to uncountable attempts to interpret and formalize it, lies in its drawing attention to the fact that mathematics is an experience, a free activity of the subject and not a formal system. As far as concerns the discussion here, it is not necessary to analyse the question of the creative subject in general, but we do need to pay attention to how Brouwer deals with the creative subject in the two proofs above. In this respect, van Dantzig correctly points out [*1949*, 350–351] that Brouwer does not use his terminology consistently throughout his 1948 paper: when he introduces assertion α, he defines it as mathematical and "untested" in general and not explicitly in relation to the creative subject; furthermore, he talks about a "positively convergent sequence" *tout court* and not about a "sequence proved positively convergent by the creative subject," and formulates the hypothesis as "if $\rho > O$ did hold" instead of "if $\rho > O$ had been proved by the creative subject S on or before the moment $t^{\cdot\prime}$." In addition, Van Dantzig underlines the fact that Brouwer had used the notion of contradiction to define negation, but contradiction is already a negative term. Hence, he concludes that it is better to avoid altogether any reference whatever to negation, because it is not a very clear notion.

Griss, on his side, does not accept the method used by Brouwer to construct the number ρ, because it seems to him to depend totally on the knowledge possessed at each instant by the specific subject who is constructing the number [*1955*, 137]:

> On n'est pas forcé d'accepter la construction du nombre p, cette construction étant si subjective qu'on ne connaît ce nombre, à moins de s'informer à tout instant auprès de M. Brouwer du nombre de choix qu'il fait et s'il sait déjà ou non démontrer l'assertion x. Je pourrais encore aller plus loin et définir une suite infinie a_1, a_2, $a_3 \dots$ de nombres rationnels pour laquelle je commence par choisir pour les a_n zéro, tout en promettant qu'à partir d'un certain n, sur lequel je ne suis pas encore décidé, je choisirai a_n différent de zéro.

In a certain sense, his criticism constitutes the completion of his definition of spread, in which he clearly states that he does not accept among the rules of construction for a spread reference to anything that is not completely determined at the moment in which the construction is begun. Yet, it is quite strange from Griss, because it is in contrast to some other assertions he had done before, which derived from his *Weltanschauung,* explained in his [*1946*]. This is based on the original datum that consciousness grasps by attaining its own fullness: the subject distinguishes himself from the object, but the one has no meaning without the other. This datum is the a priori condition of all experience, and it can be considered from three standpoints: the object in isolation from the subject, the unity between subject and object, and the linking of subject and object. In the first case we are in the domain of mathematics, in the second of mysticism, in the third of philosophy. The linking of all things implies the impossibility of distinguishing them clearly, which is also demonstrated by the impossibility of thinking two mathematical entities at the same time. Therefore, the further mathematics proceeds in this activity, the more imprecise it becomes. Vagueness is intrinsic to mathematics but also indispensable for its development, because it lies at the origin of infinite sequences: "infinite" means that it is not known when they will end (in other words, they continue indefinitely), and it is this very vagueness of mathematical entities that is responsible for the fact that it is not known at what point the sequences are to be concluded.

These assertions are in contrast to Griss' criticism to Brouwer's reply. The fact that the construction of the number ρ requires something that is not completely determined at the moment in which the construction is begun can not exclude the admissibility of such a construction from Griss' point of view, since he himself had said that vagueness is however present in infinite sequences as their very substance.

Furthermore, in my view, Griss' criticism, as well as that of van Dantzig, does not grasp the real defect in Brouwer's reply. This lies in the fact that by showing the impossibility of translating negative notions into positive ones, he merely points out the conceptual loss to mathematics if it is deprived of the definition of negation as "carried to absurdity." Such an argument in defense of this definition is particularly weak given that Brouwer had, in the name of his *Weltanschauung,* originally maintained the need for mathematics to be rebuilt, even if at the cost of drastic cuts. Moreover, he does not even consider, let alone settle, the question of principle concerning the compatibility of this definition with the intuitionistic standpoint.

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