Volume 4, no. 1 (January 1994)

T.E. Forster, *Set theory with a universal set: exploring an untyped universe*, Oxford Logic Guides, Vol. 20, Oxford, Clarendon Press, 1992, 152 + viii pages.

Reviewed by

MAURICE BOFFA

Faculté des Sciences
Université de Mons-Hainaut
7000 Mons, BELGIQUE

The author of this book is one of the few people who have been fascinated by the strangest axiomatic system of set theory ever produced: Quine's "New Foundations" (*NF*). This system is therefore the main topic of the book, even if other theories with a universal set are approached. One may read chapter 2 first, which is a good survey of the main results on *NF* and related systems (NF_3, *NFU*, *KF*). The more specialized chapter 3 (about Bernays-Rieger permutation method applied to *NF*) goes in the direction of the author's interests. Chapter 4 describes Church's and Mitchell's set theories. Chapter 1 gives the notation and some motivations. Chapter 5 contains open problems. There is a comprehensive bibliography. In the preface, the author recognizes that his book is not a monograph or a textbook, but an essay somewhat biased in the direction of his interests. Nevertheless, the book is for the moment the one place where the current research on *NF* is explained in detail.

NF got out of Pandora's box when Quine [*1937*] wanted to avoid the inconvenience that in Russell's theory of types *TT* (based on axioms of extensionality and stratified comprehension) each notion is duplicated at the different type levels. His solution was radically simple: put all the type levels at the same level, i.e. use *TT* as a one-sorted theory. So, by a mere sleight of hand, he got *NF*: the one-sorted theory generated by the axioms of *TT*. It is fascinating that *NF* is not a simple variant of *TT*. In *TT* the universes can be finite or not, well-ordered or not, but in *NF* the universe cannot be well-ordered and is therefore infinite. This famous result of Specker [*1953*] shows that the consistency strength of *NF* is at least the one of *TT*+*AI* (axiom of infinity). In his book (p. 131), Forster conjectures that *NF* is as strong as *Z* (Zermelo's set theory). My conjecture is that *NF* is not stronger than *TT* + *AI*. But since its consistency relative to a classical set theory (like *ZF*) is still an open problem, the pessimistic conjecture that *NF* is inconsistent is not unreasonable. Specker's result is the cornerstone of the development of mathematics in *NF*, since it has the happy consequence that Frege's arithmetic works in *NF*. On the other hand, it leads to a general question: which structures (linear orderings, ultrafilters, algebraic structures, ...) can be put on the universe? Nothing has been done in this direction. Another problem is to find alternative proofs of Specker's result. I know an (unsuccessful) attempt due to Hao Wang [*1953*], based on Hailperin's [*1944*] finite axiomatization of *NF*, and there is an incorrect proof in Skolem [*1962*] (p. 52). Forster's book contains a lot of results about cardinal arithmetic in *NF*, including some nice independence results due to Orey [*1964*], Henson [*1969*] and Petry [*1975*]. The following questions are still open: does *NF* prove that there are infinitely many infinite cardinals? Does *NF* prove that $\aleph_\omega$ exists? A result of Hinnion [*1979*] shows that a positive answer to the second question would imply that *NF* is stronger than *Z*. It is clear that the development of cardinal (and ordinal) arithmetic in *NF* remains a task for the future.

By definition, all (proper) axioms of *NF* are stratified, but they have unstratified consequences, like $(\exists x)(x \in x)$. Another result of Specker [*1962*] shows that the stratified theorems of *NF* are exactly the theorems of *TT+AA*, where *AA* is the set of all ambiguity axioms $\sigma \leftrightarrow \sigma^+$ (where σ is any stratified sentence and σ^+ is obtained by raising all the types in σ by 1). Specker's proof is by model theory and can be simplified via the Keisler-Shelah isomorphism theorem (Boffa [*1977*]). Kaye [*1991*] refined the result. Crabbé [*1975*; *1978*] found a proof-theoretical argument, which has been extended recently by Dzierzgowski [*1993*] in the context of intuitionistic logic. So *NF* is equiconsistent with *TT+AA*. A similar reduction holds for subsystems of *NF* like *NFU* (where the axiom of extensionality is restricted to the nonempty sets) or NF_3 (where the axioms of comprehension are those which use no more than 3 types) and has led to consistency proofs of these systems due to Jensen [*1968/69*] and Grishin [*1969*]. But the consistency of *NF* remains open. Perhaps the following fact gives an idea of the gap between *NFU* and *NF*, in terms of models of *ZF* (without the axiom of choice) with an automorphism j (Boffa [*1988*]): a model of *NFU* can be obtained from j as soon as j moves some ordinal number, and a simple compactness argument provides such an automorphism, but for *NF* we have to assume that $j(c) = 2c$ for some cardinal number c. Nobody has produced such an automorphism, but other interesting automorphisms were obtained by Cohen [*1974*]. A new approach of *NF* and *NFU* has been given by Holmes [*1991*].

Let me conclude with this nice excerpt from Forster's book (p. 2): "The view behind this book is that one should think of the paradoxes as supernatural creatures, oracles, minor demons, etc. — on whom one should keep a weather eye in case they make prophecies or otherwise divulge information from another world not obtainable by any other means."

REFERENCES

M. BOFFA. 1977. *The consistency problem for NF*, Journal of Symbolic Logic **42**, 215–220.

—. 1988. *ZFJ and the consistency problem for NF*, Jahrbuch der Kurt Gödel Gsellschaft, 102–106.

P.J. COHEN. 1974. *Automorphisms of set theory*, L. Henkin. J. Addison, C.C. Chang, W. Craig, D. Scott & R. Vaught (editors), *Proceedings of the Tarski Symposium* (Providence, American Mathematical Society), 325–330.

M. CRABBÉ. 1975. *Types ambigus*, C.R. Acad. Sc. Paris **A280**, 1–2.

—. 1978. *Ambiguity and stratification*, Fundamenta Mathematicæ **101**, 11–17.

D. DZIERZGOWSKI. 1993. *Le théorème d'ambiguïte et son extension à la logique intuitionniste*, Thèse de doctorat, Université Catholique de Louvain, Louvain-la-Neuve.

V.N. GRISHIN. 1969. *Consistency of a fragment of Quine's NF system*, Soviet Math. Doklady **10**,1387–1390.

T. HAILPERIN. 1944. *A set of axioms for logic*, Journal of Symbolic Logic **9**, 1–19.

C.W. HENSON. 1969. *Finite sets in Quine's New Foundations*, Journal of Symbolic Logic 34, 589–596.

R. HINNION 1979. *Modèle constructible de la théorie des ensembles de Zermelo dans la théorie des types*, Bull. Soc. Math. Belg. B31, 3–11.

M.R. HOLMES. 1991. *Systems of combinatory logic related to Quine's New Foundations*, Ann. of Pure and Appl. Logic **53**, 103–133.

R.B. JENSEN. 1968/69. *On the consistency of a slight (?) modification of Quine's New Foundations*, Synthese 19, 250-263.

R.W. KAYE. 1991. *A generalization of Specker's theorem on typical ambiguity*, Journal of Symbolic Logic **56**, 458–466.

S. OREY. 1964. *New Foundations and the axiom of counting*, Duke Math. J. **31**, 655–660.

A. PETRY. 1975. *Sur l'incomparabilité de certains cardinaux dans les New Foundations de Quine*, C.R. Acad. Sc. Paris **A281**, 673-675.

W.V. QUINE. 1937. *New foundations for mathematical logic*, Amer. Math. Monthly **44**, 70–80.

T.A. SKOLEM. 1962. *Abstract set theory* (= Notre Dame Math. Lectures 8), Notre Dame, University of Notre Dame.

E.P. SPECKER. 1953. *The axiom of choice in Quine's new foundations for mathematical logic*, Proceedings of the National Academy of Sciences USA **39**, 972–975.

—. 1962. *Typical ambiguity*, E. Nagel (editor), *Proceedings of the 1960 International Congress of Logic, Methodology and Philosophy of Science* (Stanford, Stanford University Press), 116–124.

HAO WANG. 1953. *The categoricity question of certain grand logics*, Math. Zeitschr. **59**, 47-56.