C.S. Peirce's Relative Product

Jacqueline Brunning
Philosophy Dept., University of Toronto

Tarski stated that the title of creator of the theory of relations belongs to Charles Sanders Peirce who showed that "... a large part of the theory of relations can be presented as a calculus which is formally much like the calculus of classes developed by G. Boole." (Tarski 1941: 73) Peirce's right to that title is seldom challenged, but it is generally held that Peirce's early efforts to formulate a theory of relations are misguided and confused. Misguided, because Peirce failed to follow the lead of De Morgan. Confused, because as Quine said of the 1870 paper, Peirce constructed a far-fetched and fantastic calculus along mathematical analogies. Peirce's 1870 paper, "Description Of A Notation For The Logic Of Relatives, Resulting From An Amplification Of The Conceptions Of Boole's Calculus Of Logic" (CP 3.45-149)[1] (hereafter NLR) is Peirce's first published paper on the algebra of relations and the one to which most of the following discussion is directed.

The standard assessment of Peirce's development of the algebra of relations is that he lacked an adequate concept of relation until, finally in 1883, he followed the lead of De Morgan, and in fact extended and completed the calculus De Morgan began.

Peirce is partly responsible for this assessment of his work. His writings are relatively unsystematic and often extremely difficult although, as Prior pointed out, Peirce "... had perhaps a keener eye for essentials than any logician before him." (Prior 1957: 111). C.I. Lewis, in a similar observation on Peirce's papers, remarked that "His papers, however, are brief to the point of obscurity ... as a consequence, the most valuable of them make tremendously tough reading".

(Lewis 1960: 106).

I suggest that the common appraisal of Peirce's work is quite wrong. In brief the explanation to be pursued is that Peirce's development of the theory of relations should be viewed as a series of attempts to find an adequate articulation for his theory of relations, rather than as a gradual clarification of the concept of relation and subsequent development of a theory. Peirce's as yet unpublished logic notebook is full of early (pre-1870) unsatisfactory attempts to find a multiplicative operation for relations. The notebook, however, does demonstrate that he had an adequate concept of relations.

In support of this thesis, I wish to to make two brief points. The first point; Peirce never followed De Morgan. He saw himself as extending the work of Boole. He was aware of De Morgan's work. Peirce was also a great admirer of De Morgan. In the opening paragraph of NLR Peirce credited De Morgan with the only significant investigation of relations. However, he repeatedly stated "It was Boole who I was chiefly thinking of in those days. My point of view remained quite opposed to some chief features of De Morgan's" (NEM 111/2: 882).[2] In another paper, Peirce stated "... I found it quite impossible to represent in syllogisms any course of reasonings in geometry or even any reasoning in algebra, except in Boole's logical algebra." (NEM IV: 334). These remarks of Peirce reflect a difference at the level of definition between De Morgan and himself. These will be discussed in the next section of this paper.

Tarski, commenting on De Morgan's work, made a similar remark. He said "... De Morgan cannot be regarded as the creator of the modern theory of relations, since he did not possess an adequate apparatus for treating the subject ... and was apparently unable to create such an apparatus." (Tarski 1941: 73)

The second point; Peirce, following Boole, was squarely within the algebraic tradition and it was natural that he borrowed from mathematics. He did! The central definitional process in the algebra of relations was taken from the Linear Associative Algebra of Benjamin Peirce, Harvard mathematician and father of Charles Peirce. This is obvious from C.S. Peirce's discussion of what he termed elementary relatives in NLR, his first paper on relations. The full extent of the influence of linear algebras on the algebra of relations is however, most explicit in Charles Peirce's 1882 paper, which is a matrix formulation of the theory of relations.

Peirce's development of the algebra of relations occurred at a particularly favoured moment in the history of logic. In 1867, Charles Peirce published two significant papers on Boole's logical algebra, and 1870 his father presented the Linear Associative Algebra to the National Academy of Sciences.

Before I show how Charles Peirce's discussion of elementary relatives in NLR borrows from the Linear Associative Algebra, I will make some brief remarks on: 1) The influence of Boole on C.S. Peirce. 2) The significance of the operation of relative product for Peirce. 3) The multiplication scheme of Linear Associative Algebras.

4) Finally, Charles Peirce's discussion of elementary relatives.

II

Charles Peirce began the 1870 paper by remarking that

> De Morgan's system still leaves something to be desired ... Boole's logical algebra has such singular beauty ... that it is interesting to inquire whether it cannot be extended over the whole realm of formal logic, instead of being restricted to that simplest and least useful part of the subject, the logic of absolute terms. (CP 3.45)

In spite of Peirce's stated intention to extend Boole's algebra, and his recognition of the fact "that De Morgan's notation cannot be applied to Boole's algebra, or to any modification thereof", (MS 527)[3] he is criticized for not following De Morgan.

De Morgan's methodology is governed by the logic of syllogism while Peirce's methodology is entirely algebraic. This algebraic model taken over from Boole is foreign to De Morgan's methods . This difference in methodology reflects a significant difference at the level of definition.

Peirce is working toward an algebra of relations from the algebra of classes. This is why his early papers appear to be concerned with relatives rather than relations and why in NLR, Peirce claimed to be giving a notation and not an algebra. Traditionally relatives were monadic predicates (terms) having an underlying relational structure. This is the way we use such relative terms as 'brother', 'mother', 'friend' , etc. in English. Obviously, relatives cannot be treated extensionally in any adequate manner since their extensions are just classes. There is an algebra of relations in the sense that extensionally, operations can be defined on relations (such as relative product); but this does not determine any algebra of relatives, for the class of lovers of servants is not a function of the classes of lovers and of servants. Peirce was not confused about the difference between a relative and a relation. It was simply that Peirce was extending Boole's algebra and Boolean algebra is a class algebra. What Peirce did not yet realize is that relations can stand on their own, and that he need not map relations onto classes and treat them as relatives. It is possible to show that in NLR, underlying Peirce's notation for relatives, is an algebra of relations developed wholly within the scope of a mapping of relations onto classes.

III

One definitional process stands out as central to Peirce's algebra of relations. This is the operation of relative product. Peirce referred to this operation as general multiplication, or simply multiplication. It is interesting to note that in NLR he goes to great lengths to show both that arithmetic and algebraic multiplication can be construed as types of this general multiplications, and to demonstrate that relative product is not a partial operation. In NLR, Peirce stated that multiplication was the application of one relation to another. He illustrated the meaning of application by example, but in the section of the paper devoted to discussion of elementary relatives, he provided a definition. A favourite Peircean example of the relative product is the product of the relation (... is a lover of ...) with the relation (... is a servant of ...) which gives the relation (... is a lover of a servant of ...).

To frame this notion precisely relations will be understood in extension, as sets of finite sequences. Lower case italics will be used for elements of the universe of discourse, and upper case italics for sequences of those elements. The relative product of any two relations R and S then will be the set (R/S) of exactly those sequences XY such that for some linking element w the sequence Xw belongs to R while the sequence wY belongs to S. In set-theoretic notation:

$$(\mathrm{R/S}) = \{XY: (\exists w)(Xw \in \mathrm{R} \,\&\, wY \in \mathrm{S})\}$$

This operation was so prominent in Peirce's thought that he treated it as the paradigm of conceptual combination. It is in fact the single most important operation in all his algebras, and even in the existential graphs.

IV

The first hypercomplex number system, quaternions, was developed by Hamilton in 1843. These numbers satisfy all the field postulates except the commutative law of multiplication. This absence of the commutative law for multiplication makes the properties of quaternions very different from those of complex numbers. In spite of numerous attempts of mathematicians to introduce quaternions into various branches of mathematics, their role remained somewhat modest. Their chief significance was the impetus they provided for the exploration of other number systems which departed even more from the properties of real and complex numbers.

Benjamin Peirce exemplified the American interest in quaternions. According to Charles Peirce, it was only after much urging on his part that his father gave up his interest in quaternions, and wrote the Linear Associative Algebra. This study merited Benjamin Peirce a place in the history of mathematics, and placed quaternions in proper perspective, as the simplest of the four-fold algebras.

In a letter to William James, Charles Peirce links his study of the algebra of relations to Benjamin Peirce's development of linear associative algebras.

> About 1869 my studies of the composition of concepts had got so far that I very clearly saw that all dyadic relatives could be combined by ways capable of being represented by addition (and of course subtraction), by a sort of multiplication such that $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$ and $(xy)z = x(yz)$ (although xy was not generally the same as yx any more than they are same in quaternions) and by two kinds of involution which I represented the following year, by x^y and ${}^xy$, which called "forward" and "backward" involution, subject to the formulae $x^{y+z} = x^y.x^z$ and ${}^{x+y}z = {}^xz.{}^yz$, $(x^y)^z = x^{(yz)}$ and ${}^x({}^yz) = {}^{(xy)}$ and $({}^xy)^z = {}^x(y^z)$. But I found my mathematical powers were not sufficient to carry me further, ... I therefore set to work talking incessantly to my father (who was greatly interested in quaternions) to try to stimulate him to the investigation of all the systems of algebra which, instead of the multiplication-table of quaternions ... had some other more or less similar multiplication table. I had hard work at first. It evidently bored him. But I hammered away, and suddenly he became interested and soon worked out his great book on "Linear Associative Algebra." (NEM 111/2: 854)

This more or less similar multiplication table became the multiplication schema of the Linear Associative Algebras and subsequently the relative product of the algebra of relations. Benjamin Peirce's Linear Associative Algebra published in lithograph version in 1871, was republished in the American Journal of Mathematics in 1881. The 1881 edition was supplemented with explanatory footnotes by

Charles Peirce in which he gave the relative form of each of the algebras. Two papers by Charles Peirce, "The Relative Forms of the Algebra", and "On the Algebras In Which Division is Unambiguous" are also included as addenda to the 1881 edition.

It is interesting that most of the assessments and reviews of Benjamin Peirce's Linear Associative Algebra also cite the contributions of Charles Peirce. For example Henry Tabor, in his classical article "On the Theory of Matrices", stated

> "Subsequent to Cayley, but previous to Sylvester, the Peirce's, especially Charles Peirce, were led to the consideration of matrices from a different point of view; namely, from the investigation of linear associative algebra involving any number of linearly independent unit". (Tabor 1890: 373)

In this same article, Tabor devoted an entire section to the discussion of Charles Peirce's system of quadrate algebras and their connection to matrices. Charles considered himself a logician not a mathematician, and the paper cited by Taber was always listed by Charles Peirce as part of his study of the algebra of relations.

The Linear Associative Algebra was a résumé of linear associative algebras known at that time and the first systematic attempt to develop a theory of hypercomplex numbers. Benjamin Peirce developed the theory sufficiently so that he was able to enumerate and classify all types of number systems of less than seven units. The linear functions or products for these algebras can be displayed in a multiplication table which defines the character of the algebra. The multiplication schema for the linear associative algebras is the definition of multiplication explicitly used by Charles Peirce in his discussion of elementary relatives in NLR. This multiplicative operation corresponds, except for notation, to our present notion of relative product.

The following list of definitions, taken from Hawkes assessment of the

Linear Associative Algebra (Hawkes: 313), appear in Benjamin Peirce's Linear Associative Algebra in somewhat different form. (Peirce, B: 104). These definitions use De. Morgan's terms: facist, faciend and factum, instead of the usual multiplier, multiplicand and product.

D1: If $\alpha\beta = \beta\alpha = \alpha$, α is an idemfactor with respect to β

D2: If $\beta\alpha = \alpha$, α is idemfaciend with respect to β

D3: If $\alpha\beta = \alpha$, α is idemfacient with respect to β

D4: If $\beta\alpha = \alpha\beta = 0$, α is a nilfactor with respect to β

D5: If $\beta\alpha = 0$, α is nilfaciend with respect to β

D6: If $\alpha\beta = 0$, α is nilfacient with respect to β

D7: If for α, a positive integer n exists so that $\alpha^n = 0$,

α is nilpotent

D8: If the non-zero number α is equal to its square

$\alpha^2 = \alpha$, α is idempotent

Every algebra has a basis, i.e. a system of elements in terms of which all the elements of the algebra can be uniquely represented in the form of linear combinations (products). After the selection of the basis the remaining units may be arranged in four distinct groups with reference to the basis. The following arrangement, which is most useful for our purposes, is taken from Spottiswoode (Spottiswoode: 154).

Group I The units are idem-factors (dd)

Group II The units are idem-faciend and nil-facient (dn)

Group III The units are idem-facient and nil-faciend (nd)

Group IV The units are nil-factors (nn)

The multiplication table for the manner in which the units of the various groups combine follows

	dd	dn	nd	nn
dd	dd	dn	0	0
dn	0	0	dd	dn
nd	nd	nn	0	0
nn	0	0	nd	nn

figure 1

Describing the general rule of multiplication Benjamin Peirce stated

> ... every product vanishes, in which the second letter of the multiplier differs from the first letter of the multiplicand; and when these two letters are identical, both are omitted, and the product is the vid which is compounded of the remaining letters, which retain their relative position. (Peirce, B 1881: 111)

We can illustrate these two laws using the table provided in figure 1. Consider the following product.

(dd) x (nd) = 0

As the first part of the quote says, the product is zero because the second letter of the multiplier, in this case d, differs from the first letter of the multiplicand, in this case n. However, when the second letter of the multiplier and the first letter of the multiplicand are identical, as in the following example

(dn) x (nd) = (dd)

These letters are omitted and the product is the first letter of the multiplier and last letter of the multiplicand, each retaining their original positions. This is the definition of relative product used by Charles Peirce in his discussion of elementary relatives in NLR.

V

Charles Peirce defined an elementary relative as one which signifies a relation "which exists only between mutually exclusive pairs (or in the case of a conjugative term, triplets, or quartettes, etc.) of individuals, or else between pairs of classes . . . ."

(CP 3. 123). He illustrated this definition by letting A:B represent an elementary relative. This gives the following system of relatives:

A:A   A:B   B:A   B:B

Peirce gave the following example of a system of elementary relatives. Let c, t, p, s denote the relatives; colleague, teacher, pupil and schoolmate. The universal extremes, the two mutually exclusive absolute terms the system requires, are u and v. In this example, u and v denote respectively, the body of teachers in a school, and the body of pupils in a school, with the understanding that these are mutually exclusive classes. The following equivalences hold.

(1) $c = u:u$

(2) $t = u:v$

(3) $p = v:u$

(4) $s = v:v$

The meanings of this equivalences are obvious. In (1) colleague is the relation of teacher to teacher; in (2) teacher is the relation of teacher to pupil; in (3) pupil is the relation of pupil to teacher; and in (4) schoolmate is the relation of pupil to pupil.

Peirce gave the following multiplication table for these elementary relatives

	c	t	p	s
c	c	t	0	0
t	0	0	c	t
p	p	s	0	0
s	0	0	p	s

figure 2

(CP 3.126)

This table is followed by a translation for the sixteen propositions represented in the multiplication table. For example, tp = c was translated as "The teachers of the pupils of any person are that person's colleagues." This product represented in terms of the equivalent ordered pairs would be

$$tp = c$$
$$(u{:}v) \times (v{:}u) = (u{:}u).$$

The product pp = 0 was translated as "There are no pupils of any persons pupils". In the equivalent ordered pairs, this product would be

$$pp = 0$$
$$(v{:}u) \times (v{:}u) = 0.$$

These products are identical to the products in Benjamin Peirce's Linear Associative Algebra. In fact, if figure 2 is recopied, and the ordered pair equivalences are added, the resulting table is identical to the multiplication table of The Linear Associative Algebra given in figure 1.

	(c) u:u	(t) u:v	(p) v:u	(s) v:v
u:u (c)	u:u (c)	u:v (t)	0	0
u:v (p)	0	0	u:u (c)	u:v (t)
v:u (t)	v:u (p)	v:v (s)	0	0
v:v (s)	0	0	v:u (p)	v:v (s)

figure 3

Charles Peirce described the two rules governing the multiplication table. First, if the last letter of the multiplier is the same as the first letter of the multiplicand they are omitted, and the product is the first letter of the multiplier and the last letter of the multiplicand with each retaining their respective places. The second rule stated that every product vanishes when the second letter of the multiplier differs from the first letter of the multiplicand. These are exactly the rules of the multiplicative operation of the Linear Associative Algebra of Benjamin

Peirce.

Much of what Quine referred to as a far-fetched and fantastic calculus is a result of Peirce's attempt to draw analogies between relations and linear associative algebras. A few examples will demonstrate how relentlessly Peirce pursued these analogies.

Peirce showed that relations are resolvable into a sum of logical quaternions. The logical quaternion for the system of elementary relatives discussed above is

$$a, c + b, t + c, p + d, s$$

where a, b, c and d are scalars and scalar multiplication is represented by the comma. (CP 3.125). Peirce then added, as if to emphasize the link between elementary relatives and quaternions, "... any relative may be regarded as resolvable into a logical sum of logical quaternions." (CP 3.127).

Peirce also provided the relative forms for a series of algebras taken from Benjamin Peirce's Linear Associative Algebra. He then stated that "... all such algebras can be interpreted on the principles of the present notation .... In other words, all such algebras are complications and modifications of the algebra of (156)". (CP 3.130) The algebra of (156) is the algebra of elementary relatives of NLR that is given above in figure 2.

Another example of an extended analogy between relations and linear associative algebras is Peirce's discussion of a quaternion relative. Peirce's example of a quaternion relative is

$$q = xi + yj + zk + wl$$

where x,y, z and w are scalars. (CP 3.131) Peirce also gave the scalar form of q, the vector form of q, the tensor form of q and the conjugate form of q, along with complicated relative interpretations for each of these forms. Trying to make these plausible, Peirce noted: "The conception of multiplication we have adopted is that of the application of one relation to another. so, a quaternion being the relation of one vector to another, the multiplication of quaternions is the application of such a relation to a second." (CP 3.76).

It is clear that these mathematical analogies with their extended relative interpretations are tedious, difficult, and as Quine suggested, far fetched. However, it is not surprising that Charles Peirce went to the trouble to draw them. They show his considerable ingenuity, but more importantly, they are an attempt by Charles Peirce to both exploit the linear algebras for further notational gains, and to justify the adoption of the multiplication operation of the linear algebras as the relative product of the algebra of relations.

In subsequent papers on the algebra of relations, Peirce refined and developed the notions already implicit in NLR. He abandoned his use of relatives almost immediately. The construction of elaborate mathematical analogies became much less prominent in Peirce's subsequent papers on the algebra of relations. However, he continued to exploit the relation between linear associative algebras and the algebra of relations. This resulted in his 1882 paper, "Brief Description of the Algebra of Relations." (CP 3.306-322), which is a matrix formulation of the theory of relations.

NOTES

1. References to the Collected Papers of Charles Sanders Peirce, eds., C. Hartsborne, P. Weiss and A. Burks will be preceeded by 'CP' and will follow the convention of volume number followed by a period and the paragraph numbers.

2. References to The new Elements of Mathematics ed. C. Eisele will be cited as NEM and will follow the convention of volume number followed by a colon and the page numbers.

3. Manuscript references to the microfilm edition of the Charles Sanders Peirce Papers (Harvard University) are cited as Ms followed by the manuscript number.

References

Eisele, Carolyn, ed. The New Elements of Mathematics by Charles S. Peirce. 4 volumes. Hague: Mouton Press, 1976.

Hawkes, H.D. "On Hypercomplex Number Systems." American Journal of Mathematics. 24 (1902), 312-30.

Lewis, C.I. A Survey of Symbolic Logic. New York: Dover Publications, 1960.

Peirce, Benjamin, "Linear Associative Algebra," American Journal of Mathematics. IV (1881), 97-229.

Peirce, Charles. The Collected Papers of Charles Sanders Peirce. Ed. Charles Hartshorne, Paul Weiss, and Arthur Burks. 8 volumes. Cambridge: Harvard University Press, 1931-1958.

Charles Sanders Peirce Papers. Microfilm Edition, 30 reels. Harvard University Library Microreproduction Service, 1963-1966.

Prior, Arthur. Time and Modality. Oxford: Clarendon Press, 1957.

Quine, Willard Van Orman. "Review of the Collected Papers of Charles Sanders Peirce, Vol. IV: The Simplest Mathematics." ISIS XXII (1934), 551-3.

Spottiswoode, William. "Remarks on some Recent Generalizations of Algebra." Journal of London Mathematical Society, 1872, pp. 147-64.

Taber, Henry. "On the Theory of Matrices." American Journal of Mathematics. XII (1890) 337-95.

Tarski, Alfred. "On the Calculus of Relation." The Journal of Symbolic Logic. 6, No. 3 (1941), 73-89.