Haskell B. Curry, *Foundations of Mathematical Logic* (NY, McGraw-Hill, 1963, reprinted by Dover, 1977, 1984).

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This book represents a two-semester course that Curry taught as a first graduate course in logic at the Pennsylvania State University in the late 1950s and early 1960s. The contents differ from today's first graduate course in logic; a better title might be "Foundations of Constructive Proof Theory" or "Foundations of Gentzen-Style Proof Theory". The book was written in 1959-1961, just a few years too soon to incorporate the ideas of Prawitz (1965).

Curry's reason for writing this book when he did are stated in the preface to Curry et al. (1972): after completing Curry and Feys (1958), he felt he needed the results of this book for the then projected second volume of Curry and Feys (1958). As it turned out, proofs of the cut elimination theorem in Chapters 14-16 of Curry et al. (1972) are all based on proofs given in this book.

The main part of the book does not start until Chapter 5. Chapter 1 is a brief discussion of the nature of mathematical logic in very general terms. Chapters 2-3 are introduction to formal systems and "epitheory" (i.e., metatheory) respectively, and are revisions of Chapters 1-2 of Curry and Feys (1958). Chapter 4 is on algebraic logic, and is a revision of Curry (1952). The rest of the book covers the elementary proof theory of the first-order predicate calculus with special emphasis on the techniques of Gentzen, and constitutes a major revision of Curry (1950). Chapter 5 deals with the positive fragment of the propositional calculus (i.e., with $\wedge$, $\vee$, and $\supset$, but without negation). Chapter 6 deals with the full propositional calculus with negation, Chapter 7 deals with first-order logic, and Chapter 8 deals with modal logic.

Chapter 5 is really the most important in the book. It begins with a general discussion of the positive propositional calculus, a discussion which includes Curry's approach to semantics. Since Curry has a reputation as a strict formalist with no interest in the meaning of the symbols being used, it will probably surprise most logicians to learn that Curry had any ideas on semantics. This reputation of Curry's is probably based on the fact that he was not interested in the usual kind of model theory. But he did have definite ideas on the meaning of the logical connectives and quantifiers: he thought of propositional and predicate calculus as formalizing the elementary metatheory of a certain kind of elementary formal system, and he explained the basic rules for the connectives and quantifiers on this basis. These explanations are given in the first section of each of Chapters 5-8; they depend on the material in Chapters 2-3. What he means by an elementary formal system is one in which the rules all have the form

$$\frac{E_1 \quad E_2 \quad \dots \quad E_n}{E_0}$$

where $E_0, E_1, E_2, \dots, E_n$ are all what he calls *elementary statements*. In other words, no rules have discharged assumptions. Curry's model for such a formal system is an equation calculus for combinatory logic or λ-calculus, where the elementary statements are equations (which assert convertibility). In propositional and predicate logic, the atomic formulas are taken to be the elementary statements of the elementary formal system whose metatheory is being formalized. These elementary formal systems are called *post systems* in Prawitz (1971).

The second section of Chapter 5 deals with the natural deduction formulation (Curry calls it the "T-formulation") and a Hilbert-style formulation (i.e., in which the only rule of inference is modus ponens [and possibly, in predicate logic, universal generalization]; Curry calls it the "H-formulation") of positive propositional logic. There are two systems considered: the absolute system (TA and HA), which is constructive and is defined (in the case of TA) by taking the standard natural deduction introduction and elimination rules for the connectives $\wedge$, $\vee$, and $\supset$, and the classical system (TC and HC), which is obtained from the absolute system by the addition of Peirce's law. The latter takes the form of the scheme

$$((A \supset B) \supset A) \supset A$$

in HA and of the rule

$$\frac{\begin{array}{c}[A \supset B] \\ A\end{array}}{A}$$

in TA. Then, in the third section, he introduces Gentzen's L-systems and gives the formulations of the L-systems LA and LC. There are singular and multiple formulations of each system: the multiple formulation of LA is really a mixed system, since the rule for $\supset$ on the right is restricted to being singular. The singular form of LC is obtained by adding a rule for Peirce's law, which Curry writes as follows:

$$\frac{\mathfrak{X}, A \supset B \Vdash A}{\mathfrak{X} \Vdash A.}$$

(Curry uses capital German letters where Gentzen used capital Greek letters for sequences of formulas.) There are also minor variations of each formulation. All of these systems are formulated without the rule Cut; the rule is shown to hold as a metatheorem in what Curry calls the "elimination theorem" (in the next section). The rest of this section is devoted to showing informally how it is possible to find a proof by working backwards from the sequent to be proved and to proving some elementary metatheorems about L-systems. (Notes: 1) There is a tendency to refer to L-systems as "sequent calculi", but strictly speaking, an L-system is a particular kind of sequent calculus; one in which there is, for each connective and quantifier, one rule for introducing it on each side, and the only rule [or almost the only rule] that removes a formula is Cut. 2) Curry expected the naming scheme of this book to be consistent with that of Curry and Feys (1958) and Curry et al. (1972); he chose the name "T-formulation" in preference to Gentzen's name "N-formulation" because he wanted to reserve the letter "N" for "natural number". Note that "T" is the second consonant of the word "natural".)

The last two sections of Chapter 5 contain proofs of the standard metatheorems for L-systems. These include the elimination theorem, the inversion theorem (which says that under certain circumstances if a provable sequent can be the conclusion of a given rule, there is a proof of it in which it is the conclusion of that rule), the separation theorem (which says that the only introduction rules used in a proof are those for the connectives and quantifiers which appear in the conclusion), the equivalence of different formulations of the same system, and other similar theorems. There is also a formalization of the process of searching backwards for a proof in a *proof tableau*. (Note: The proof of the elimination theorem for the multiple formulation of LM is the only one I know of for a mixed system. Cut elimination for mixed systems can be tricky: see López-Escobar (1983). Curry's proof works only for mixed systems with the following property [which Curry failed to state explicitly]: if the rule for a given connective or quantifier on the left fails to be invertible, then the rule for the same connective or quantifier on the right must be singular. López-Escobar's system failed to satisfy this condition because the rule for $\forall$ on the left is not invertible.)

Chapter 6 deals with negation. Curry considers two notions of negation: absurdity (an elementary statement is *absurd* if every elementary statement can be deduced from it) and refutability (an elementary statement is *refutable* if an elementary statement known to be false can be derived from it). Systems involving refutability usually come with elementary statements that are postulated to be false; these postulates are called *counter-axioms*. For each kind of negation, there is a choice about whether to postulate the law of the excluded middle (which, when interpreted constructively, means that all statements (in the appropriate universe) are decidable. This would appear to give us six different kinds of negation, but in fact two of them turn out to be equivalent, leaving us with five systems. The L-formulations are:

- LM Absolute refutability (Minimal logic), formed by adding introduction rules on both sides for negation to LA.
- LJ Absolute absurdity (Intuitionistic logic), formed from LM by adding *ex falso quodlibet* .
- LD Decidable refutability, formed from LM by adding excluded middle.
- LE Classical refutability, formed from LC the way LM is formed from LA.
- LK Classical absurdity, formed from LE by adding *ex falso quodlibet* Decidability holds for this system, which is the classical propositional calculus.

These L-formulations are introduced in the second section of Chapter 6. For each system, there are three types of formulations: an F-formulation, in which the formula F (which is now usually written '$\perp$') is postulated and $\neg A$ is defined to be $A \supset F$, an N-formulation, in which $\neg$ is a primitive connective and F is not postulated, and an FN-formulation in which both $\neg$ and F are postulated. In the case of LD, the rule expressing the law of excluded middle is

$$\frac{\mathfrak{X}, \neg A \mid\vdash \mathfrak{Y} \qquad \mathfrak{X}, A \mid\vdash \mathfrak{Y}}{\mathfrak{X} \mid\vdash \mathfrak{Y}.}$$

There is a convention that in a singular system, $\mathfrak{Y}$ consists of one formula. This rule is a special case of Cut. Curry points out in the preface to the Dover reprinting of 1977 that this rule is not analogous to the rule for Peirce's law in Chapter 5 and that it might be better to change one of the two rules to make them analogous. The standard theorems about L-systems and the equivalence of different formulations of the same system are proved in this section. The third section deals with the T- and H-formulations of negation, and the fourth section deals with special properties of classical negation. (Note: The system LE is due to Saul Kripke; it is from a paper that he submitted to Westinghouse Science Talent Search in February, 1958, which has never been published [and of which Curry did not keep a copy]. The system has an interesting semantics: it uses truth tables with the possibility that F may take the value "true".)

Chapter 7 does the same thing for the first-order quantifiers. This turns out to be relatively straightforward, although some people may think that the specification of formulas of the second subsection is more detailed than necessary. There are seven quantified systems, one for each of the unquantified systems. The definition of proof tableaux of Chapter 5 (which is ignored in Chapter 6) is extended to the full first-order systems in this chapter. There is also a section with come classical metatheorems, including prenex normal forms, the Herbrand-Gentzen theorem, the Skolem normal form, and a proof of the completeness of LK (formalizing the elementary metatheory of the empty elementary formal system). The last result is almost the only nonconstructive result in the book.

Chapter 8, which is short, gives L- and T-formulations of S4 necessity.

Each chapter ends with a supplementary section with historical information and brief discussions of topics related to the material treated in the text. These sections are a valuable source of information on the history of Curry's own ideas.

Although the book was written to be a graduate level textbook, it should be used with some caution, since it is difficult to read. This is partly because Curry was not a good expository writer. (Curry knew this about himself: he once criticized an early version of Seldin (1975), which is an expository paper, for sounding too much like him!) He modified proofs to make them as general as possible even when that made them more difficult to understand; he would have done better to give a proof of a relatively simple case and then indicate how the proof can be modified to cover additional cases. He also had a habit of referring to a displayed formula by a number instead of repeating it no matter how short the formula and how many pages the reader has to turn back to find it. In addition, he tended to use an unusual vocabulary of his own design that nobody else ever used. His purpose was reasonable: to avoid disputes about the use of words. For example, his use of the prefix "epi-" instead of "meta-" is a result of the fact that Kleene, in a review of one of his papers from the early 1940s, objected to his use of the prefix "meta". (Kleene's objection was that the use of the prefix implied that the underlying formalism was based on the assumption that the formal objects were assumed to be strings of symbols, whereas in Curry's conception of a formal system, they can be the elements of any inductively generated set. This objection may have been valid in the early 1940s, but by the late 1950s and early 1960s, most logicians were using the prefix "meta-" in a sense so close to Curry's use of "epi-" that Curry would probably have done better to stick with "meta-" and introduce a remark or a footnote at an appropriate place explaining that his use of the prefix might differ from that of some others.) The problems with exposition even affect material based on drafts of others: even though Chapters 1-2 of Curry and Feys (1958), on which Chapters 2-3 of this book are based, were revised by Curry from a draft written by Robert Feys, by the time Curry got through with them they looked like his other writings.

Another difficulty that some students will have with this book is Curry's attitude toward foreign languages: he worked very hard to be able to read them, and he expected his readers to either do the same or else to know what they are missing. In one case (p. 89), he introduces a quotation from Hilbert in German without a translation:

> Hilbert has, of course, definite reasons for preferring a syntactical representation, viz., the concreteness mentioned at the end of Sec. C6. His own statement is as follows ...:

[Quotation in German omitted]

> This is a point well taken. But it simply argues that one must have the possibility of a syntactical representation, not that one must actually exhibit it. ...

In my opinion, it would be useful to have a translation in English, at least in a footnote, especially for students whose native language is not Indo-European.

Nevertheless, there is much useful material in this book, including the difficult Chapters 2-3. Much of this material is not, to my knowledge, available elsewhere. The consideration in parallel of two positive systems and five systems with negation is a useful alternative to the usual treatment which treats all nonclassical logics as variants of the "standard" logic. Any readers who do manage to make their way through this book will be well rewarded for their effort. The book is also useful as a reference book, and its extensive bibliography is a useful guide to the literature up to the time it was written.

Furthermore, the semantical discussion of this will probably make at least as much sense to most computer scientists as traditional model theory. It is interesting that computer scientists have recently become interested in both combinatory logic and λ-calculus and, in addition, in the work of Curry generally. This interest contrasts with the fact that for so many years Curry was considered by most logicians to be something of an oddity working on the fringes of the subject. I suspect that this interest by computer scientists is closely related to Curry's ideas on semantics, as presented in this book and to the fact that unlike most other logicians, Curry came to logic from applied mathematics rather than pure mathematics or philosophy. (Curry's first graduate program was an engineering program at M.I.T. Before he switched to pure mathematics, he earned an M.A. in physics. From 1942 to 1946, he put logic aside to do applied mathematics for the war effort of the U.S.A., and while he was doing this he became part of the ENIAC project.) I think this is why Curry never placed as much importance in set-theoretic models as most other logicians and why his criteria for the acceptability of a formal system were so pragmatic. That this should appeal to computer scientists, for many of whom logic is applied mathematics, can hardly be a surprise.

The logic community owes Dover a debt of gratitude for keeping this book in print.

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