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Marcus Giaquinto
The Search for Certainty
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REVIEW

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In *The Search for Certainty* Marcus Giaquinto presents an historically informed and philosophically rich account of the major attempts at providing a justification for classical mathematics. Mathematical knowledge had been regarded as the paradigmatic example of certainty for a long time, but a need for its justification began to emerge in the second half of the nineteenth century. It became all the more pressing with the discovery of several paradoxes within the theories that were intended to serve as foundations, and some spoke of a "foundational crisis" in mathematics. Giaquinto's book relates and discusses the fascinating enterprise of securing mathematics from the threat posed by the paradoxes.

The book is divided into six parts and, despite the fact that it is not intended to be a history, the material is presented in chronological order. The background for the later developments is provided in the first part, which presents the quest for clarity and rigor in nineteenth century mathematics. This foundational endeavor led to the development of a logical calculus, work on the arithmetization of analysis and on the axiomatization of number systems, and finally to the development of a theory of transfinite classes. It is in this context that the paradoxes emerged that cast doubt on what had been proposed as the foundations of mathematics. Thus, these foundations were in need of reconsideration and the search for certainty began.

In the second part of the book the class paradoxes named after Burali-Forti, Cantor, and Russell are presented carefully together with a discussion of Cantor's, Frege's and Russell's responses to them: Cantor introduced the notion of absolutely infinite multiplicities, Frege tried to amend his logical axioms, and Russell invented type theory to block the paradoxes. As Giaquinto emphasizes, however, none of these responses provided an acceptable solution. Here the difference between *blocking* a paradox and *solving* it comes into play: A solution requires independent reasons for believing in the principles that are put forward to block the paradoxes. For example, simply asserting that certain classes are "too big" to be sets does not constitute a solution as long as no additional justification is given for the proposed limitation of size. Thus, *ad hoc* principles, like Cantor's, although successful in blocking the known paradoxes are not considered to count as a solution to the problem of providing satisfactory foundations.

In Part III "The Language Paradoxes and *Principia Mathematica*," Giaquinto turns to two other kinds of paradoxes, namely definability paradoxes (*e.g.*, least indefinable ordinal, Berry's, Richard's) and truth paradoxes (*e.g.*, Russell's, liar), and he discusses Russell's and Ramsey's attempts to solve them. Russell realized that intuitive evidence for the basic assumptions does not guarantee freedom from contradictions. Thus, he abandoned Frege's approach of attempting to provide epistemological foundations for mathematics, and he argued instead that our confidence in the assumptions is justified by their consequences. However, independent reasons for the principles that he employed to block the paradoxes, like the axiom of reducibility, were again lacking, or the proposed solutions were too narrow in scope to provide a basis for all of classical mathematics.

In the wake of these developments another fundamental shift of emphasis in the search for certainty was proposed by Hilbert. Rather than trying to provide a justification for the *truth* of mathematics, he aimed at ascertaining the *reliability* of classical mathematics relative to a small and unproblematic (finitistic) part of it. The development of axiomatic set theory and the very promising results that were achieved prior to 1930 in connection with Hilbert's Programme are the topic of Part IV of the book under review.

The publication of Gödel's incompleteness theorems in 1931 radically changed the face of foundational research in mathematics. These and other major results in twentieth century logic are presented in Part V, and their relevance to Hilbert's Programme is discussed with the result that "the search for certainty through Hilbert's Programme cannot succeed" (p. 196). Thus, the search for certainty in the period covered in this book is characterized by the defeat of the foundational programmes of Frege, Russell, and Hilbert, and by various shifts of emphasis resulting from these developments. Frege aimed at establishing the certainty of the truth of mathematics, but he could not overcome Russell's paradox. Russell himself, who weakened Frege's goal by aiming at justifying our confidence in the truth of mathematics, was not successful either, since his theory is based on principles that are no more evident than those that they were supposed to justify, and thus are themselves in need of justification. Finally, Hilbert's Programme, with the original goal of exhibiting the reliability of classical mathematics, was halted by Gödel's second incompleteness theorem.

If Giaquinto's book ended at this point the reader would be left with the impression that the search for certainty failed. However, in the concluding part of the book Giaquinto explains how more recent developments (reverse mathematics, Feferman's predicativism) succeeded in partially achieving Hilbert's aims, and he argues that the iterative conception of sets justifies our confidence in the consistency of set theory. Thus, he is able to conclude on a soothing note: "The final balance, then, is positive. Though we cannot be *certain* of the reliability of *all* of classical mathematics, we can be certain of the reliability of a significant part of it, and we can be confident in the reliability of all of it" (p. 229). In other words, the foundational crisis, if there ever was one, has been overcome, and philosophers of mathematics can now turn their attention to other challenging topics. Giaquinto suggests investigating alternatives to the set-theoretic framework for the study of abstract structures, and exploring the nature of mathematical understanding and the growth of mathematical knowledge.

Giaquinto presents the various views he discusses sympathetically and does not, in general, question their fundamental assumptions, *e.g.*, that mathematics is reducible to logic or set theory, or that there is a single theory to which all of mathematics can be reduced. Surprisingly, Giaquinto does not even mention the well-known difficulties of representing the natural numbers within set theory, as put forward by Richard Dedekind and Paul Benacerraf. Foundational programmes that do not accept all of classical mathematics as worthy of justification, like Brouwer's, are from the outset beyond the scope of this book. However, I think that the conception of "classical mathematics" as a closed and well defined subject matter that needs to be justified is an idealization, since it is not clear-cut what constitutes classical mathematics. The foundational work of Cantor, Zermelo, and others introduced notions and principles that were not unanimously recognized as belonging to mathematics at all, so that the foundational enterprise itself changed the content of what it set out to rescue. This book documents tellingly how philosophers of mathematics were forced to revise their views in light of new developments in mathematics to which they themselves contributed to a considerable extent, and how many of the underlying assumptions were not vindicated by the results of the investigations they sparked off. An open, growing picture of mathematics results, like that of a blooming tree that continues to grow at its branches as well as at its roots.

A book like this one, which depends essentially on mathematical advancements and technical results, requires the author to be extremely careful in navigating between the Scylla of assuming too much, and thus rendering the text too difficult for non-experts, and the Charybdis of assuming too little, and thus overwhelming the reader with too much introductory material. Giaquinto does an excellent job in sailing through this narrow channel. He is able to present the essential ingredients of the arguments at a level that is accessible to non-experts yet does not trivialize them. Whenever possible he introduces new material with reference to previous discussions and illustrates definitions by examples as well as by examples that are similar, but that do not fall under the definitions. Relevant definitions are often repeated later in the text so that one does not have to go back and look them up again, which makes the reading flow smoothly. Arguments that are omitted from the text are often presented in an appendix or in the endnotes, where the technicalities are explained in detail and are accompanied by references to the relevant literature. Although many of the arguments are not new, they gain impact and cogency by being collected and arranged systematically. The only portion of Giaquinto's book where I see some, albeit small, scope for improvement is the index, which could have been made more comprehensive and uniform. For example, there are no references to the class abstraction principle, soundness, and conservativeness, and Grelling's paradox is listed under "Grelling," but not under "paradoxes" as are the others.

In sum, *The Search for Certainty* is a superb synoptic account of the intense and fruitful work that went into clarifying the foundations of mathematics. As such, it fills the gap in the literature that Giaquinto reports to have noticed when he was a student of logic, and it does so in an excellent manner. I envisage it as being used as a textbook for philosophy of mathematics courses and as complementary literature for courses in set theory and logic. *The Search for Certainty* is certainly appealing to many historically minded philosophers and mathematicians of a wide range of expertise.

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