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André Delessert
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REVIEW

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In the Introduction to his book Delessert observes that, very amazingly, Gödel's most important results—the completeness theorem of 1930 and the two incompleteness theorems of 1931—have been very coldly received by the mathematical world. By that he means that they have had hardly any influence on mathematical *practice*. With his book, more than sixty years after the publication of Gödel's papers, Delessert intends to find a fresh way to emphasize their revolutionary nature.

The book has two parts, entitled "Le nombre sous l'ancien régime" and, very significantly, "Non! Sire, c'est une révolution". The five chapters contained in the first part provide an historical sketch of the concept of number, going from Plato's number-as-Idea notion to the number-as-set conception of Dedekind and Cantor. In extreme synthesis, Delessert distinguishes three main historical phases. In the first one, the classic period, the number concept is strictly linked to the concept of universe, which, in its turn, is seen as a well-proportioned and well-ordered cosmos. Within this framework the task of mathematics is to exhibit the order and proportion of the universe. In the second period, the modern age, numbers become tools, instruments exploited by physics in the inquiry of natural laws. The nineteenth century, the third phase, marks a new, deep, turning point. "Tolérer des géométries prétendues non euclidiennes, c'est rejeter la tutelle des sciences de la nature. C'est autoriser les mathématiques à s'installer dans un univers fictif. Le même sentiment s'exprime quand les novateurs prétendent que, pour maîtriser la notion de nombre, il faut considérer comme objects mathématiques la totalité des nombres naturels ou celle des nombres réels. [...] De plus, la notion mathématique d'*ensemble*, qui va se faufiler sous le couvert de l'infini actuel, a des propriétés si nouvelles et si paradoxales que beaucoup craignent de voir le discours mathématique sombrer dans un véritable délire." (pp. 93-94). This period could be called "ontological" because mathematics focuses on the very nature of its own objects.

The second part of Delessert's book consists of six chapters, and is devoted to retracing the main threads of what happened after the birth of cantorian set theory: axiomatization and formalization of various mathematical theories, the emergence of first-order theories, and the fundamental metatheorical results of the 1920s and 1930s, together with some philosophical reflexions on them. I delay some remarks about Delessert's exposition of these topics because it seems more to the point to hint firstly at what according to Delessert makes Gödel's results revolutionary. He stresses the difference between a strong and a weak meaning of the word "finite": the strong one concerns the aggregates of signs which can be actually written down; the weak notion, on the other hand, occurs when we qualify a set as "finite", meaning that it can be mapped uniquely onto a natural number. This notion is regarded as "weak" because it depends on the assumption of the actual existence of the (infinite) set of the natural numbers. There is an important link between this distinction and the one between the domain of numerals (that is to say, the natural numbers which can be graphically represented within a formal language), and the domain of the natural numbers. This latter distinction is between the domain "de l'espace et du temps où se déroulent les actions mentales du mathématicien et celui des êtres mathématiques, accessibles à la seule pensée, sur lesquels portent ces actions. La locution 'et cetera' [occurring, for instance, when one writes '0,1,1+1, *etc.*'] est un indicateur qui nous signale qu'on passe de l'un à l'autre." (p. 199). Moreover, "[d]ans une théorie du premier ordre, les preuves doivent être effectivement écrites. Le fonctionnement de la logique du premier ordre élémentaire s'inscrit dans le monde du numéral, qui est non mathématique. Nous verrons bientôt que cette remarque doit être prise à la lettre. En revanche, les objects mathématiques auxquelles se rapporte cette logique sont dans le monde des ensembles, des totalités infinies actuelles. Le théorème de complétude jette un pont entre ces deux mondes." (p. 200). Building this bridge becomes necessary in the proof of the completeness theorem, where the totality of the formulas must be taken into account. Given this distinction, Delessert interprets the formalization process as the merging of mathematical objects into the domain of numerals; consequently, the effect of the completeness and incompleteness theorems is the drawing of a deep dividing line between the two domains. From

Delessert's point of view the completeness theorem belongs to the non-numeral domain, and its meaning consists in showing that formalization is true to the mathematical world. The incompleteness theorems, on the contrary, belong to the domain of numeral, and they show that this domain constitutes a *closed* world: meaning that there are questions which it allows to formulate but not to answer. This qualification can sound rather strange, but what Delessert means is, roughly speaking, that the Gödel number of the consistency proof of **P**eano **A**rithmetic will not be a numeral, but an inaccessible element of a non-standard model of **PA**. Reference, here, is to the (fourth) consistency proof of PA Gentzen provided in 1943 by proving that transfinite induction up to ϵ_0 is unprovable in **PA**. This consistency proof "est de longueur transfinie, ce qui éloigne évidemment de la logique du premier ordre et des mathématiques élémentaires. Mais cela confirme le fait que le domaine de validité des théorèmes d'incomplétude se situe au niveau des numéraux." (p. 209). So, much more than the existence of essentially different models of various mathematical theories (a fact already available as a consequence of the Löwenheim-Skolem theorems), what the revolutionary nature of Gödel's results properly consists in is the fact that they stress the two distinct levels in which mathematical facts are partitioned, making necessary at the same time trying to build a new global picture of mathematical practice.

Delessert's book is written in a not too technical style, and many details are overlooked. So, it would not make sense to make a point of every shortcoming. Sometimes, however, they could engender misunderstandings in the reader. I limit myself to only some of these. On page 167 we are told that intuitionistic logic is three-valued: a proposition can be true, false, or neither true nor false. This is an old point, held by Barzin and Errera in 1927 (in *Sur la logique de M. Brouwer*) and rebutted by Heyting in the 1930s: the fact is, simply, that intuitionistic logic has two truth-values, but it is verification-centered, and not truth-centered. Equally ambiguous is the way in which the completeness theorem is described (a point which is relevant for his previously recorded interpretation of the theorem). According to Delessert, it shows that first-order theories are able to cope with any need of mathematical reasoning: "le théorème de complétude [...] établit la réussite de la formalisation hilbertienne des mathématiques. Il montre que la logique du premier ordre est celle des mathématiques. Celles-ci n'ont pas besoin de recourir à la logique du second ordre, comme l'ont parfois cru d'excellents mathématiciens." (p. 186). I think this is a puzzling way to frame the question; anyway, it seems that the existence of theories which can be formalized only at the second order level acts as a sort of "modus tollens" for Delessert's point. Minor flaws in the treatment of the completeness theorem are on page 188 (where it is said that the compactness theorem is a *step* in the Henkin-style proof of the completeness theorem, and a *corollary* in the Gödel-style proof: of course, facts are the other way round), and on page 192 (where, contrary to the fact, the proof of the completeness theorem is qualified as *constructive*). Last, a more substantial problem concerns Tarski's theorem on truth: it is correctly formulated (p. 210), but then (on the same page) it is described in a way which may bring about some confusion. In fact, we are told that this theorem "permet d'affirmer qu'en général il n'existe pas de moyen formel de déterminer si un énoncé quelconque pris dans une théorie mathématique arbitraire est *dérivable* dans cette théorie." (my emphasis).

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