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*The Reason's Proper Study.*

*Essays Towards a Neo-Fregean Philosophy of Mathematics*

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## ON NEO-FREGEANISM

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In the first section of this paper, I will argue that the neo-Fregean program, as expressed in this volume<sup>1</sup>, is flawed by a basic inconsistency. In section (2) I will state my objections to the method employed by Frege himself in the definition of number. In the last section, I will hint at a revision of what I call “Frege’s method”.

### 1. THE INCONSISTENCY OF THE NEO-FREGEAN PROGRAM.

Since B. Russell pointed out in 1902 that a contradiction was derivable in Frege’s system of “basic laws of arithmetic”, several logicians and philosophers – including Frege himself – have been interested in repairing the Fregean construction. Neo-Fregeanism – presented in this book – is the most recent attempt to salvage Frege’s project. Obviously, the phrase “neo-Fregeanism” indicates a double commitment: to revise

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<sup>1</sup>The volume is organized into four main parts: I. *Ontology and abstraction principles*, II. *Responses to critics*, III. *Hume’s principle*, IV. *On the differentiation of abstracts*, V. *Beyond number theory*. Part I includes three essays by Bob Hale: *Singular terms (1)*, *Singular terms (2)*, *Grundlagen §64*; one by Crispin Wright: *Why Frege does not deserve his grain of salt: a note on the paradox of ‘the concept horse’ and the ascription of Bedeutungen to predicates*; and one by both authors: *Implicit definition and the a priori*. Part II includes two papers by Wright: *Field and Fregean platonism*, and *Critical notice of Michael Dummett’s Frege: Philosophy of mathematics*; and two papers by Hale: *Is platonism epistemologically bankrupt?*, and *Dummett’s critique of Wright’s attempt to resuscitate Frege*. Part III exhibits four papers by Wright: *On the harmless impredicativity of  $N^=$  (Hume’s principle)*, *Response to Dummett*, *On the philosophical significance of Frege’s theorem*, and *Is Hume’s principle analytic?* Part IV includes one paper, by both authors: *To bury Caesar...* . Part V also has only one paper, by Hale: *Reals by abstraction*.

the flawed, original system, but only within the limits of what might still be regarded as Fregean.

The neo-Fregeans read Frege as successively considering two possible definitions of number: the *contextual* and the *explicit*. The first one, also called *Hume's principle* by the neo-Fregeans, is the following biconditional: the number of a concept F = the number of a concept G iff F and G stand in a relation such that there is a bijection between the individuals falling under F and those falling under G. Such a bijection Frege calls *equinumerosity*. Thus, briefly, the biconditional can be presented as follows:  $N'F = N'G$  iff F and G are equinumerous. (To present numbers under the description "the number of concept F" does not involve any loss of generality: for Frege numbers are essentially linked to concepts). The explicit definition is as follows:  $N'F =$  the *extension* of the concept "being equinumerous with F".

While Frege, according to Hale and Wright, "abandons" the contextual definition, which he initially considers, and adopts the explicit one, the neo-Fregeans do precisely the opposite: they retain the contextual definition, or Hume principle, and abandon the explicit one. Their reason is that the latter brings in sets (extensions), and with sets, the Russell contradiction.

Both Frege and the neo-Fregeans want to answer the question "What is number?" (for the neo-Fregeans, *cf.* p. 15, the end of the second paragraph: "...setting it [number] up as the concept of a genuine sort of object"). The difference is that Frege thinks that the Hume principle is *insufficient* to answer that question (and this is why he moves to the famous explicit definition), whereas Hale and Wright claim that "Hume's Principle suffices to explain the concept of number as a sortal concept" (p. 15), and emphatically support a "contextual explanation of the fundamental concepts of arithmetic" (p. 1).

The claim that the Hume principle is sufficient to yield information on the nature of number is, in my view and *pace* the neo-Fregeans, *a priori* (*i.e.*, by mere inspection of Hume's principle) false. For example, if we are interested in understanding the nature of number 5, perhaps described as the number of the concept *fingers of my left hand*, briefly  $N'FL$ , all we learn from the Hume principle is that  $N'FL = N'X$ , where X is any concept equinumerous with FL, for instance FR: the concept *fingers of my right hand*. This says nothing about the nature of the number 5, except that 5 is somehow related to a certain concept. If one believes that the singular terms of the form "the number of the concept F" have a denotation, or as the neo-Fregeans put it, that numbers are genuine objects, then *as far as the Hume principle is concerned*, the number of the concept F could be identical to a set of apples (pleasing

the empiricists), or to a set of dots in pure space or to a sequence of recurring sounds in pure time (pleasing the Kantians), or to the extension of the concept “being equinumerous with F” (pleasing Frege), or indeed to *any arbitrarily chosen object*, for example Julius Caesar (empiricism again, at least with regard to Caesar’s body). The only information provided by the Hume principle is the one that follows from the unique restriction it imposes on the otherwise absolute arbitrariness of the choice of a denotation: if there is a bijection between the extensions of concepts F and X, and  $N/F = \text{Caesar}$ , then  $N/X = \text{Caesar}$  too.

That the Roman emperor is not excluded from qualifying as a candidate for being a number, appears to be, for Hale and Wright, a major problem. Indeed, they regard *the Caesar problem* as a, if not the, central problem of neo-Fregeanism. However, the Caesar problem is, in my view, a pseudo-problem, simply created by the insistence on aiming at two inconsistent goals with regard to the semantics of the singular terms of the form “the number of the concept F”. These are: 1) to adopt pure contextuality (Hume’s principle as the unique fountain of arithmetical knowledge), 2) to view numbers as genuine objects and consequently to assign a denotation to the singular terms in question.

If the philosopher of arithmetic pledges to be totally contextual with regard to the semantics of the singular terms of the form “the number of the concept F”, and does not plan to assign any denotation to them (equivalently, if the philosopher wants to use *only* the Hume principle as the source of arithmetical knowledge), then there is no problem: he got what he wanted. Such a philosopher will not have to even consider the possibility that  $5 = \text{Caesar}$ , because he will not have to consider the possibility that 5 is anything. If, however, the philosopher of arithmetic pledges to answer the question “What is number?”, and assumes that the answer is to assign a denotation to “5”, then he will find himself in the situation in which Frege found himself in sections §62-67 of *Grundlagen*, and will have to either transcend pure contextuality, adopting some explicit definition (as Frege did) or rethink the very significance of the biconditional called “Hume’s principle” and restart the project in an entirely different way (as suggested below).

It makes sense to speak of “the Caesar problem” *only* within the context of a tentative, exploratory search, as recorded in sections §62-67 of Frege’s *Grundlagen*. One can forgive Frege for not immediately realizing, while walking through new territory, that it is inconsistent to think of the biconditional, now called Hume’s principle, as *the* fountain of insight on the nature of number. For a moment, the famous biconditional may have appeared to him as a possibility in the march

towards logicism, and during that moment one may appropriately say that Frege encountered “the Caesar problem”, or rather “the England problem” (which is his actual example in *Grundlagen* §66). Again, during that moment Frege could have said: “the England problem is the central problem of my philosophy of arithmetic”. It did not take much, however, for Frege to see that “the England (or Caesar) problem” was just a way of saying that the Hume principle is not an option in the march towards logicism. Outside this exploratory trial-and-error scenario, and especially for those who already know the results of Frege’s tentative moves, it does not make sense to speak of a Caesar problem.

Contrary to this, Hale and Wright insist that the Caesar problem exists, that it is “one of the hardest” problems they “must solve” (pp. 15-16), that a solution is needed (pp. 340-345), and that the problem is “not intractable” (pp. 15-16). Essay 14 of the volume, titled *To bury Caesar...*, in more than sixty pages (pp. 335-396) “explores what seem to us to be the most promising lines along which a solution might be found” (p. 16).

The last statement of the long essay *To bury Caesar...* on the burial of the Roman emperor is that the Caesar problem “has effectively gone away” (p. 396). Not that the entire essay constitutes a demonstration of this statement; in fact, the essay reveals much trial and error, ups and downs, and even moments of philosophical despair, such as “Can anything be rescued from the rubble?” (p. 385). The alleged “solution” of the alleged “Caesar problem” emerges towards the end of the essay and resorts to the old notion of ontological category.

Before looking into Hale and Wright’s particular categorial strategy, it should be pointed out that philosophers have always divided entities into classes, or categories. Frege, for example, following many others, divided everything into sensible and non-sensible. Such categories can help to rule out Caesar as a candidate for being a number (if one believes that Caesar has, or had, a body, and numbers are non-corporeal). But to rule out, say, the archangel Gabriel from being a number, a finer categorization is needed, for example the division of non-sensible entities into those who have an intellect and those without an intellect. By pursuing this path, one may accumulate a long sequence of negative statements or answers to the question “What is number?”. A huge *arithmetica negativa* (similar to the *theologia negativa*) would emerge. These categorial “reductions” of the problem concerning the nature of number may end or not, and may be objectionable in any other sense, but what matters here is that they would amount to adding, to the sheer contextuality of Hume’s principle, a collection of explicit definitions rather than just a single one (as Frege did).

Now, the way in which Hale and Wright apply the notion of ontological category is not the one just described. They conceive the ontological categories in such a way that each category has “a criterion of identity distinctive of it” (p. 389), and the desired category which contains numbers (and nothing but numbers) appears to be straightforwardly grasped as the category of “objects identifiable by means of the Hume principle”, rather than painfully approximated through successive “reductions”, as in the above mentioned *arithmetica negativa*. Hale and Wright’s categorial strategy does not seem to bring in any crypto-explicit definitions, or to be endlessly negative; however, this is because it does not really yield *any information at all* on the nature of numbers. When contemplating the alleged category of “objects identifiable by means of the Hume principle” it appears that *anything at all* can be included in it, for example angels. The neo-Fregean categorial strategy leaves us exactly where we started: in the Hume principle. All we know is that, if angel A is assigned as denotation of “the number of the concept *fingers of my left hand*”, then angel B, assuming  $A \neq B$ , cannot be appointed as reference of “the number of the concept *fingers of my right hand*”.

So little (actually: absolutely nothing) is known about the nature of numbers, that Hale and Wright are led to consider the danger of identifying such items with members of other categories (*e.g.*, persons). They are inclined to believe that “no object can belong to more than one category” (p. 393) but they cannot conclusively prove such a grand ontological claim. In view of this situation, they end up with the remark that, if the possibility of true cross-categorial statements cannot be ruled out, then “the Caesar problem becomes just an instance of a problem that afflicts *all* sortal concepts ... and there is no special cause to see the vulnerability of Hume’s principle to the problem as marking a deficiency in the understanding of numerical terms” (p. 396). Here both Fregeans and neo-Fregeans may recall the Latin phrase quoted by Frege as his first reaction to the bad news of the Russell contradiction: *solatium miseris, socios malorum habuisse* (it is good to know that one has partners in the suffering) – but Frege did not use this Latin phrase as a way out of the contradiction.

The final proclamation, at the end of the long essay *To bury Caesar...*, that “the Caesar problem has effectively gone away” (p. 396), conceals the fact that neo-Fregeanism fails to preserve either of the two components of Frege’s philosophy of arithmetic that it wants to preserve: platonism and logicism (p. 1). Room continues to be left for construing numbers any way one wishes: as sets of apples (empiricism), as sets of *a priori* dots, as sequences of *a priori* recurring sounds

(Kant), *etc.* Such a “permissive” philosophy of arithmetic cannot be submitted as “logicism”, much less as Frege’s logicism. On the other hand, insofar as singular terms of the form “the number of the concept F” continue to lack a proper denotation, they are mere *façons de parler*, which signals nominalism (in the philosophy of arithmetic), not compatible with platonism.

The main question turns out to be the following: How can excellent philosophers be dazzled by the Hume principle to the extreme that they fail to perceive its incurable poverty? One answer is that the neo-Fregeans have been encouraged by finding Hume’s principle to be proof-theoretically very fruitful: “Hume’s principle, added to a suitable system of second-order logic, suffices for proof of the Dedekind-Peano axioms” (p. 5). This achievement is proudly referred to, by neo-Fregeans, as *Frege’s theorem*, relative to whose consistency the expectations are high: “we now have as much assurance as it seems reasonable to demand” (p. 5). However, the profound and decisive explanation stems, I believe, from elsewhere, namely from the *terminology* that has been used, and continues to be used, in order to refer to the Hume principle. The Hume principle is called an *abstraction* principle. “Abstraction” and cognate words count among the most frequently occurring in Hale and Wright’s volume. Now, there is a certain magic about the word abstraction. To begin with, it suggests that somehow in Hume’s principle, or behind it, there is an operation (precisely: abstraction) which *produces* special entities called “abstracta”. By referring to Hume’s principle as abstraction, the incurably sterile biconditional seems – illusion! – to become semantically fertile. Unfortunately, the sterility remains, since the “abstractive” terminology, in connection with the Hume principle, is entirely inappropriate. Although inappropriate, that terminology is so deeply rooted in the history of logic in the 20th century, that one can understand that excellent philosophers end up expecting, from the barren Hume’s principle, the offsprings that it *cannot* yield.

## 2. THE ARBITRARINESS OF FREGE’S METHOD<sup>2</sup>.

Frege, contrary to the neo-Fregeans, does say what is number (explicit definition) but his way of doing this is not less liable to serious objections. First of all, one has to be clear on what is Frege’s definition of number. Is it just the explicit definition? The answer is no. Frege’s definition of number exemplifies a procedure which has two stages. The

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<sup>2</sup>For this section 2, and section 3, I refer to my publications listed in the bibliography.

first stage consists in stating the biconditional called nowadays Hume's principle: the number of the concept F = the number of the concept H iff there is a bijection between the objects falling under F and those falling under H. The second stage consists in producing an explicit definition, which assigns to the singular terms "the number of the concept F", "the number of the concept H", *etc.*, appropriate entities as their denotation, where "appropriate" means that the assignments are compatible with the biconditional, and nothing else. Thin as it is, this requisite establishes an *essential link* between the explicit definition and the biconditional of the first stage. Frege *can* adopt the explicit definition *because* it is compatible with what now is called the Hume principle. This is not said aloud by Frege – it is just whispered, towards the end of the first paragraph of §68: "Versuchen wir also...": "Let us *therefore* try..." That is, because of having shown that the assignment of "the extension of the concept *equinumerous to the concept F*" as denotation of the singular term "the number of the concept F" is *compatible with the Hume principle*, Frege proceeds to actually state his famous definition.

In describing the Fregean way of defining number, Hale and Wright write: "Frege rapidly convinces himself that it [the Caesar problem] cannot be solved without abandoning the attempt at a contextual explanation altogether. In *Grundlagen* §68 he does just that, switching to an explicit definition of numbers in terms of extensions of concepts (roughly, sets)" (p. 338). Also: "Frege quickly decides that ... a different form of definition is needed, and opts for his famous explicit definition" (p. 3). In this account of Frege's attempt to define number, rather common in the Fregean literature, the above mentioned crucial point of the linkage of explicit definition and Hume principle, and their fusion into *one* method, is not apparent. Frege did not "abandon altogether" the Hume principle, as said by Hale and Wright. On the contrary, he *continued to use* the Hume principle as *the platform* on which the explicit definition would be constructed.

Combining, as indicated, the explicit definition with the so-called Hume principle constitutes a procedure that deserves being called *Frege's method*. This method is applied by Frege twice: in the definition of number and in the introduction of sets (*Wertverlauf*), and it is rather plausible to imagine that it tacitly underlies Frege's theory of *Bedeutung*. Frege presents the method as something rather new for logicians, but not entirely new (meaning, probably, "for mathematicians", *Grundlagen*, end of §63). In Carnap's *Meaning and Necessity* the presentation of the method reaches its official and entirely general form, but also, alas, it reveals its philosophical defects and poverty. One

may wonder, with Husserl, how can anyone think that Frege's method means an advancement of logical theory<sup>3</sup>. The method is regrettably frivolous: the choice of entities at the second stage *lacks any justification*. The frivolity is perhaps not apparent in Frege but becomes irritatingly obvious when Carnap says that, after having completed the first stage, one should "look around" for suitable entities (suitable = compatible with the biconditional), which inspired my referring to Frege's method as "the looking around method" ("circumspection method" from *circum-spicere*, may sound less irreverent).

Thus, a philosopher or any ordinary person interested in learning *what* is, for example, 5, or the number of the concept *fingers of my left hand*, will be disappointed both by Frege himself as much as she was disappointed by the neo-Fregeans. The neo-Fregeans, in their pure contextuality plan of basing everything upon Hume's principle, say nothing; Frege gives an answer, but unjustified. Why should 5 be the extension of the concept *being equinumerous with the concept fingers of my left hand*? No insight is available here, except that the choice is compatible with Hume's principle, and convenient for the logicist program.

There is a further complication with regard to the method applied by Frege in his attempt to analyze number that adds some excitement to the history of logic and philosophy. In a strange tale of two methods, if not of two cities, the orbit of Frege's (looking around) method intersected the orbit of *another* method: Peano's *abstraction method*. Peano had the insight that, given a domain  $a, b, c \dots$ , for example the universe of fractions:  $1/2, 2/3, 3/6 \dots$ , with an equivalence relation  $x \sim y$ , for example: having equal cross-products, one could *do abstraction* in the sense of "leaving out" any predicate that was not invariant with respect to the given relation, and "retaining" only the invariant predicates. (I borrow the phrases "leave out" and "retain" from Locke.) For example, in our talking about  $1/2$ , *while performing abstraction*, we refrain from saying " $1/2$  is a fraction that cannot be further simplified", because this is not true of  $3/6$ . During this abstraction, equivalent fractions become no longer distinguishable, which is the right path towards the rationals. However, these incipient gestures made by Peano towards genuine abstraction did not survive the clash with the looking-around method. The latter was preferred by mathematicians, and the abstraction method vanished, to reappear later on, only in the works of Herrmann Weyl and Paul Lorenzen. Now, the

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<sup>3</sup>"Ich kann nicht einsehen, dass diese Methode eine Bereicherung der Logik bedeute", 1891, p. 134, in the section titled *Freges Versuch*, ch. 7.

collision of the two methods had a very strange secondary effect: at a certain point, the Fregean method began to be designated by means of the terminology used originally by Peano for his method, namely by the word “abstraction” or related phrases (from the early “definition by abstraction” and “principle of abstraction” to the most recent “logical abstraction”). Of course, this use of the term “abstraction” to designate Frege’s looking-around method is a pseudo-use of the word.

Now, it is not unknown that names have a power of their own. After one century of *wrongly* calling Frege’s looking-around method “abstraction”, it is easy to begin to believe that it *is* abstraction. It is also natural to believe that the first stage of the method is the crucial and essential one for the alleged abstraction. The Hume principle easily becomes surrounded by a very suggestive, mysteriously promising, and philosophically attractive halo: it begins to look like an operation which generates certain products: the number of the concept F, the number of the concept G . . . , which are accordingly regarded as “abstract entities”. In sum, the spurious terminology used in connection with Frege’s method, or with its first, neo-Fregean half: Hume’s principle, helps to conceal the frivolity of the method and in particular the total sterility of Hume’s principle.

### 3. REVISING FREGE’S METHOD.

With all great philosophers of the past, subsequent generations have reached a point where the question arises: What should we keep, and what should we discard from the legacy of this great thinker? In Frege’s case, I believe that his greatest philosophical contribution lies in his radical revolution in the theory of predication (an accomplishment closely related to his logical creation, especially quantification). With regard to his philosophy of arithmetic program, the two main pillars – definition of number and the issues related to arithmetical induction (ancestral) – need serious revision. Here I will briefly comment on how the method followed by Frege in the definition of number (and of *Wertverlauf*), *i.e.*, “Frege’s method”, could be revised.

A careful look at Frege’s method – “looking around” – shows that it secretly includes the potential of being reconstructed in terms of genuine abstraction. In fact, behind the irritating frivolity of Carnap’s instruction for the second part of the method: “look around for suitable entities”, genuine abstraction lurks. In emphasizing that *anything* is welcome as long as it is compatible with the biconditional of the first stage, the followers of the method are actually saying that the differences among the indefinitely many candidates *do not matter* (as long

as the biconditional of the first stage is respected). This, obviously, signals genuine abstraction. The followers of the looking around method may *say* that they dislike genuine abstraction (Frege, Dummett) and the neo-Fregeans (and Dummett) may adopt pseudo-uses of the term, but what they unknowingly do is genuine abstraction. To be sure, the abstraction is neither properly recognized nor adequately expressed: it needs explicit and technical recognition – which is not the same as calling “abstraction” Frege’s method or proclaiming that its first half, Hume’s principle, is an “abstraction principle”. To put it in Hegelian dialectical terminology, the truth of the looking-around method (full Fregean or half, neo-Fregean), lies in the abstraction method. In order to bring to the surface this secret truth, Frege’s method should be revised in a way that the genuine abstraction involved – the leaving out and the retaining – is properly expressed. There would continue to be a universe of discourse:  $a, b, c \dots$  with an equivalence relation  $x \sim y$ , but rather than beginning with Hume’s principle (or any other biconditional of its form), one would end with it. Hume’s principle would become a true thesis, including singular terms (“the number of the concept F”, *etc.*) which have *already* a denotation: an abstractum.

To be sure, this revision of Fregean doctrines might not be Fregean, given that abstraction may not favor platonism, and then the result would not qualify as neo-Fregeanism. Also, the elusive nature of abstracta may deter even good philosophers from pursuing this path (it is easy to describe the procedure of “retaining” and “leaving out” but it is difficult to exactly describe the nature of what is *retained*, that is, of the abstractum, or abstract entity). If for this or other reasons abstraction appears as unworkable, and must be abandoned, then the last hope for curing Frege’s looking-around method vanishes as well. But then the misleading, pseudo-uses of the beautiful word “abstraction” should be avoided too.

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