The Review of Modern Logic
Volume 9 Numbers 3 & 4 (December 2003–August 2004) [Issue 30], pp. 21–26.

AN AXIOMATIC CHARACTERIZATION OF LINEAR ORDERS

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ABSTRACT. In this paper we propose an axiomatic characterization of the set of linear orders using the concept of a choice rule, which assigns to each ordered pair of feasible alternatives and a reflexive binary relation, exactly one element from the feasible set.

1

The use of binary relations is ubiquitous in common language. While the more obvious use of binary relations occurs in descriptive sentences (as for instance: Mr. A is a "neighbor of" Ms. B), it is also used in framing declarative sentences (as for instance: "Go home!"). The reason is that implicit in the formation of a declarative sentence is an act of choice. Presumably, the person who decided to declare "Go home!" had the choice of declaring "Don't go home!", and arrived at the conclusion by choosing one of two or more alternatives. The person's declaration was therefore associated with an act of choice, where pronouncing one statement was considered by the individual to be more desirable than pronouncing another. In a way, the same choice theoretic perspective can be invoked in the context of descriptive sentences, even if the act of choice involves choosing a binary relation from a given set of binary relations. An obvious alternative to describing Mr. A as a "neighbor of" Ms. B is to describe Mr. A as "not a neighbor of" Ms. B. While a descriptive sentence may actually be a statement of (or about) a binary relation, what we are trying to emphasize here is that the statement itself is the result of an act of choice.

Proposition 1 of [Rubinstein 1996] provides an axiomatic characterization of the set of linear orders defined on a finite set of alternatives, using methods of mathematical logic. Here, we propose an axiomatic characterization of the set of linear orders using the concept of a choice

Received 7/24/2002. Revised 3/22/2003.

rule, which assigns to each ordered pair of feasible alternatives and a reflexive binary relation, exactly one element from the feasible set.

[Rubinstein 1996] is concerned with descriptive sentences embodying one or more aspect of a binary relation, and it is in this context that Proposition 1 of that paper has been presented. The questions that Proposition 1 naturally lead to are for instance: "Why does every natural language have comparatives (bigger, brighter etc.) and superlatives (biggest, brightest etc.) built into its syntax? Why are these constructions based on linear orderings?" Our approach is that the construction of a sentence, or for that matter any linguistic exercise, involves an act of choice, possibly based on a binary relation. Since the purpose of language is to communicate, the choice of language is determined by what the user of that language wants to communicate. The existence of comparatives and superlatives in natural language can therefore be viewed as providing greater flexibility in communication, by enlarging the set of alternatives from which choices can be made. In this approach the natural question that arises is the following: Given a set of alternatives, and a set of binary relations, under what conditions is it possible to associate with each feasible set of alternatives and a binary relation a unique feasible alternative, so that the rule which does so, respects a permutation in the nomenclature of the alternatives? Such rules, if one exists, are called neutral and any collection of binary relations which permits a neutral choice rule is called identification friendly. The purpose of invoking neutral choice rules is to render linguistic exercises meaningful. Thus for instance, if it is widely understood in a community that the meaning of the sentence "Mr. A is a 'neighbor of' Ms. B", as we understand it in English, is best expressed by the sentence "Mr. A as 'not a neighbor of' Ms. B", then to convey the meaning of the first descriptive sentence, as it is understood in English, the community would use the second descriptive sentence.

In a sense, the purpose of this paper is intrinsically related to the classical philosophical discussions on "natural kinds", which Chapter 1 of [Rubinstein 2000] briefly alludes to. The discussion on "natural kinds" as in [Goodman 1972], [Quine 1969], [Watanabe 1969], inquires into the "factors that confirm an inductive argument". The "riddle of induction" centers around the question: Why do we choose a particular statement to communicate a context, if that statement has been a valid mode of communicating a similar context in the past? For example, if the statement "Emeralds are green" has been a valid mode of communicating the color of emeralds up to this moment, why do we expect that this same statement will continue to be a valid mode of communicating the color of emeralds ever afterwards?

Our first result states that a collection of binary relations is identification friendly if and only if it is the set of linear orders (i.e. rankings of alternatives). Our second result describes all possible neutral choice rules on the domain of linear orders. Such rules satisfy the following property: given any two sets of feasible alternatives containing the same number of elements, the alternatives selected in both situations must have the same rank within their respective feasible sets. This requirement allows for the linear orders to be different in the two situations.

It is worth investigating the problem of associating with a given class of neutral choice rules the minimum number of observations required to completely identify a member of that class. A consequence of our second result is that this number cannot exceed the cardinality of the underlying set of alternatives.

A related paper, [Kalai 2001], poses a rather interesting hypothesis: "choices of individuals as modeled in economic theory are statistically 'learnable' from 'a few examples': namely a number of examples which is at most a polynomial in the number of alternatives. For such class of choice functions learnability appears to reflect (in a concrete and quantitative way) the structural nature of individual choice as modeled in theoretical economics".

2

Let X be a finite, nonempty set. Given any nonempty subset A of X, let $[A]$ denote the collection of all nonempty subsets of A. Thus in particular, $[X]$ denotes the set of all nonempty subsets of X.

A binary relation R on X is said to be (a) reflexive if $\forall x \in X$: $(x,x) \in R$; (b) complete if $\forall x,y \in X$ with $x \neq y$, either $(x,y) \in R$ or $(y,x) \in R$; (c) transitive if $\forall x,y,z \in X$, $[(x,y) \in R$ & $(y,z) \in R$ implies $(x,z) \in R]$; (d) antisymmetric if $[\forall x,y \in X$, $(x,y) \in R$ & $(y,x) \in R$ implies $x = y]$. Given a binary relation R on X and $A \in [X]$, let $R \mid A = R \cap (A \times A)$. Let Ω denote the set of all reflexive binary relations and let L denote the set of all reflexive, complete, antisymmetric and transitive binary relations. If $R \in \mathrm{L}$, then R is called a linear order. Given a binary relation R, let $\mathrm{P}(R) = \{(x,y) \in R \mid (y,x) \notin R\}$, $\mathrm{I}(R) = \{(x,y) \in R \mid (y,x) \in R\}$ and $\mathrm{N}(R) = \{(x,y) \in X \times X \mid (x,y) \notin R \text{ and } (y,x) \notin R\}$. $\mathrm{P}(R)$ is called the asymmetric part of R, $\mathrm{I}(R)$ is called the symmetric part of R and $\mathrm{N}(R)$ is called the noncomparable part of R. Given a binary relation R on X and $A \in [X]$, let $\mathrm{G}(A,R) = \{x \in A \mid \forall y \in A\colon (x,y) \in R\}$. If $R \in \mathrm{L}$, then for all $A \in [X]$, $\mathrm{G}(A,R)$ is a singleton.

A nonempty subset D of Ω is called a domain. A choice rule on a domain D is a function $f\colon [X] \times D \to X$ such that:

(i) $\forall A \in [X]$ and $R \in \Omega$: $f(A, R) \in A$;
(ii) $\forall R, Q \in \Omega$ and $A \in [X]$: $[R \mid A = Q \mid A]$ implies $[f(A, R) = f(A, Q)]$.

Let $\mathrm{g}\colon [X] \times \mathrm{L} \to X$ be defined as follows: $\forall A \in [X]$ and $R \in \Omega$, $\mathrm{G}(A, R) = \{\mathrm{g}(A, R)\}$.

Let Π be the set of all one to one functions from X to X. Given $R \in \Omega$, $A \in [X]$ and $\pi \in \Pi$, let $\pi(A) = \{\, \pi(x) \mid x \in A \,\}$ and $\pi(R) = \{\, (\pi(x), \pi(y)) \mid (x, y) \in R \,\}$.

A domain D is said to be admissible if $[R \in D]$ implies $[\pi(R) \in D$, for all $\pi \in \Pi]$.

The following Lemma is obvious:

Lemma 1. *Let $D \subset \mathrm{L}$ and suppose D is an admissible domain. Then $D = \mathrm{L}$.*

However, L is not the only admissible domain. For instance the domain consisting of the single binary relation $X \times X$ is clearly admissible.

A choice function f on an admissible domain D is said to be neutral if, for all $(A, R, \pi) \in [X] \times D \times \Pi$: $f(\pi(A), \pi(R)) = \pi(f(A, R))$.

An admissible domain D is said to be identification friendly if there exists at least one neutral choice rule f on D.

Theorem 1. *An admissible domain D is identification friendly if and only if $D = \mathrm{L}$.*

Proof. Clearly, L is an admissible domain. Further, the choice rule $\mathrm{g}\colon [X] \times \mathrm{L} \to X$ defined above is neutral. Thus L is identification friendly.

Let D be any identification friendly domain and let $R \in D$. Let f be a choice rule on D such that for all $(A, R, \pi) \in [X] \times D \times \Pi$: $f(\pi(A), \pi(R)) = \pi(f(A, R))$. Let $x, y \in X$ with $x \neq y$ and let $A = \{x, y\}$. Suppose towards a contradiction that $(x, y) \notin \mathrm{N}(R) \cup \mathrm{I}(R)$. Let $\pi\colon X \to X$ be defined as follows: $\pi(x) = y$, $\pi(y) = x$ and $\pi(w) = w$ for all $w \in X \setminus \{x, y\}$. Thus, $\pi(A) = A$ and $\pi(R) \mid A = R \mid A$. Hence, by the definition of a choice function, $f(\pi(A), \pi(R)) = f(A, R)$. However, by our assumption on f, $f(\pi(A), \pi(R)) = \pi(f(A, R))$. Thus, $\pi(f(A, R)) = f(A, R) \in A$, which is not possible.

Thus, $[x, y \in X,\ x \neq y]$ implies $[(x, y) \notin \mathrm{N}(R) \cup \mathrm{I}(R)]$. Thus, R is complete and antisymmetric.

Towards a contradiction suppose that R is not transitive. Then there exist $x, y, z \in X$ with $x \neq y \neq z \neq x$ such that $(x, y) \in R$, $(y, z) \in R$ but $(x, z) \notin R$. Since R is complete, it must be the case that $(z, x) \in R$. Let $A = \{x, y, z\}$ and let $\pi \colon X \to X$ be defined as follows: $\pi(x) = y$, $\pi(y) = z$, $\pi(z) = x$ and $\pi(w) = w$ for all $w \in X \setminus \{x, y, z\}$. Thus, $\pi(A) = A$ and $\pi(R) \mid A = R \mid A$. Hence, by the definition of a choice function, $f(\pi(A), \pi(R)) = f(A, R)$. However, by our assumption on f, $f(\pi(A), \pi(R)) = \pi(f(A, R))$. Thus, $\pi(f(A, R)) = f(A, R) \in A$, which is not possible. Hence, R is transitive. Thus, $R \in \mathrm{L}$. Thus, $D \subset \mathrm{L}$. Since D is an admissible domain, by Lemma 1 it follows that $D = \mathrm{L}$. □

It might be worth investigating the class of all neutral choice rules that can be defined on L. Towards that end we proceed as follows: Given $A \in [X]$ and $R \in \mathrm{L}$, let $\mathrm{f}_1(A, R) = \mathrm{g}(A, R)$. Having defined $\mathrm{f}_k(A, R)$ for $1 \leq k < \#A$ (where for $A \subset X$, $\#A$ denotes the cardinality of A) let $\mathrm{f}_{k+1}(A) = \mathrm{g}(A \setminus \{\, \mathrm{f}_1(A, R), \ldots, \mathrm{f}_k(A, R) \,\}, R)$. A cardinality based rank rule is a choice rule f on L such that for all $k \in \{\, 1, \ldots, \#X \,\}$, there exists $i(k)$ satisfying the following property: for all $A \in [X]$ with $\#A = k$, $f(A, R) = \mathrm{f}_{i(k)}(A, R)$.

Theorem 2. *A choice rule on* L *is neutral if and only if it is a cardinality based rank rule.*

Proof. Clearly a cardinality based rank rule is neutral. Hence suppose f is a neutral choice rule and let $A, B \in [X]$, $R, Q \in \mathrm{L}$ with $\#A = \#B = k$.

Clearly, $\#(X \backslash A) = \#(X \backslash B)$. If $\#(X \backslash A) = \#(X \backslash B) > 0$, then let ρ be any one to one function from $X \setminus A \to X \setminus B$. Let $A = \{\, x_1, \ldots, x_k \,\}$ and let $B = \{\, y_1, \ldots, y_k \,\}$, where $(x_i, x_{i+1}) \in \mathrm{P}(R)$ and $(y_i, y_{i+1}) \in \mathrm{P}(Q)$ for all $i \in \{\, 1, \ldots, k-1 \,\}$. Let $\pi \colon X \to X$ be defined as follows: $\pi(x_i) = y_i$ for all $i \in \{\, 1, \ldots, k-1 \,\}$ and $\pi(x) = \rho(x)$ if $X \setminus A \neq \varnothing$ and $x \in X \setminus A$. Clearly, $\pi(A) = B$. Suppose $f(A, R) = x_i$. Since f is neutral, $f(B, \pi(R)) = f(\pi(A), \pi(R)) = \pi(f(A, R)) = \pi(x_i) = y_i$. However, $B \mid Q = B \mid \pi(R)$. Thus, $f(B, Q) = f(B, \pi(R)) = y_i$. This proves the theorem. □

Acknowledgment. I want to thank Ariel Rubinstein for suggesting the approach to the paper (which involved rewriting and extending an earlier version) and for helpful comments. I would also like to thank Gil Kalai for useful suggestions and a referee of this paper for suggesting the linguistic questions that Proposition 1 of [Rubinstein 1996] lead to and which this paper tries to answer.

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