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Review of
CRAIG SMORYŃSKI, *LOGICAL NUMBER THEORY I, AN INTRODUCTION*

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Smoryński's account of what he calls 'logical number theory' is an entertaining account of some of the germs in mathematical logic and 'arithmetic', *i.e.*, the theory of the natural number $\mathbb{N}$, up to around 1970. The mathematics is sometimes very beautiful, the presentation of it is always personal and highly idiosyncratic, and there are a great number of remarks putting the material in its historical context. It is intended to be read by anyone with sufficient mathematical background, such as an advanced undergraduate, or a beginning postgraduate. Such a reader will find a combination of material from logic and number theory with some interesting historical disgressions.

The title suggests that the book is concerned with number theory first and foremost, and in particular those aspects of it that are best studied using tools from a logician's toolbox. The author has therefore been able to make a rather free (and often refreshingly different) choice of material. Typical here is the discussion of the Fueter-Pólya theorem very early on in the book. This theorem concerns polynomials $P(x, y)$ that define bijections from $\mathbb{N}^2 \to \mathbb{N}$. One such polynomial, called the *pairing function* was defined by Cantor, and is essentially $\langle x, y\rangle = (x^2+2xy+y^2+3x+y)/2$. (Here, unlike, Cantor, we take the convention that 0 is a natural number, so $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and the definition of Cantor's pairing function has been modified accordingly.) The Fueter-Pólya theorem states that the only quadratic polynomials $P(x, y)$ over the reals defining a bijection $\mathbb{N}^2 \to \mathbb{N}$ are the Cantor pairings $\langle x, y\rangle$ and $\langle y, x\rangle$.

Of course, the pairing function or something like it is essential for any study of logical number theory, but the Fueter-Pólya theorem, however interesting it is, is not at all necessary, and is not referred to later on in the book. Nevertheless, the issues it raises are fascinating and

Smoryński gives a list of open problems, one of which asking whether the assumption that $P(x, y)$ be quadratic is necessary.

In fact, Smoryński does not give the full proof of the theorem just stated — he gives an elementary case-by-case analysis proving a weaker form of the theorem, and then quotes two results of number theory — one being Lindemann's theorem that e^{θ} is transcendental for algebraic $\theta \neq 0$ and the other concerned with polynomial approximations of the number of lattice points — to prove the main result. The decision to do this seems to be the right one, since even this 'shortened' discussion of the pairing function runs to some 28 pages.

The reviewed volume is organized into three long chapters on: arithmetic definability; Hilbert's tenth problem; and formal theories of arithmetic. A second volume is promised, which would include amongst other things, Gödel's second incompleteness theorem, nonstandard models and the Paris-Harrington theorem, but to date (six years after the publication of volume I) this sequel has not appeared yet.

The contents of the chapters can be summarized as follows. Chapter I contains background on polynomials, the pairing function, the Chinese remainder theorem and Gödel's β function, primitive recursion, the Ackermann function, computability, arithmetic definability and the arithmetic hierarchy. Chapter II contains the negative solution of Hilbert's tenth problem on the solubility of diophantine equations, firstly via Σ_1 definability of r.e. predicates, exponential diophantine equations and the Pell equation, then again via binomial coefficients and register machines. A welcome discussion here concerns the (still open) question concerning solubility of diophantine equations in the rationals and its equivalence to the problem of solving homogeneous equations over the integers. Chapter III starts with an outline of Hilbert's programme and an introduction to first-order logic via languages, structures and a Gentzen-style sequent calculus. The rest of this chapter contains proofs of the decidability of the Presburger(-Skolem) arithmetic of $\mathbb{N}$ with order and addition, and of Skolem arithmetic of $\mathbb{N}$ with multiplication (for this, Cegielski's axiomatization is presented), and then (in contrast) the incompleteness and undecidability results of Gödel, Rosser and others. It is pleasing to see Julia Robinson's interpretation of $\mathbb{N}$ in the ordered field of rationals appear in a text of this type.

Smoryński's style is characterized by a great number of lengthy — some would say indulgent — digressions. In the main, these are historical remarks concerning the mathematics being discussed. Certainly, I enjoyed these historical digressions hugely. Smoryński has read very widely, and offers many insights into the historical background of his subject. (I should state that I am not sufficiently expert to be sure that the balance of his historical remarks is entirely fair to all the parties he refers to, in particular in the questions of attribution of various theorems.) In an ideal world, every mathematician would pay as much attention to the history of his subject — but alas time is too short, or general interest in such matters is too slight (especially by the funding authorities, perhaps).

Occasionally Smoryński's digressions are not quite as tangential as they seem. For example, the machinery of finite differences, which takes so much space at the very beginning of Chapter I, is apparently only present to make the simple point that the standard convention for writing a polynomial as a sum of powers with varying coefficients, $\sum_{i=0}^{n-1} a_i X^i$, is not always the most convenient for all possible calculations. But to the author's credit, this machinery is used profitably later on in the chapter. It is somewhat irritating, however, that these digressions and juxtapositions of seemingly unrelated material obscure the overall direction of the book. His style will not be to everyone's taste, which is a pity as he has a great deal of interesting things to say.

Smoryński states in his preface that he 'would not hesitate to use [the book under discussion] as an introductory logic textbook in a mathematics department'. As such he is presenting a quite refreshing view of logic in the undergraduate curriculum as supporting mathematics, presenting new methods, and giving an important methodological view of mathematics. Whether his illustrations are intended for students or professionals is another matter. For example, the notion of 'definability' is introduced with the briefest mention of two classical nondefinability results, Lindemann's transcendence theorem and a reference to Galois theory that may seem obscure to many readers.

The official definitions from logic that students typically find difficult, such as the notions of a first-order formula, a sentence, and of bound and free variables, are put off as long as possible. Given sufficient time in the undergraduate curriculum and provided the students being taught have sufficient background mathematical knowledge and motivation (two major provisos), this is a good starting point from which to teach logic. But one of the effects of organizing the book in this way is that its pace is rather uneven: much of the sections on the more traditional aspects of logic are mathematically trivial (but perhaps notationally complex), but these intersperse some much more elegant and (for the typical undergraduate) sophisticated mathematics in quite an unexpected and sometimes disconcerting way.

I agree with Smoryński that, as part of a mathematics degree, logic should be taught in some context where interesting and nontrivial mathematical conclusions can be drawn. Typical applications and exercises in standard logic textbooks rarely succeed in making the case for logic as part of mathematics. A case in point is the familiar use of the compactness theorem to reduce the problem of giving a four-colouring of an infinite planar graph to the finite case, when quite clearly all that is required is König's lemma! On the other hand, nonstandard models and nonstandard methods are quite within the grasp of those undergraduates who are able to follow the proof of the insolubility of Hilbert's tenth problem, and nonstandard number systems present elegant and entirely adequate applications of first-order logic. Unfortunately, although Smoryński does present the completeness and compactness theorems, none of these applications — in particular applications to nonstandard models — appear, these apparently having been relegated to the forthcoming (?) volume II.

Unless a lecturer gives a course on the nonalgorithmic solubility of diophantine equations (which is presented in this book in a particularly clear fashion), I doubt if this book really can be used as a student textbook on elementary logic or 'logical number theory'. There is too much detail in some places, and rather too many other topics omitted (or postponed for volume II) for students to see the overall shape of the subject.

The final issue I would like to discuss here is whether there really is a subject that should be called 'logical number theory' and if so what it is, and whether this book really does serve as a useful introduction.

Smoryński, in his preface, invites comparisions with analytic number theory, which is of course a major area of study in mathematics, and he makes his wish that 'number theorists' should dip into his book to see what logic has to offer number theory quite clear. Certainly there is a lot to offer here already, most notably the chapter on Hilbert's tenth problem. Not so celebrated, perhaps, but nevertheless relevant to this subject sitting on the boundary between logic and number theory, are Presburger and Skolem arithmetic. (As it happens, I was slightly disappointed with the presentation of quantifier elimination for these theories — more modern treatments using back-and-forth seem rather more enlightening and less pedestrian.) On the other hand, Smoryński clearly sees Hilbert's programme and the Gödel incompleteness results as part of 'logical number theory'. While not belittling their importance, I would not choose to put such a large emphasis on them as he does here. Certainly, Gödel's results show the possibility of very straightforward number theoretic statements being independent of certain formal systems, or even of stating the consistency of formal systems. With the solution of Hilbert's tenth problem, we even see that these statements can be put in a form expressing the nonsolubility of specific diophantine equations, but even so no such statement of genuine number theoretic interest has been found. Of course, following from the Paris-Harrington theorem, plenty of independent statements of *combinatorial* interest are now known, but that is another matter altogether.

It seems to me that, if there is a significant contribution at all to number theory from logic, it has come from quite a different direction. Most importantly of all, logic has supplied us with (or has given us the tools to analyse) a vast number of interesting number systems. Of obvious relevance are nonstandard models of arithmetic, not just of the theory of the natural numbers with addition and multiplication, but of Preburger and Skolem arithmetic and decidable extensions of these, and of other weak systems such as the systems of *open induction* where one can study the effect of algebraic axioms on the solubility of diophantine equations. Going slightly beyond this, but still (at least to me) within the scope of 'logical number theory' are the p-adic number fields (with Macintyre's quantifier elimination result giving the basic tools required to analyse them) and the pseudofinite fields of Ax and others. Tarski's work on real closed fields has been given new relevance recently with Wilkie's model completeness result for exponential fields and many interesting (and difficult) decidability and transcendence questions that this raises. But a text on 'logical number theory' which aims to give a newcomer to logic the necessary background and flavour of some of these topics would be quite a different sort of book than the one reviewed here.

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