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Review of
F. R. DRAKE AND D. SINGH, *INTERMEDIATE SET THEORY*

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This book is intended as a college text intermediate between a minimal undergraduate introduction to set theory and any beginning graduate text in the subject. It is also intended as an introduction to a minimum amount of foundational aspects that every mathematician should know about. The book is a welcome addition to the project of filling in the missing detail of beginning graduate set theory texts, weighing in on the side of definability as opposed to that of combinatorics.

This review will outline the contents of the text, being more detailed in the parts that are "intermediate", and end with some final comments. Even though the first seven chapters are in some sense conventional, the plan of attack should be remarked on: even though the formal theory is introduced first, a more naïve approach to sets is employed before coming back to work in the formalization.

Chapter 1 (Some of the history of the concept of sets) introduces the history of set theory from Cantor's creation to the axiomatizations, discusses some of the paradoxes and subsequent type structures. Chapter 2 (First-order logic and its use in set theory) presents the syntax and semantics of LST (the language of set theory) and the addition of new terms to LST. Chapter 3 (The axioms of set theory) presents the ZF (Zermelo Fraenkel) axioms as well as arguments for them. Chapter 4 (Cardinals) covers countable and uncountable sets as well as the arithmetic of cardinal numbers from a naïve (i.e., non-axiomatic) standpoint. Chapter 5 (Order relations and ordered sets) continues a naïve stance towards orderings, some properties of ordered sets, lattices, Boolean algebras, and well-ordered sets. Many exercises develop the material, including the arithmetic of linear orders, but still from the non-axiomatic standpoint.

Chapter 6 (Developing mathematics within ZFC) begins a return at the axiomatic level to everything that precedes this chapter. Beginning with the representations of the natural numbers, the rational numbers, and the real numbers (Dedekind cuts) in ZFC, we continue to an axiomatic presentation of ordinals in ZF, culminating in the basic properties of ordinals, transfinite induction, and cardinals as initial ordinals. Along the way we have the Peano axioms for the natural numbers, Dedekind's second-order theorem of the uniqueness of the natural numbers, and a discussion of second-order and first-order theories.

Chapter 7 (The axiom of choice) proves various equivalences between the forms of choice, taking "every set of disjointed nonempty sets has a choice-set" as primitive. Equivalent forms investigated include the existence of a choice function, the multiplicative axiom, uniformization, the well-ordering theorem, as well as the Hausdorff's maximal principle and Zorn's lemma. Simple consequences of AC are proven. A nice collection of exercises take the student further: maximal filters and ideals, König's lemma, closed unbounded sets, stationary sets, Fodor's theorem, and the axiom of determinacy.

And now for the heart of this intermediate text. The authors take the high road through this material, giving the reader quite a panorama for the 65 pages involved. Chapter 8 (Constructible sets and forcing) is where the modeling of first-order logic inside ZF begins. Gödel's class of constructible sets L is defined, absoluteness and reflection are introduced, and it is proved that L is a model of ZFC + GCH. The approach taken is that of representing first-order definability in ZF via Gödel set coding of formulas, defining (the set representing a) formula sequence and (the set representing) satisfaction inductively, but the alternative presentation (Gödel's later approach, based on work by Bernays) of closing a set under iterations of eight relatively simple functions is also discussed. Finally, forcing with partial orders is presented: beginning with a countable transitive $\in$-model M of ZFC as "ground model", partial orders P (in M) and generic filters G (usually not in M) are defined, the truth and definability lemmas are stated, and then the latter are used to prove that the generic extension M[G] is a model of ZFC. Three forcing partial orders are presented to illustrate various generic extensions: one Cohen real is added to show that $V \neq L$ is consistent, enough Cohen reals are added to show that $\neg$ CH is consistent, and finally an uncountable cardinal is collapsed to show that cardinals are not always preserved. Before the latter generic extension is presented, a definition of chain conditions, the delta system lemma, and preservation of cardinalities are presented. After the concrete applications above, the proofs of the definability and truth lemmas are given, using strong forcing. Left to the exercises are iterated forcings, embeddings of partial orders, and Martin's Axiom. Models for the independence of AC are then presented: these are models strictly between the ground model M and the generic extension M[G] (forcing alone will not destroy AC), defined using groups of automorphisms of the forcing partial order. A particular symmetric model N with a Dedekind-finite set is constructed, where the corresponding forcing partial order is that for adding ω many Cohen reals. There are further exercises for a model having an amorphous set and a model having a countable family of pairs of Dedekind-finite sets. The chapter ends with a brief section on Boolean-valued models as introduced by Solovay and Scott.

The book ends with alternative axiomatizations for sets and with a link to questions in the philosophy of mathematics. Chapter 9 (Miscellaneous further topics) introduces variant set theories that allow proper classes (von Neumann-Bernays and Morse-Kelley), axioms of extent, other presentations of set theory (Montague-Scott, Quine, Ackermann), discusses mathematics without the axiom of foundation, and ends, appropriately, with remarks on the philosophy of mathematics. An appendix gives the constructions which are assumed to be known by all students at this level, bringing various notations in line with the ones used in the text. The bibliography, although two inconsistent alphabetizations are used for its ordering, is quite good: there is a wide variety of sources, and there is a source for anything the student might want to follow up on.

Bringing the constructible sets, definability, and forcing earlier to undergraduates, *Intermediate Set Theory* both bridges a gap at the intermediate level for the student of set theory as well as offers some important foundational background for the general student of mathematics. The alternating approach of philosophy – axioms – mathematics (starting in chapter 4) – formalization (beginning in chapter 6) makes good pedagogical sense. This reviewer's only criticisms are minor. For one, there are some typographical errors, some notations that are not listed in the index, and many incorrect references to section and exercise numbers (a LaTeX counter gone astray?). All of these editorial oversights can be rectified in subsequent printings. For another, more than once this reviewer felt that the authors were a bit terse in their explanations. Sometimes a single extra sentence could help confirm the reader's conceptualization or help summarize what has been proved. All in all, this is a carefully written book which will be very useful for courses or for independent study at the juncture for which it was intended.

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