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Paolo Mancosu (editor)
From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s
New York: Oxford University Press, 1998
x + 637 pp. ISBN 0195096312

Dirk van Dalen
Mystic, Geometer, and Intuitionist: The Life of L. E. J. Brouwer
Vol. 1: *The Dawning Revolution*
Oxford: Clarendon Press, 1999
464 pp. ISBN 0198502974

REVIEW

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The first two decades of the 20th century saw the emergence of several kinds of nonclassical logic: modal logic by C. I. Lewis in 1918, three-valued and many-valued logic by Jan Łukasiewicz and Emil Post independently of each other in 1920, and intuitionistic logic, which made its first inchoate appearance in L. E. J. Brouwer's doctoral dissertation of 1907 and then was elaborated during the 1920s and 1930s. Both of the books under review are concerned with intuitionistic logic, but in neither of them is it the primary concern. We begin with van Dalen's book.

Mystic, Geometer, and Intuitionist, the first volume of what will eventually be a two-volume biography of Brouwer, examines his life and work up to 1925. Intuitionistic logic appears in the chapter on Brouwer's dissertation and in the last chapter on intuitionism in the early 1920s. Van Dalen intimates that "Brouwer was, for reasons that will become clear in Volume 2, considered both a great man and an embarrassment in Holland ... [like] a noble family hiding the eccentric old baron in an isolated wing: a man to be praised but not to be seen" (p. vi–vii). This biography, very rich in previously unpublished material (particularly correspondence), is first rate, and is quite likely to become the definitive biography of Brouwer. At the moment it is the only full-length biography.

Brouwer's interests were very broad, a fact fully reflected in this biography, which treats his interest in artists and vegetarianism, in Flemish politics and personal religion, and in a misanthropic idealism somewhat like Schopenhauer's. At the age of 17, he wrote: "I am struck by the loathsomeness in the world that surrounds me, and is part of me ...; I detest most people" (p. 20). The early chapter "Mathematics and Mysticism" deals in part with philosophical lectures which he gave at Delft in 1905 and which became his first book, *Life, Art, and Mysticism*. In it he argues that man's ultimate goal is the mystical one of total introspection, but at the same time he foreshadows the ecological theme that man's destruction of the natural world will lead to his own destruction. Those lectures spelled out his extremely traditional and chauvinistic views on "woman" as a temptress and as an illogical being, who ought to be a servant of the male and to have no aspirations of her own.

Brouwer's doctoral dissertation (1907), the subject of the next chapter, was constrained by his supervisor, the mathematician D. J. Korteweg, who repeatedly cut out various philosophical parts. Mathematics, in Brouwer's eyes, is a tool of sin (because it is used to dominate nature and people) unless it is turned to higher development: the free unfolding of the self (p. 81). In his notes for the dissertation he wrote intriguingly: "One should refuse to do mathematics, but since this point has been reached, one should refuse to do the next step, that is, mathematical logic" (p. 82). As the dissertation neared completion, he insisted in a letter to Korteweg that mathematical reasoning, which consists of mental constructions, is not the same as logical reasoning, which necessarily uses language. Moreover he argued in the dissertation that mathematics is independent of logic, whereas logic depends on mathematics. (There are intimations of C. S. Peirce's views here, but Brouwer seems to have known nothing of Peirce.) For Brouwer, mathematics was language-independent.

It may surprise the reader to learn that in the dissertation (1907) Brouwer regarded the Law of the Excluded Middle as true but vacuous, like the proposition "if P, then P". Only a year later, he reversed course when he published an article, "The Unreliability of the Logical Principles", concluding that the Principle of the Excluded Middle is equivalent to Hilbert's doctrine that every mathematical problem is solvable. He now doubted that the Principle of the Excluded Middle is valid for infinite sets, giving his first counterexamples to this principle. One of them was the proposition that in the decimal expansion of π

there is a numeral which occurs more often than the others. Since it is not (now) known whether this proposition is true or false, then the Principle of the Excluded Middle is not true. Later he gave many other such counterexamples. All of them, like this one, are time-dependent and knowledge-dependent.

Besides intuitionism, Brouwer is known above all for his work on classical topology (particularly for the invariance of dimension and for his fixed-point theorem), the subject of the next two chapters. Then follow several chapters on his career, and questions surrounding nationalism, and his attempts to undo the Allied boycott of German scientists. A final chapter treats intuitionism in the early 1920s, the interest of the German set-theorist A. A. Fraenkel in intuitionism, and the first work of Brouwer's student (and eventual successor) Arend Heyting. Late in his life, Brouwer confessed that "I don't like mathematics and it basically bores me" (p. 195).

All in all, van Dalen's biography is extremely impressive for the vast range of sources, especially the use of previously unpublished letters. Yet at the same time the book has a serious scholarly flaw. The whole point of scholarly apparatus is to permit any scholar to check the accuracy and interpretation of any claim by going to the original document, be it published or unpublished. This is especially the case for unpublished correspondence. Unfortunately, van Dalen completely fails to tell the reader where letters are to be found. He quotes, among others, letters to Hilbert from Koebe, letters to Weyl from Klein, as well as letters between Brouwer and numerous people. No clue is given as to where these letters are kept. Perhaps one could guess that the letters to Brouwer are kept in the Brouwer Archive, which van Dalen was instrumental in establishing. But the reader is not even told where this archive is located — at the University of Amsterdam, perhaps? The reader has to be content with a slightly cryptic sentence in the preface: "The list of documents and their sources will appear in Volume 2." It remains to be seen if that list will include all the letters.

Let us now turn to Mancosu's book, *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s.* The first part of its title is perplexing, since it is unclear in what way Brouwer leads to Hilbert. Certainly Hilbert's early foundational work preceded Brouwer's. Perhaps the title simply means that Mancosu decided to begin his book with a section on Brouwer and end it with a section on Hilbert. But even this interpretation remains unsatisfactory since, although the book begins with Brouwer, it does not end with Hilbert. The second half of the title informs us that the book will be about the foundational debate in the 1920s. However, the author's preface soon lays that illusion to rest. The book will be only about part of this debate during the 1920s. It will exclude all research on logicism and related matters. (Thus Wittgenstein, Ramsey, and Russell will be left out.) Further, it will exclude all research on the foundations of set theory. (So important articles by Fraenkel, Skolem, von Neumann, and Zermelo will be omitted.) In fact, the only part of the foundational debates of the 1920s to be discussed is the debate between formalism and intuitionism, with Hermann Weyl in the role of convert to the latter. Mancosu characterizes this formalist-intuitionist debate as "central". But this is already a value judgment, and one that needs a justification. Such a justification is not forthcoming.

Let us consider the matter from the standpoint of the mathematician. Intuitionism had essentially no influence on the working mathematician (outside of the Netherlands) until the 1970s. By contrast, intuitionistic logic did have some influence on logicians (such as Alonzo Church at Princeton) late in the 1920s.

Yet here we run into an idiosyncrasy of Mancosu's book. It was intended to include only material which had not previously been translated into English. So, *a priori*, Mancosu excluded the work of logicians, such as Church, who wrote in English, however relevant and timely their work may be. Naturally, such a criterion skews the material included in the book. But then Mancosu himself does not abide by his own criterion; he includes two articles which have previously been translated into English in a fine source book from the same publisher, *viz.*, William Ewald's *From Kant to Hilbert: Readings in the Foundations of Mathematics*. The two articles were by Hilbert, "The New Grounding of Mathematics" (1922) and "The Grounding of Elementary Number Theory" (1931). Although Mancosu notes in passing that these two articles have previously been translated, he does not mention that two other articles in his book have previously been translated as well, *viz.*, Brouwer's "Mathematics, Science, and Language" (1928) and "The Structure of the Continuum" (1928), also in Ewald's book.

According to Mancosu's preface, his book originated from a pedagogical need: the need for texts in English to teach a course at Oxford on philosophy of mathematics and particularly on the formalist-intuitionist debate. If that was the real purpose of the book, then it is immaterial that four of the articles have previously been translated. But, granting this, then it is also immaterial whether any of the selections have previously been translated into English, or whether they were originally written in English. Then the choice of selections would be wide open, and should be made on their merits rather than by a purely mechanical criterion such as their previous non-translation.

Of the four sections of the book (1. Brouwer, 2. Weyl, 3. Bernays and Hilbert, 4. Intuitionistic Logic), clearly the fourth will be of most interest to logicians. Yet this is the slenderest of the four. Its content is reduced even further because two of the articles consist of the Dutch and German versions of Brouwer's "Intuitionist Splitting of the Fundamental Notions of Mathematics" (1923). This section would have benefited by including some of Rolin Wavre's articles of 1924 and 1926, defending intuitionistic logic, and some of Paul Lévy's replies (1926, 1927) defending the Law of the Excluded Middle. And surely Church's 1928 brief but enlightening article "On the Law of the Excluded Middle" deserves inclusion, even if it happened to be written in English. Mancosu was aware of all of these articles, and discussed them briefly in his introduction to the fourth section, but still decided to omit them.

One article in the first section is useful for giving a sense of just how different Brouwer's continuum is from the set of all real numbers. The real numbers are ordered by the usual relation $<$. But Brouwer's continuum is not ordered. The real numbers are separable, *i.e.*, they have a countable subset S (*e.g.*, the rational numbers) such that between any two real numbers there is a member of S. Brouwer's continuum is not separable. Finally, any two real numbers are either identical or are distinct; this is not true in Brouwer's continuum.

Probably the most useful selection in the fourth section was Heyting's "The Formal Rules of Intuitionistic Logic" (1930), which put this kind of logic on a firm basis. Now it could be investigated mathematically by those who had not been converted to intuitionism. It would have been even more useful to have included Heyting's "The Formal Rules of Intuitionistic Mathematics" (1930), which appeared in the same journal immediately after the other.

However, there can be no real excuse for not including Kolmogorov's ground-breaking article "On the Principle of the Excluded Middle" (1925). Mancosu omits it merely on the ground that it was not well known at the time. But the depth of the content of the article, which Hao Wang explains beautifully in his introduction to its translation in van Heijenoort's *From Frege to Gödel: A Source Book in Mathematical Logic*, renders its inclusion essential. Although it appeared five years earlier, it anticipated Heyting's formalization of intuitionistic logic, and more besides. It is ironic, then, that Mancosu ends his book by including a short 1931 article by Kolmogorov which interprets intuitionistic logic as a calculus of problem-solving.

The number of typographical errors in Mancosu's book is extremely small. Still, there is one error worth mentioning. In a translated article, Brouwer mentions "the arithmetization of these [non-Euclidean] geometries by Riemann, Beltrami, Cayley and Klein" (p. 54). In two other articles, Bernays refers to "Klein in his Erlangen Program" (pp. 192, 235). The index refers to all three occurrences of "Klein" as Fritz Klein, the mathematician who during the 1930s revived the theory of lattices which Dedekind had established in the 1890s. But that was not the Klein whom Brouwer and Bernays had in mind, and could not be since Fritz Klein's work was published after theirs. The famous Klein who was involved with non-Euclidean geometry and who in 1872 set forth the well-known Erlangen Program was surely Felix Klein, Hilbert's senior colleague at Göttingen.

To sum up, van Dalen's book is a valuable addition to a logician's library. Mancosu's book is better left for occasional use in your university library. Both books, despite their limitations, are worth reading and pondering.

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