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Review of

JAAKKO HINTIKKA,
THE PRINCIPLES OF MATHEMATICS REVISITED

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In this engaging, provocative manifesto, Professor Hintikka breaks the ground for what he hopes will be a revolution in logic, in much the same "critical and constructive" spirit as motivated Bertrand Russell almost a century ago in *The Principles of Mathematics*. (Other than the title, and the inspiration Russell provides Hintikka in his attempt to change the future direction of mathematical logic, there is no explicit connection between the two works.) In *The Principles of Mathematics Revisited*, Hintikka proposes what could be characterized roughly as a paradigm shift from a proof-theoretic to a model theoretic understanding of the principle role mathematical logic should play in mathematics. Traditionally, the emphasis in logic has been on the formulation and investigation of "pure relations of logical consequence." Hintikka would instead have logicians focus more on what he calls logic's "descriptive function": its use as a general language for the formulation and study of mathematical theories and their models. When we give priority to what can be *said* using logic, in contrast to what can be *proved* using it, Hintikka believes we will discover that many widely accepted ideas—on such issues, for example, as truth definitions, the role of set theory in logic, and the most appropriate form of negation—can be revised with far-reaching positive consequences for the state of modern logic. For Hintikka, this elevation of model theory to its deserved primary role represents the final stage in model theory's long road towards rehabilitation following the doubts about its status raised by Frege, Russell, Wittgenstein, Carnap, and others in the earlier decades of the twentieth century.

Since the emphasis is to be on model theory , Hintikka naturally starts with our intuitions about truth and their bearing on the definition of truth in a model. He first outlines the main features of a game-theoretical semantics (GTS), in which the truth or falsity of a sentence in a given model depends on which of two players, the "verifier" or the "falsifier," has a winning strategy in a game played according to a few simple rules for dealing with the usual logical constants. The constants are eliminated from the sentence "from the outside in," until an atomic sentence is reached whose truth or falsity can be determined immediately from the interpretation of the model. For Hintikka, GTS is proof that one can have a semantics that is both truth-conditional and verificationist at the same time (though he takes care to explain how this does not quite vindicate constructivism's standard idea of what verification means). Hintikka also believes that GTS adds greater substance to Gödel's famous claim for the primacy of our mathematical intuitions in determining what should be assumed true in set theory. For example, GTS "resoundingly vindicates" the axiom of choice, since $\forall x \exists y S(x, y)$ is true in GTS if and only if $\exists f \forall x S(x, f(x))$ is. But the main importance of GTS for Hintikka's purposes is that it shows us exactly what is wrong with standard first-order logic, and how to fix it.

For the most part, first-order predicate logic derives its power, and hence its reputation as "standard," from its use of strings of quantifiers to express functional relationships between variables. From the beginnings of predicate logic, with Frege, it was therefore assumed as a matter of course that quantifiers had to be *dependent*, as in defining a function: $\forall x \exists y S(x, y)$ means there exists a *y depending on x* such that $S(x, y)$. Hintikka calls this "unfortunate" assumption Frege's Fallacy. When GTS is applied to standard logic, we see that quantifier dependence is nothing more than *informational* dependence: that is, the possession by each player of perfect information about all previous moves made in the game. But in GTS, there is nothing that necessarily presupposes such knowledge on the part of the two players. A GTS-based logic in which the players were ignorant of certain past moves would contain quantifiers that do not depend on previous quantifiers. Along these lines, Hintikka develops what he calls Independence-Friendly (IF) first-order logic. In the notation he has chosen, $(\forall x)(\exists y/\forall x)S(x, y)$ means that for all x, there is a y independent of x such that $S(x, y)$.

Hintikka calls IF first-order logic "our true elementary logic." It is as easy to understand as traditional standard logic. It requires no special semantics and shares with standard logic a number of traditional nice properties, including compactness, the downward Löwenheim-Skolem property, and Beth's Theorem—often in stronger forms. But its major advantage, particularly when the emphasis is on what can be said using logic, is its much greater expressive power. This power derives from the fact that in GTS a player's claim to having a winning strategy is not bound by the necessity for perfect information. Consequently, many concepts not expressible in standard first-order logic can be defined in IF logic, or in an extended version of IF which allows contradictory negation. These include such ideas as that of well ordering, power set, open set, and continuity, as well as the principles of mathematical and transfinite induction.

These results lead Hintikka to re-evaluate some of the assumptions behind Tarskian truth definitions, among which the most important is probably the assumption that semantic attributes such as truth are *compositional*. From this idea—that the truth of an expression depends on the truth of its constituent expressions—it follows that truth is independent of context. Since the constituents of a sentence are not sentences but formulas, Tarski had to define the additional notion of satisfaction of a formula, resulting in a definition of truth somewhat less direct and less natural than one might hope for. Given that IF logic is non-compositional and expressions in it are context-dependent, Tarski-style truth definitions are not possible in IF logic, but Hintikka does not see this as a disadvantage. He claims that the very existence of a simple and powerful logic like IF casts doubt on the necessity for defining truth compositionally, challenging the normative role played in logic up to now by Tarski-style truth definitions.

It is no surprise that a game-theoretical approach to truth definition is the natural one for IF logic. Hintikka shows how such a truth definition can even be formulated in the language of IF first-order logic itself, thus avoiding Tarski's famous impossibility result. Since it is no longer necessary to define truth in a metalanguage, the model theory of first-order logic is freed from the reliance on higher-order logic and set theory that for so long has been assumed unavoidable, with all of its accompanying problems and paradoxes. Hintikka sees this as reinforcing his claim that model-theoretic ideas are the truly basic ones.

However, IF logic has its quirks. It must be extended if one is to have the usual contradictory negation found in first-order logic: in the basic version of IF the law of the excluded middle does not hold, since if neither the verifier nor the falsifier has a winning strategy a sentence is neither true nor false. Once again, as Hintikka takes care to explain, this does not necessarily add force to the intuitionist or constructivist positions. Hintikka has no illusions that IF will make everyone happy, at least not all of the time. There seems to be a trade-off, for example, between model-theoretic and proof-theoretic power. You can say a lot more in IF than in standard first-order logic, but it appears to be more difficult to prove things.

Indeed, is this not how things really are in mathematics? One of Hintikka's main arguments for IF is that it is in fact truer to the way we do mathematics. Hintikka claims that present-day logic, with its basis in a beautiful proof theory, is the result of abandoning the idea that logic can have any major role to play in mainstream mathematical research. Rather than developing a logic that mathematicians can use, logicians have left the main highway to construct a magnificent edifice on a remote side road where hardly anyone else ever goes. The reception of Hintikka's recommendations for the future of logic will obviously depend on how willing logicians are to accept this characterization of what they have been doing for the past several decades. But whether or not he wins very many of them to his point of view, this highly readable, wide-ranging book deserves a great deal of attention and debate.

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