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THE ENIGMA OF THE INFINITESIMAL: TOWARD CHARLES L. DODGSON'S THEORY OF INFINITESIMALS [1]

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1. Introduction

In 1888 Charles L. Dodgson published *Curiosa Mathematica, Part I. A New Theory of Parallels* (hereafter CMI) containing his novel alternative Euclidean parallel postulate: in every circle the inscribed equilateral hexagon is greater [in area] than any one of the segments which lie outside it. In Appendix II, "Is Euclid's Axiom True," he discusses infinitely large and infinitely small magnitudes, particularly infinitesimal lines and strips, and infinitesimal angles and sectors. In this section he develops a theory of infinitesimals that although flawed, contains elements that ultimately were addressed in the rigorous theory of infinitesimals Abraham Robinson created more than a half century later.

To provide a context for Dodgson's work, the paper begins with a survey of the main lines of thought about infinitesimals in the nineteenth century in analysis and in geometry, including the incomplete and divergent view held by Charles S. Peirce.

2. Infinitesimals in Analysis.

In the period between Gottfried Leibniz (1646–1716) and Abraham Robinson (1918-1974), roughly 300 years, infinitesimals were used in mathematics without being properly understood. In the first 200 years or so after the invention of the calculus infinitesimals as numbers were sometimes confused with the number zero. Alternatively, mathematicians following Augustin Cauchy (1789-1857) regarded them as variables with zero as their limit. The lack of a precise definition of the real number system was the principal stumbling block. Once this piece was put into place in the 1870s, the evolution of the calculus into a

[1] This paper is an expansion of the invited address, "Charles L. Dodgson's Theory of Infinitesimals" given at the 1998 Midwest Conference on the History of Mathematics held at Iowa State University.

rigorous theory that had been anticipated by Cauchy and Bernhard Bolzano (1781-1848) and completed by Karl Weierstrass (1815-1897) removed infinitesimals from formal mathematics. By eliminating the intuitively appreciated dynamic component of continuous motion from limiting processes, substituting instead an abstract static epsilon - delta definition, taking the limit of an expression containing the infinitesimal dx involves just a variable dx whose limit is 0 taking on nonzero values. There is no need to distinguish an infinitesimal from any other variable.

In this way infinitesimals, which cannot be real numbers, were circumvented so that in their disguised use the Archimedean axiom of the real number system was obeyed. But infinitesimals themselves continued to be used informally. In geometry where differentials were represented geometrically, ambiguity of expression was typical in arguments that employed them because infinitesimals beyond the first order were routinely ignored. In analysis x and $x + dx$ were treated as if they were equal, allowing in certain situations one to be used in place of the other but not uniformally so. According to Bolzano, "we have infinitely many infinitely small quantities, with one standing to another in any prescribed ratio, ... [and] that among the infinitely small quantities, we have ... an infinite number of different orders." [6, pp. 109 ff.]

Cantor's vociferous antagonism to infinitesimals also contributed to the denial of infinitesimals as numbers. Cantor was critical of the work with infinitesimals by such mathematicians as Paul du Bois-Reymond (1870-1), Otto Stolz (1882-3), Giuseppe Veronese (1889), Rodolfo Bettazzi (1890), and Tullio Levi-Civita (1892-3), and their attempts to develop rigorous theories.

As Joseph Dauben noted, Cantor's belief in the linearity of the number system and the essential Archimedean property of the reals made the existence of infinitesimals impossible. In a letter to Weierstrass in 1887 Cantor wrote,

> Non-zero linear numbers (in short, numbers which may be thought of as bounded, continuous lengths of a straight line) which would be smaller than any arbitrary small finite number do not exist, that is, they contradict the concept of linear numbers. [10, p. 130]

3. Non-Archimedean Geometry.

In 1890 Cantor corresponded with Veronese (1854-1917) and rejected his infinitesimal elements too. A year later Veronese published his

Fondamenti di Geometria containing the earliest example of a non-Archimedean ordered field[1]. Veronese was motivated by the question of the existence of straight line segments not satisfying the Archimedean axiom.

The usual interpretation of the Archimedean property (axiom) is that for any segments AB, CD there is a finite natural number n such that if CD is chosen a unit of length, then the length of AB is at most n times CD. However, if we reverse the roles of AB and CD, then with AB as the unit of length, the length of CD must be at least 1/n times AB. In other words, if one segment is chosen as a unit of length, no other segment can be infinitely small with respect to this unit. So infinitesimal numbers are prohibited in a geometry that employs the Archimedean property[2]. When the underlying complete ordered field is non-Archimedean, the corresponding geometry has magnitudes (segments, angles, areas, *etc.*) that are infinitely small.

Veronese showed that the continuum postulates given by Dedekind, Cantor and Weierstrass implicitly contain properties not required by the continuum; specifically, that the Archimedean axiom can be separated from the continuum. In his *Fondamenti*, Veronese constructed infinite and infinitesimal segments as the elements in a non-Archimedean ordered field. But his development was flawed and his student Levi-Civita, approaching the problem arithmetically, completed the construction in 1892-3. Generalizing his construction, Levi-Civita introduced infinitesimals in a consistent way by constructing, in 1898, a non-Archimedean totally ordered field with formal power series as elements[3].

In the appendix to *Fondamenti*, Veronese commented on the first of the non-Archimedean systems, by Du Bois-Reymond, extended by Stolz beginning in 1883, which produces an alternative theory of the continuum based on a different idea of an infinitesimal, the system not being an ordered field. Stolz, who later developed his own theory

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[1]Two years earlier, in 1889, Veronese published "Il continuo rettilineo e l'assioma V d'Archimede" in *Atti Della Reale Accademia Dei Lincei, Memorie* (Della Classe Di Scienze Fisiche, Matematiche E Naturali) Roma, 6, 603–24 containing a construction of infinite and infinitesimal line segments.

[2]To develop his theory of proportion, Euclid excluded infinitely large and infinitely small magnitudes from his geometry, tacitly assuming in Book V that magnitudes obey the Archimedean axiom. He did not explicitly employ the axiom until Pr.1 of Bk. X.

[3]In 1907 Hans Hahn, extending the work of Levi-Civita, formulated a rigororous theory of infinitesimal numbers but the theory was not useful for analysis because the usual properties of functions like $\ln x$, $\sin x$, e^x are not retained.

of infinitesimals, also showed that the Archimedean property can be proved using Dedekind's continuum postulate and therefore that the existence of infinitely small straight line segments was impossible.

4. Charles S. Peirce's Theory of Infinitesimals.

Charles Sanders Peirce (1839-1914) held an altogether different view of infinitesimals, one closer to the idea of a smoothly continuous quantity, a continuum in the small. As Dauben wrote of Peirce's ideas beginning in 1881,

> The difficulty in describing the continuity of the real line ... [is] reduced to the fact that numbers *per se* could never account for continuity. Numbers expressed nothing but the order ... of discrete objects. Nothing discrete could possibly be multitudinous enough to account for the continuum. [11, p. 130]

This interpretation of Peirce's infinitesimals is supported in a recent paper by John Bell [3]. A different interpretation is the one given by Stephen H. Levy, who asserts that

> Peirce's theory of infinitesimals is a theory of an extended real number system. His infinitesimals correspond roughly to the non-standard elements ... of a nonstandard model for analysis. [31, p. 135]

Levy adds that unlike Abraham Robinson's non-standard model, in Peirce's incipient theory there is a largest infinitesimal, of order 1. [31, p. 140]

Levy writes that Peirce, a proponent of Cantor's theory of the infinite, used that theory to argue for the acceptability of infinitesimals, stating that infinitesimals "...'involve no contradiction'..." and "...'lend themselves to mathematical demonstrations'." [31, p. 136]

5. Infinitesimals in Geometry.

The discovery of the non-Euclidean geometries focused attention on the behavior of parallel lines at infinity. For example, in hyperbolic geometry if a right triangle contains an angle of zero degrees, the hypotenuse and one side are asymptotic straight lines. When Nikolai Lobachevsky (1793-1856) constructed the first non-Euclidean geometry in 1840, later named hyperbolic by Christian Felix Klein (1849-1925), he used analytic methods to express relations of magnitude in his geometry, regarding infinitely small magnitudes as continuous variables having limit zero.

Eugenio Beltrami's work of 1866 and 1868 provided a concrete metrical interpretation of hyperbolic geometry, a model of the geometry on an ordinary surface (bounded plane). Points of the surface represented on an auxiliary plane lie in the interior of the unit circle; ideal points (points at infinity on the surface) lie on the circle. Straight lines are represented by chords; parallel straight lines by chords intersecting in a point on the circle.

In this interpretation the asymptotic character of parallel infinite straight lines, and consequently the relationship between such lines and the Archimedean property was not immediately apparent. That Beltrami's model was not entirely satisfactory to other mathematicians concerned with the foundations of geometry is evidenced by the work of Wilhelm Killing (1847-1923) who beginning in 1880, focusing on the analytic treatment of non-Euclidean space forms, considered such forms as being representable by a system of n-tuples of real numbers, each element of the tuple varying continuously. His approach naturally involved the classification of infinitesimal motions, but he proceeded in the customary way by ignoring second and higher order infinitesimals, effectively treating them as approximate real numbers. [21, pp. 303ff.]

Beltrami's work, unfortunately, had the effect of masking the rôle of the Archimedean property because in his model the ordinary projective properties of the surface also are valid. Beltrami's method allows both Euclidean and hyperbolic geometries to be considered, without any parallel postulate being assumed, within the fabric of (metrical) projective geometry — a method in which the ideal elements play a major role in the sense that, as Arthur Cayley had showed in 1859, the set of basic metrical objects, the absolute of the plane, includes the line at infinity [5, p. 163][4].

Building on Beltrami's and Cayley's work, Klein developed non-Euclidean geometry (1871, 1873) by considering a quadratic form "near" the imaginary spherical circle given in Cartesian coordinates and involving an arbitrarily small parameter ε. When ε is positive the non-Euclidean geometry is hyperbolic. The determinant D of this form given by

[4]In the projective model real points are located in the interior of the absolute conic (an infinite plane); ideal points, located on the conic, are determined by the pencils of parallel lines; ultraideal points, exterior to the conic, are determined by the non intersecting lines. (These lines exist only in hyperbolic geometry and can be considered to complete the projective plane.) [39, p. 155]

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\varepsilon \end{vmatrix} = -\varepsilon$$

vanishes only when $\varepsilon = 0$, *i.e.* when the form represents the imaginary Euclidean spherical circle. Klein implicitly is using infinitesimal numbers in representing D. He wrote,

> "Our assumption then amounts to this, that we replace the quadratic form whose determinant vanishes by a quadratic form whose determinant is positive or negative (but arbitrarily small in absolute value)." [26, p. 180]

Klein then develops a general distance measure for any quadratic surface and goes on to show that it yields the Euclidean metric when $\varepsilon = 0$. His complicated argument depends in large part on the handling of ε. By not allowing it to become 0, but only to become very small, certain expressions can be be neglected and certain approximations made, *e.g.* $\arcsin \varepsilon = \sin \varepsilon$, for small ε. [26, pp. 182-3]

But Klein makes no connection between the need for infinitesimal numbers and the non-Archimedean nature of the non-Euclidean geometries he develops[5]. The importance of the Archimedean property in connection with hyperbolic geometry was not established until 1900 by Max Dehn when he showed that in this geometry (which he called Semi-Euclidean), the Archimedean property necessarily does not hold.

David Hilbert (1862–1943) who was concerned with the foundations of both arithmetic and geometry, established the theorem that the real number system is an Archimedean ordered field satisfying an axiom of (arithmetic) completeness. In fact, by constructing a non-Archimedean ordered field in the second edition of his *Grundlagen der Geometrie* in 1903, Hilbert demonstrated that a non-Archimedean geometry logically could exist. [18, p. 111]

6. Charles L. Dodgson's Theory of Infinitesimals.

The relationship between continuous geometric magnitudes (line segments, angles, areas) and infinitesimals was the basis of Dodgson's

[5]Klein discussed the role of the Archimedean axiom as a continuity axiom in his description of a well known non-Archimedean system, that of horn-shaped angles, much later, in the 1920s.

(1832–1898) attempt to understand the nature of infinitesimals. Although Dodgson had no contact with any of the work on non-Archimedean systems being published at the time, his writing is philosophically closest to that of Veronese[6]. Dodgson and Veronese both believed that the Archimedean axiom does not extend to infinite space because there is no way to observe two segments obeying it, but that one can establish the Archimedean axiom between two finite segments because it can be observed. Throughout the appendices of CMI, Dodgson maintains the distinction between finite and infinitely large magnitudes. Euclid never assumed the Archimedean property for any geometric magnitudes. He only assumed that straight lines behaved in an Archimedean way, suggesting either that infinite straight lines behave like finite straight lines, or that infinite straight lines are excluded from propositions dealing with measurement. Dodgson concurred with the latter.

Dodgson believed that infinitesimals were numbers and that they were needed to distinguish among numbers that seemed to be equal but were not really so. In correspondence concerning a problem in probability in 1886 he argued that

> "when an event is possible, its chance of happening is not zero." [38, p. 218]

In connection with this point, Dodgson claims that the question of whether or not a convergent infinite series reaches its limit is equivalent to the question of whether an infinitesimal is or is not equal to zero. He used the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$ to illustrate his reasoning. The limit of this series is 1, and its nth partial sum is $1-\frac{1}{2^n}$. He states,

> "Hence, if when n is infinite, the series reaches its limit, the infinitesimal 2^{-n} must be equal to zero." [38, p. 218]

Dodgson carries this argument over to a strip infinite in length whose area can be shown to be finite. He claims its width must be infinitesimal because if it were finite, the area of the strip would be infinite.

To understand what Dodgson is saying, consider the problem of tossing a coin endlessly and the probability of obtaining a head on any toss. Since there is a 50% chance of obtaining a head on the first toss, and a 25% chance of obtaining a head on the second toss, there is a $\frac{1}{2}+\frac{1}{4}$ chance of obtaining a head on either the first or the second or both tosses. Continuing in this way, the chance of obtaining a head in the

[6]Dodgson recorded the beginning of his work on infinitesimals in a diary entry dated 29 March 1885: "I am now writing on 'Infinities and Infinitesimals'." [19, v.2, p. 433].

first three tosses is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$; in the first four tosses it is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, *etc.*

Expressing the terms of the series in binary form (where $\frac{1}{2}=0.1$, $\frac{1}{4}=0.01$, $\frac{1}{8}=0.001$, *etc.*), we see that the chance of obtaining a head in the first four tosses is $0.1+0.01+0.001+0.0001=0.1111$. Extending the partial sums to express the chance of obtaining a head on *some* toss produces 0.111111... But perhaps *every* toss will produce a tail, its chance of occurring being the infinitessimal 0.000000000...1 The limit of the series, 1, is really the sum of these two probabilities, the partial sum 0.11111... and the infinitesimal 0.00000...1. [31, p. 130]

Dodgson was not concerned with the foundations of the calculus and the attendant controversy over limits vs. infinitesimals. His interest was in the rôle of infinitesimals in geometry and he needed a method to construct them. His belief in the necessity of their existence is rooted in the two ways he interprets continuity. The first is the continuity implicit in the Archimedean axiom; the second is the continuum of numerical values as one moves from infinite values through finite values through infinitesimal values to zero. His method of constructing two linear magnitudes not obeying the Archimedean axiom, employs the notion of infinite area. The method is the content of Appendix II of CMI. A substantial revision is incorporated in the third edition (1890).

We should first recall that the transition in the meaning of "existence" in mathematics: from existence by construction to existence as consistency was going on during this time and did not emerge as the standard until the turn of the century when it is apparent, for example, in Hilbert's *Grundlagen der Geometrie* (1899).

Dodgson's basic idea of the number system was an extension of the notion of "denseness" to the existence of an extended real number system that would include infinitesimal and infinite numbers obeying the same laws as ordinary real numbers. Contrary to published opinion, Dodgson considered non-Euclidean geometry as a possible model of real space, and he understood that measurement in this geometry depended on the denial of the Archimedean axiom:

> "... given a line and a Point not on it, a whole 'pencil' of Lines may be drawn, through the Point, and not meeting the given Line.... After drawing one such line the others [will] make with it angles which are *infinitely small fractions of a right angle.*" [13, p. 51]

Dodgson needed a way of apprehending the behavior of parallel straight lines at infinity. In CMI he describes the construction of two linear magnitudes not obeying the Archimedean axiom in this way:

Beginning with an infinite strip whose width is one inch and whose area is infinite, he claims that if the sides of the strip are allowed to approach each other, the area remains infinite (until the sides coincide and the area becomes zero) because a sequence of infinite areas having a finite ratio to each other is generated.

Now Dodgson reverses the process and starting with an infinite strip whose area he (erroneously) assumes is one square inch, he concludes that the width of the strip must be infinitely small because the finite area of the strip is the product of an infinite length H and an infinitesimal width $\frac{1}{H}$[7].

To construct different orders of infinitesimals of smaller size, he first cuts off a one inch piece from the infinite strip whose area he believes is one square inch, claiming its area must be a first order infinitesimal since no multiple of the area of this short strip can equal the one inch area of the infinitely long strip. Now he seeks the value of the width of the short strip (whose length is infinite) and concludes it must be a smaller infinitesimal than the width of the infinite strip because otherwise the short strip would have a finite area. Continuing in this way yields smaller and smaller infinitesimals of the third, fourth, *etc.* orders.

This book, although flawed, is to my knowledge, the earliest publication linking the necessity of a non-Archimedean number system containing infinite and infinitesimal numbers with a geometry that includes infinitely large and infinitely small magnitudes. Moreover, in the third edition, in proving the Euclidean parallel postulate from the Archimedean axiom — by restricting the statement of the twelfth (parallel) axiom to apply only to a finite angular defect, Dodgson establishes the equivalence of segments that satisfy the Archimedean property to segments that are finite, *i.e.* commensurable segments.

Dodgson offers additional evidence that physical space indeed may be non-Euclidean:

> "If a Pair of Lines make, with a certain transversal, two interior angles on the same side of it together less than two right angles, then, so long as the defect is *finite*, there is no doubt that the Lines intersect … when the

[7]Dodgson correctly states that the area of a plane is an infinity of higher order than the area of an infinite strip whose width is finite. Also, he understands that if two lines (infinite) intersect in a finite angle, the area of one of the sectors created is an infinity of higher order than the area of the infinite strip. One should note that during this time Henri Lebesgue's (1875-1941) theory of area, the content of his thesis (1902), had not yet been published.

> 'defect from two right angles' becomes an Infinitesimal of the *first* order, the lines may possibly intersect, but can only do so at an infinite distance; and that, when the defect has become an Infinitesimal of the *second* order, the Lines have ceased to intersect." [13, 3rd ed., p. 55]

7. Some Twentieth Century Developments.

Work toward a rigorous theory of infinitesimals continued into the twentieth century. By 1951 Alfred Tarski had demonstrated that real closed ordered fields are first order indistinguishable from the ordered field of real numbers $\mathbb{R}$, *i.e.* $\mathbb{R}$ does not admit an algebraic extension to an ordered field that is more inclusive, but it is not unique.

In 1961 Abraham Robinson was able to establish non-Archimedean real closed extensions of the system of real numbers (the non-standard or hyperreal number systems) incorporating infinite and infinitesimal numbers. Building on earlier work by Tarski (1935), Thoralf Skolem (1934), and A.I. Malcev (1936), among others, Robinson's results in model theory, a term due to Tarski, resolved the inconsistency in Leibniz's concept of the continuum.

That inconsistency is this: Leibniz not only assumed two quantities can be considered equal if they differ by an infinitesimally small amount relative to them (nonlinearity), but also that the laws of arithmetic hold for infinitesimal as well as for finite quantities (linearity).

Robinson's results based on properties of predicate logic provide a consistent theory in respect to classical mathematics by establishing that x and $x+dx$, although not equal, are equivalent in a well-defined sense and can be substituted for each other in some, but not in all relations. In other words, if $\mathbb{K}$ is the set of sentences in a formal language L which is true in $\mathbb{R}$, then there exists a proper (non-Archimedean) extension $^*\mathbb{R}$ of $\mathbb{R}$ that is a model of $\mathbb{K}$, provided that the sentences of $\mathbb{K}$ refer only to the admissable entities of a given type. [36, p. 538]

Statements about real numbers, when reinterpreted according to the rules for extending the theory to $^*\mathbb{R}$, are valid for the members of $^*\mathbb{R}$. $^*\mathbb{R}$ is a totally ordered non-Archimedean field containing $\mathbb{R}$ as a proper subfield, and containing infinitely small numbers.

Robinson's rigorous theory of infinitesimals, to which the operations of real numbers apply, distinguishes between the standard part of the finite real number a, the uniquely determined standard real number infinitely close to a, and an infinitesimal number a, one whose absolute value is less than any positive standard real number. In Robinson's theory each standard real number is surrounded by numbers infinitely close to it, while the infinitesimals form a set whose standard part is zero[8].

In Robinson's theory the following are progressively smaller nonzero infinitesimals, the equivalence classes of:

$$\{1, 1/2, 1/3, \ldots, 1/n, \ldots\},$$
$$\{1, 1/4, 1/9, \ldots, 1/n^2, \ldots\},$$
$$\{1, 1/2, 1/4, \ldots, 1/2^n, \ldots\}.$$

These infinitesimals are different from those that result from the theory put forth by Leibniz who considered infinitesimals as being divided into classes of successive orders of infinite smallness. For example, for the first order differential dx, all other first order differentials stand in finite ratio to dx. Generally, all nth order differentials stand in finite ratio to dx^n, and the set of infinitesimals consists only of these classes of differentials [7, p. 83].

8. Conclusion.

Since the time of Euclid, concerns about magnitude and number have been important in the foundations of mathematics. Beginning in the last thirty years of the nineteenth century the relationship between numbers and the magnitudes used in Euclidean geometry together with the implications of the Archimedean axiom became clarified. Ultimately these issues and those involving infinitesimals came together in Robinson's nonstandard analysis.

Dodgson's inchoate theory marks one step in the historical evolution of the concept of infinitely small numbers. His approach, in which infinitesimals are formulated as elements in a non-linear (non-Archimedean) number system underlying a non-Euclidean geometry, is the earliest attempt to establish this connection.

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