L^p improving multilinear Radon-like transforms

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Abstract

We characterize (up to endpoints) the k-tuples $(p_1, \ldots, p_k)$ for which certain k-linear generalized Radon transforms map the product $L^{p_1} \times \cdots \times L^{p_k}$ boundedly into $\mathbb{R}$. This generalizes a result of Tao and Wright.

1. Introduction

Let $n \geq 2$, $k \geq 2$, and for $1 \leq j \leq k$, let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a smooth submersion (i.e. $D\pi_j$ has maximal rank at each point). Without loss of generality, $\pi_j(0) = 0$, $1 \leq j \leq k$. We define an operator S, acting on k-tuples of functions on $\mathbb{R}^{n-1}$ by

$$(1.1) \qquad S(f_1, \ldots, f_k) := \int_{\mathbb{R}^n} \prod_{j=1}^k f_j \circ \pi_j(x) a(x) \, dx.$$

Here $a \in C_0^\infty(\mathbb{R}^n)$ is a cutoff function with $a(0) \neq 0$ whose support will be contained in a small neighborhood V of 0. We are interested in k-tuples $(p_1, \ldots, p_k)$ for which S satisfies

$$(1.2) \qquad \left| S(f_1, \ldots, f_k) \right| \leq C \prod_{j=1}^k \|f_j\|_{L^{p_j}(\mathbb{R}^{n-1})},$$

where C is a finite constant which depends on the π_j, the p_j, and a, but not on the f_j.

This issue has been resolved (or resolved up to endpoints) in some special cases.

First, when $k = 2$, Tao and Wright in [22] characterized up to endpoints the pairs (p_1, p_2) such that (1.2) is satisfied. These bounds were reproved by Christ in [4] using partially alternative techniques. More recently, in [8], Gressman used the techniques in [22] and [4] and several new ideas to establish restricted weak-type bounds at the endpoint when the fibers of the π_j are polynomial curves.

Second, when $k = n$ and the kernels of the differentials $d\pi_1, \ldots, d\pi_n$ span the tangent space to $\mathbb{R}^n$ at every point, the bound (1.2) holds with $p_j = n - 1$, $1 \le j \le n$. When the π_j are linear, this is the Loomis–Whitney inequality ([12]). When the π_j are not linear, the bound was proved by Bennett, Carbery, and Wright in [3] (when the π_j are in $C^3(\mathbb{R}^n)$). An alternative proof, using induction on scales, has recently been given in [1] by Bejenaru, Herr, and Tataru (perturbed Loomis–Whitney in dimension 3) and as a special case of the result in [2] by Bennett and Bez. The advantage of these two more recent results is that they give more quantitative bounds and treat a lower regularity case ($\pi_j \in C^{1,\beta}(\mathbb{R}^n)$) than [3].

There is a significant gap between the special cases treated in [22] and [3]. In the first case, curvature of the fibers of the π_j plays a role in determining the L^p bounds, but the theory is restricted to bilinear operators. In the second case, multilinear operators are allowed, but curvature plays no role. Our goal in this article is to fill in this gap by characterizing (up to endpoints) the k-tuples $(p_1, \ldots, p_k)$ such that the bound (1.2) holds for all values of k and smooth submersions $\pi_1, \ldots, \pi_k : \mathbb{R}^n \to \mathbb{R}^{n-1}$. Our techniques are an adaptation of the method of refinements developed by Christ in [5] and later applied in [22], [4], and [3] as well as many other articles. In particular, many of our arguments are adapted from [4] and [22], though some new details are needed for the multilinear case. To the extent possible, we have tried to follow the outline from those two works.

Notation. The notation we will employ is relatively standard, and in particular largely matches that in [22] and [4]. As has become common in the harmonic analysis literature, we will hide constants in two types of symbols. If A and B are two non-negative real numbers, then $A \lesssim B$ if there exists a (large) constant C such that $A \le CB$, and $A \ll B$ if there exists a (small) constant c such that $A \le cB$. The chief difference between the symbols is in their use, $\lesssim$ typically appearing in the conclusion and $\ll$ in the hypothesis of a statement. For instance, the statement, "If $A \ll B$, then $f(A) \lesssim f(B)$," may be read as, "There exist constants c and C such that whenever $A \le cB$, we have $f(A) \le Cf(B)$." Finally, $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The dependence of the implicit constants will be specified as needed.

2. Restricted weak type inequalities and L^p-improving operators

We start by making some preliminary reductions in the k-tuples $(p_1, \ldots, p_k)$ under consideration. For the purposes of this section, all implicit constants depend on a, the π_j, and the p_j.

Suppose that $\sum_{j=1}^{k} p_j^{-1} \leq 1$. Let V be a bounded set which contains $\mathrm{supp}(a)$. Then by Hölder's inequality,

$$|S(f_1, \ldots, f_k)| \lesssim \prod_{j=1}^{k} \|f_j \circ \pi_j \cdot \chi_V\|_{L^{p_j}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{k} \|f_j\|_{L^{p_j}(\mathbb{R}^{n-1})},$$

where the last inequality follows from our assumption that π_j is a submersion and by the boundedness of V.

Now suppose that $p_i < 1$ for some $1 \leq i \leq k$. For $0 < \delta \ll 1$, let $f_i = f_\delta$ be the characteristic function of the ball of radius δ centered at 0 in $\mathbb{R}^{n-1}$. For $j \neq i$, let f_j be the characteristic function of the ball of radius 1 centered at 0 in $\mathbb{R}^{n-1}$. Then since $a(0) \neq 0$, and since π_j is a submersion,

$$|S(f_1, \ldots, f_k)| \sim \delta^{n-1}$$

and

$$\prod_{j=1}^{k} \|f_j\|_{L^{p_j}} \sim \delta^{(n-1)/p_i}.$$

Letting $\delta \to 0$, (1.2) cannot hold.

Henceforth, we will consider only those k-tuples $p = (p_1, \ldots, p_k)$ satisfying

(2.1) $$\sum_{j=1}^{k} p_j^{-1} > 1 \quad \text{and} \quad 1 \leq p_j \leq \infty \quad \text{for } 1 \leq j \leq k.$$

We say that S is L^p-improving if it satisfies (1.2) for some k-tuple p satisfying (2.1). This terminology is motivated by the case when $k = 2$ and the

fibers of π_1 and π_2 are transverse near 0. In this case, (1.2) is equivalent to boundedness of the operator T defined by

$$Tf(x) = \int_{\pi_2^{-1}\{x\}} f(y)\, a(y)\, d\sigma_x(y)$$

from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2'}(\mathbb{R}^n)$. (Here $d\sigma_x$ is equal to arclength measure on $\pi_2^{-1}\{x\}$.) Additionally, (2.1) is equivalent to $p_2' > p_1$, and since $L^{p_2'}(V) \subset L^{p_1}(V)$ for V bounded, we think of Tf as lying in a better space than f.

We will focus on establishing restricted weak type inequalities, i.e. in proving (1.2) in the case when each f_j is the characteristic function of a Borel set. We note that in order to prove that S is of restricted weak type $(p_1, \ldots, p_k)$, it is enough to show that whenever $\Omega \subset \mathbb{R}^n$ is a Borel set, we have

$$(2.2) \qquad \int \chi_\Omega(x)\, a(x)\, dx \lesssim |\pi_1(\Omega)|^{1/p_1} \cdots |\pi_k(\Omega)|^{1/p_k},$$

with the implicit constant independent of Ω. This can be reformulated as a lower bound on Ω as follows. We assume that $\Omega \subset \mathrm{supp}(a)$, and for $1 \le j \le k$, we define

$$(2.3) \qquad \alpha_j := \frac{|\Omega|}{|\pi_j(\Omega)|}.$$

We observe that by the coarea formula, α_j is approximately equal to the average size (in terms of euclidean arclength) of the intersection of the fibers of π_j with Ω. Additionally, under the assumption (2.1), (2.2) is equivalent to the inequality

$$(2.4) \qquad \alpha^b := \alpha_1^{b_1} \cdots \alpha_k^{b_k} \lesssim |\Omega|,$$

where $b = b(p) = (b_1, \ldots, b_k)$ is defined by

$$(2.5) \qquad b_i = \frac{p_i^{-1}}{\sum_{j=1}^k p_j^{-1} - 1}, \quad 1 \le i \le k.$$

It is this inequality that we will try to prove.

3. Vector fields and statement of results

Associated to each of the submersions is a (nonunique) C^∞ vector field X_j on $\mathbb{R}^n$ which is nonvanishing and tangent to the fibers of π_j.

In [6], Christ, Nagel, Stein, and Wainger proved that if $k = 2$, then S is L^p-improving if and only if the X_j and their iterated Lie brackets span the tangent space to $\mathbb{R}^n$ at each point where $a \ne 0$. In Section 5, we will prove the necessity portion of this theorem for $k > 2$:

Theorem 1. *If S is L^p-improving, then the X_j and their iterated Lie brackets span the tangent space to $\mathbb{R}^n$ at each point where $a \neq 0$.*

The spanning of the tangent space by the iterated Lie brackets is known as the Hörmander condition.

The results of [6] characterized L^p-improving operators but did not show for which pairs (p_1, p_2) the inequality (1.2) holds. In [22] Tao and Wright determined, up to endpoints, which pairs (p_1, p_2) as in (2.1) satisfy (1.2). A later proof of this theorem, along the lines of [22] but with some simplifications, is due to Christ in [4]. The bulk of this article will be devoted to showing that the result of Tao and Wright extends to the case $k \geq 2$.

To state the result, we review a few definitions from [22], generalized to the multilinear setting.

A *word* is a d-tuple $w \in \{1, \ldots, k\}^d$ for some $d \geq 1$, and W denotes the set of all words. If $w \in W$, its degree is the k-tuple whose j-th component is the number of entries of w which equal j. Finally, to each $w \in W$ is associated a vector field X_w, defined by the recursive equation

$$X_{(w,j)} := [X_w, X_j].$$

For example, if $k \geq 3$,

$$X_{(1,2)} = [X_1, X_2] \qquad X_{(1,2,3)} = [[X_1, X_2], X_3].$$

Crucially, by antisymmetry and the Jacobi identity, each iterated Lie bracket of the X_j may be written as a linear combination (with constant coefficients) of vector fields X_w (cf. [9, Lemma 4.4]). For instance,

$$[[X_1, X_2], [X_3, X_4]] = -X_{(1,2,4,3)} + X_{(1,2,3,4)}.$$

Therefore the Hörmander condition is equivalent to the statement that the vector fields X_w, with $w \in W$, span the tangent space to $\mathbb{R}^n$ at each point where $a \neq 0$.

Given an n-tuple $I = (w_1, \ldots, w_n)$ of words, we denote by λ_I the determinant

$$\lambda_I(x) := \det(X_{w_1}(x), \ldots, X_{w_n}(x)),$$

and by $\deg(I)$ the k-tuple

$$\deg(I) := \deg(w_1) + \cdots + \deg(w_n).$$

Finally, we recall one more definition before stating the main theorems. The Newton polytope P of the vector fields $X_1, \ldots, X_k$ is the closed convex hull of the set of points

$$\big\{(b_1, \ldots, b_k) \geq \deg(I) : I \in W^n \text{ and } \lambda_I(0) \neq 0\big\}.$$

Here $(b'_1, \ldots, b'_k) \geq (b_1, \ldots, b_k)$ if $b'_i \geq b_i$ for each $1 \leq i \leq k$. Hence if $P \neq \emptyset$, then P has nonempty interior. Recall the definition (2.5) of $b(p)$ for $p = (p_1, \ldots, p_k)$ satisfying (2.1). The next theorem sharpens Theorem 1.

Theorem 2. *If $p \in [1, \infty]^k$ satisfies (2.1), and $b(p)$ does not lie in P, then S is not even of restricted weak-type $(p_1, \ldots, p_k)$.*

The above theorem and the next one almost completely characterize those k-tuples p for which S satisfies (1.2).

Theorem 3. *If $p \in [1, \infty]^k$ satisfies (2.1) and $b(p)$ lies in the interior of P, then, provided $\mathrm{supp}(a)$ is contained in a sufficiently small neighborhood of 0, S satisfies (1.2), i.e. S is of strong-type $(p_1, \ldots, p_k)$.*

We have as a corollary to Theorems 1, 2, and 3 the following.

Corollary. *The operator S is L^p-improving if and only if the iterated Lie brackets of the X_j span the tangent space to $\mathbb{R}^n$ at each point where $a \neq 0$.*

We will give a more geometric formulation of Theorem 3 later; see Theorem 4.

4. Related work

There is an extensive bibliography in [22] to which we direct the interested reader. We will focus here on some more recent results.

Endpoint bounds

The main theorems here and in [22] and [4] do not establish boundedness of the operator T at the Lebesgue endpoints (the points $p = (p_1, \ldots, p_k)$ with $b(p)$ equal to a vertex of the Newton polytope). At least in the case of real analytic π_j, it is likely that the proof and not the operator is at fault for this omission. See remarks in [4, Section 11] for some conjectures and remarks related to endpoint bounds in the case $k = 2$.

In a few special cases of the Tao–Wright theorem, strong-type endpoint bounds are known. When $k = 2$, one example of our operator is $S(f_1, f_2) := \langle Tf_1, f_2 \rangle$, where

$$Tf(x) := \int f(x - \gamma(t)) \, a(x, t) \, dt.$$

Here $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a parametrized curve. When $\gamma(t) = (t, \ldots, t^n)$, the author has proven endpoint bounds for T in high dimensions in [19], sharpening the restricted weak-type result due to Christ in [5] and extending lower dimensional results of Littman in [11], and Oberlin in [14], [15], [16]. In another recent article [7], Dendrinos, Laghi, and Wright have established

strong-type endpoint bounds for convolution with affine arclength measure along polynomial curves γ in dimension 3, extending work of Oberlin in [17], and this result was generalized to higher dimensions by the author in [20]. The work [10] of Laghi is in a similar vein.

Finally, in [8] Gressman has settled the question of restricted weak-type boundedness at the endpoint in the polynomial case of the Tao–Wright theorem. The arguments in [8] extend to the multilinear setting with no difficulty.

Another multilinear operator

In [3], Bennett, Carbery and Wright have proved the following non-linear generalization of the Loomis–Whitney inequality: If $k = n$ and the vector spaces $X_1, \dots, X_n$ span the tangent space to $\mathbb{R}^n$ at 0, then for a having sufficiently small support (containing 0), $|S(f_1, \dots, f_n)| \lesssim \prod_{j=1}^n \|f_j\|_{L^{n-1}(\mathbb{R}^{n-1})}$. The articles [2] and [1] contain proofs of this generalization by induction on scales and establish the generalization of the Loomis–Whitney inequality under much lower regularity assumptions (though [2] is more general, addressing for instance certain cases wherein the fibers have unequal dimensions). Our result gives a partial generalization of this bound (in the perturbed Loomis–Whitney case). On the one hand, our methods cannot be used to obtain the endpoint $(n-1, \dots, n-1)$ as this corresponds to $b = (1, \dots, 1)$. On the other hand, even in the case $k = n$, our theorem establishes new bounds for certain $\pi_1, \dots, \pi_n$.

Multi-parameter Carnot–Carathéodory Balls

Independently of this work, in [21] Street has generalized the work of Nagel–Stein–Wainger and Proposition 4.1 of [22] for multi-parameter Carnot–Carathéodory balls, as well as analyzing the situation when the iterated Lie brackets span a proper subspace of $\mathbb{R}^n$. This work does not however address certain issues which we use to bound the multilinear operator.

5. Proof of Theorem 1

In [6], Christ, Nagel, Stein, and Wainger consider an operator defined as in (1.1) when $k = 2$ and the π_j are codimension m submersions with $0 < m < n$. It is rather simple to adapt the arguments in [6] to our circumstances to produce a short, relatively self-contained proof of Theorem 1. We record this proof here both for the convenience of the reader and to provide some useful geometric intuition for later on.

Proof. Let V be the vector space spanned by $\{X_w(0) : w \in W\}$, and let r be its dimension. Since each X_j is nonzero, we have that $r \geq 1$, and for the proof of the proposition, we may assume that $r < n$. Let $w_1, \ldots, w_r \in W$ be chosen so $\{X_{w_1}(0), \ldots, X_{w_r}(0)\}$ is linearly independent, and let $Y_j = X_{w_j}$.

For $t \in \mathbb{R}^r$, define

$$t \cdot Y := \sum_{j=1}^{r} t_j Y_j, \quad \Gamma(t) := \exp(t \cdot Y)(0).$$

Then Γ has rank r at 0, so it is an embedding of a small neighborhood of 0 in $\mathbb{R}^r$ onto an r-dimensional submanifold $0 \in M \subset \mathbb{R}^n$. Heuristically, the X_j lie along M, so considering a δ-neighborhood M_δ of M, for each $1 \leq j \leq k$, $\pi_j(M_\delta)$ is of size δ^{n-r}, which is proportional to the size of M_δ. Letting δ tend to 0, we obtain the requirement $\sum p_j^{-1} \leq 1$. Unfortunately, this heuristic is misleading, since we only have information about the Lie brackets of the X_j at 0, and we will have to use the techniques of [6] to obtain more quantitative information; in the terminology of [6], the submanifold M will be invariant under the X_j to infinite order at 0.

We will use a quantitative version of the Baker–Campbell–Hausdorff formula, which is stated in [6] (for instance). Let $V_1, \ldots, V_m$ and $W_1, \ldots, W_m$ be smooth vector fields on $\mathbb{R}^n$. Then for $v, w \in \mathbb{R}^m$, if we define $v \cdot V := \sum v_j V_j$ and $w \cdot W$ analogously, then

$$\exp(v \cdot V) \exp(w \cdot W)(0) = \exp\Big(\sum_{k=1}^{N} c_k(v \cdot V, w \cdot W) \Big) + O(|v|^{N+1} + |w|^{N+1}).$$

Here each c_k is a homogeneous Lie polynomial of degree k.

For $s' \in \mathbb{R}^r$ and $s'' \in \mathbb{R}$ sufficiently close to 0 and $1 \leq j \leq k$, we define $G_j(s) = G_j(s', s'') = \exp(s'' X_j) \Gamma(s')$. Since $\{X_w(0) : w \in W\} \subset \mathrm{span}\{Y_1, \ldots Y_r\}$, we may write

$$\sum_{k=1}^{N} c_k(s' \cdot Y, s'' X_j) = P_1^j(s) \cdot Y + \sum_{|\beta|=1}^{N} s^\beta W_{\beta,1}^j,$$

where P_1^j is a vector-valued polynomial with $P_1^j(0) = 0$ and each $W_{\beta,1}^j$ is in the span of $\{X_w : w \in W\}$ and satisfies $W_{\beta,1}^j(0) = 0$.

We assume inductively that

$$G_j(s) = \exp(P_m^j(s) \cdot Y + \sum_{|\beta|=m}^{N} s^\beta W_{\beta,m}^j)(0) + O(|s|^{N+1}),$$

where P_m^j is a vector-valued polynomial, $P_m^j(0) = 0$, each $W_{\beta,m}^j$ is in the span of $\{X_w : w \in W\}$, and $W_{\beta,m}^j(0) = 0$. The implicit constants in the $O(\cdot)$ notation are allowed to depend on N (via the C^N norms of the X_j). Since $W_{\beta,m}^j(0) = 0$ for each β, m,

$$G_j(s) = \exp\Big(P_m^j(s) \cdot Y + \sum_{|\beta|=m}^{N} s^\beta W_{\beta,m}^j\Big) \circ \exp\Big(-\sum_{|\beta|=m}^{N} s^\beta W_{\beta,m}^j\Big)(0) + O(|s|^{N+1}).$$

We apply the Baker–Campbell–Hausdorff formula and our inductive assumption $P_m^j(0) = 0$ to see that the right side equals the exponentiation of

$$P_m^j(s) \cdot Y + \sum_{k=2}^{N} c_k\Big(P_m^j(s) \cdot Y + \sum_{|\beta|=m}^{N} s^\beta W_{\beta,m}^j, -\sum_{|\beta|=m}^{N} s^\beta W_{\beta,m}^j\Big) =$$

$$= P_m^j(s) \cdot Y + \sum_{|\beta|=m+1}^{N} s^\beta \tilde{W}_{\beta,m+1}^j + O(|s|^{N+1}).$$

Here the $\tilde{W}_{\beta,m+1}^j$ are in the span of $\{X_w : w \in W\}$ but do not necessarily vanish at zero. We can add the "error" in the $\tilde{W}_{\beta,m+1}^j$ (the part that does not vanish at 0) to $P_m^j(s) \cdot Y$ and write

$$G_j(s) = \exp(P_{m+1}^j(s) \cdot Y + \sum_{|\beta|=m+1}^{N} s^\beta W_{\beta,m+1}^j)(0) + O(|s|^{N+1}),$$

where P_{m+1}^j is a vector-valued polynomial satisfying $P_{m+1}^j(0) = 0$ and the $W_{\beta,m+1}^j$ are in the span of $\{X_w : w \in W\}$ and satisfy $W_{\beta,m+1}^j(0) = 0$. Proceeding by induction,

$$G_j(s) = \exp(P_{N+1}^j(s) \cdot Y) + O(|s|^{N+1}),$$

where $P_{N+1}^j(0) = 0$, and we may assume that P_{N+1}^j is of degree less than $N+1$.

Consider coordinates $x = (x', x'') \in \mathbb{R}^r \times \mathbb{R}^{n-r}$ on a neighborhood of 0 in $\mathbb{R}^n$ so that $M = \{x'' = 0\}$. Define $\gamma_j(x, t) := e^{tX_j}(x)$. We will also write $\gamma_j = (\gamma_j', \gamma_j'')$ in the above coordinates.

Let N be a fixed positive integer. Since Γ is a local parametrization of M, and since x'' gives the distance from the point x to M, we have shown that for each x', t sufficiently close to 0 and each $1 \leq j \leq k$,

$$\gamma_j''(x', 0'', t) = O(|x'|^{N+1} + |t|^{N+1}).$$

More generally, since $\gamma_j(x, 0) = x$,

$$\gamma_j(x, t) = \big(x' + O(|t|), x'' + O(|t||x''|) + O_N(|t|(|x'|^N + |t|^N))\big).$$

Let $\delta > 0$. Let $M_{\delta,N} = \{(x', x'') : |x'| < \delta, |x''| < C_N \delta^N\}$. Then if $x \in M_{\delta,N}$, and $|t| < c\delta$, $\gamma_j(x,t) \in M_{2\delta,N}$. Therefore $\delta|\pi_j(M_{\delta,N})| \lesssim |M_{2\delta,N}|$, i.e., $|\pi_j(M_{\delta,N})| \lesssim C_N \delta^{r-1} \delta^{N(n-r)}$. On the other hand, supposing $\sum p_j^{-1} > 1$, we can find N so that $r + N(n-r) < (\sum p_j^{-1})((r-1) + N(n-r))$, and letting $\delta \to 0$ (with this N fixed), $|M_{\delta,N}| \lesssim \prod |\pi_j(M_{\delta,N})|^{1/p_j}$ cannot hold with a constant independent of δ. ∎

6. Multi-parameter Carnot–Carathéodory Balls I

We will spend much of Sections 6 and 8 reviewing properties of multi-parameter Carnot–Carathéodory balls which can be readily deduced from the work of Tao–Wright and Nagel–Stein–Wainger, though some new details will be needed. An independent and far more in-depth discussion of these objects may be found in [21].

If $\delta_1, \ldots, \delta_k$ are sufficiently small positive numbers, and $x \in \mathbb{R}^n$ is sufficiently close to 0, then we define the multi-parameter Carnot–Carathéodory ball $B(x; \delta_1, \ldots, \delta_k)$ to be the closure of the set of all points

$$e^{t_N \delta_{j_N} X_{j_N}} e^{t_{N-1} \delta_{j_{N-1}} X_{j_{N-1}}} \cdots e^{t_1 \delta_{j_1} X_{j_1}}(x),$$

where $N \geq 0$ is an integer, $1 \leq j_i \leq k$, for each $1 \leq i \leq N$, and $\sum_{i=1}^N |t_i| \leq 1$.

Essentially, this is the set of points which can be reached by starting at x, then flowing first along one vector field $\delta_{j_1} X_{j_1}$, then along another, and so on, for total time less than or equal to 1.

We now give a heuristic proof of Theorems 2 and 3. We caution the reader that the equations in the following two paragraphs are not quite true or are false without additional assumptions, and require some proof at the very least.

For Theorem 2, we consider $\Omega = B(0; \delta_1, \ldots, \delta_k)$, with $0 < \delta_j \ll 1$. Since

$$e^{t\delta_j X_j}\Omega \subset B(0; 2\delta_1, \ldots, 2\delta_k)$$

whenever $|t| \leq 1$ and since

$$|B(x; 2\delta_1, \ldots, 2\delta_k)| \sim |B(x; \delta_1, \ldots \delta_k)|,$$

we have that $\frac{|\Omega|}{|\pi_j(\Omega)|} \gtrsim \delta_j$. Thus if (2.4) holds, we must have that $\delta^b \lesssim |\Omega|$. As

$$|B(x; \delta_1, \ldots, \delta_k)| \sim \sup_I \delta^{\deg(I)}|\lambda_I(x)|$$

(roughly), we must have $b \in P$.

For Theorem 3, let $\Omega \subset \operatorname{supp}(a)$. Then for $1 \leq j \leq k$, α_j represents the average of the 1-dimensional Hausdorff measure of the set $\pi_j^{-1}\{y\}$ for $y \in \pi_j(\Omega)$. Hence we expect that for a 'good' point $x \in \Omega$ and for $1 \leq j \leq k$,

$$\left| \{ t \in \mathbb{R} : e^{tX_j}(x) \in \Omega \} \right| \sim \alpha_j.$$

Iterating, Ω contains a set which 'looks like' the Carnot–Carathéodory ball $B(x; \alpha_1, \ldots, \alpha_k)$, so whenever $\lambda_I(x) \neq 0$,

(6.1) $$\alpha^{\deg I} \sim \alpha^{\deg I} |\lambda_I(x)| \lesssim |B(x; \alpha_1, \ldots, \alpha_k)| \leq |\Omega|.$$

Thus, from the definition of P, we obtain (2.4), and by real interpolation, Theorem 3.

There are certain technicalities involved in the proof of Theorems 2 and 3, for instance a correct analogue of (6.1), for which we will not be able to use the Carnot–Carathéodory balls themselves. The remainder of this section will be devoted to translating Tao and Wright's discussion of certain sets and mappings associated to the Carnot–Carathéodory balls from the bilinear to the multilinear setting. We will return to the Carnot–Carathéodory balls themselves in Section 8.

We begin by reviewing some notation from [22]. For the remainder of this section $\varepsilon > 0$ will be a small parameter, and K will be a large parameter. We will be more specific about these quantities later on. For the purposes of this section, all implicit constants depend on ε and on the π_j.

We let $\delta_1, \ldots, \delta_k$ be positive numbers which satisfy the smallness and the nondegeneracy conditions

(6.2) $$\delta_j \leq c_{\varepsilon,K}, \quad 1 \leq j \leq k$$
(6.3) $$\delta_i \lesssim \delta_j^\varepsilon, \quad 1 \leq i, j \leq k.$$

We remark that the nondegeneracy condition is necessary for the balls $B(x; \delta_1, \ldots, \delta_k)$ to satisfy the doubling property of [13] and [18, Ch. 1], for instance. Indeed, there is an example in [4] of a pair of vector fields X_1 and X_2 which satisfy the Hörmander condition but have the property that there is no universal constant C such that

$$|B(0; 2\delta_1, 2\delta_2)| \leq C |B(0; \delta_1, \delta_2)|$$

for all sufficiently small δ_1, δ_2.

By Theorem 1, we may assume that there exists $I_0 \in W^n$ with $\lambda_{I_0}(0) \neq 0$. Shrinking V, we may in fact assume that $|\lambda_{I_0}| \sim 1$ throughout V. We let $d := \sum_{j=1}^k (\deg I_0)_j$, and define $\mathbf{I}$ to be the finite set

(6.4) $$\mathbf{I} := \left\{ I \in W^n : (\deg I)_j \leq \tfrac{d}{\varepsilon}, 1 \leq j \leq k \right\}.$$

If $I \notin \mathbf{I}$ and $\delta = (\delta_1, \dots, \delta_k)$ is as above, then $(K\delta)^{\deg I} \lesssim (K\delta)^{\deg I_0}$. We define $\Lambda = \Lambda_{K\delta}$ by

$$(6.5) \qquad\qquad \Lambda(x) = ((K\delta)^{\deg I} \lambda_I(x))_{I \in \mathbf{I}}.$$

Since $|\Lambda(x)| \gtrsim (K\delta)^{\deg I_0}$, we have that

$$(6.6) \qquad\qquad (K\delta)^{\deg I}|\lambda_I(x)| \lesssim C_I |\Lambda(x)|,$$

for all $I \in W^n$ and $x \in V$.

With $x_0 \in V$ fixed, we choose $I_{x_0} = (w_1, \dots, w_n) \in \mathbf{I}$ so that

$$(6.7) \qquad\qquad (K\delta)^{\deg I_{x_0}}|\lambda_{I_{x_0}}(x_0)| \sim |\Lambda(x_0)|.$$

Our goal is to gain a basic understanding of the mapping $\Phi = \Phi_{x_0, K\delta}$ defined by

$$(6.8) \qquad \Phi(t_1, \dots, t_n) = \exp\left(\sum_{j=1}^{n} K^{-1}(K\delta)^{\deg w_j} t_j X_{w_j}\right)(x_0)$$

for t near 0 in $\mathbb{R}^n$. By smallness of the $K\delta_j$, the domain of definition of Φ may be taken to be uniform in $x_0 \in V$ and δ.

Since

$$(6.9) \qquad |\det D\Phi(0)| = K^{-n}(K\delta)^{\deg I}|\lambda_{I_{x_0}}(x_0)| \sim |\Lambda(x_0)| \neq 0,$$

the mapping Φ is a diffeomorphism on a neighborhood U of 0. For $t \in U$ and $w \in W$, we define Y_w to be the pullback by Φ of $K^{-1}(K\delta)^{\deg w} X_w$:

$$(6.10) \qquad Y_w(t) = (D\Phi(t))^{-1}[K^{-1}(K\delta)^{\deg w} X_w(\Phi(t))].$$

Although we are suppressing this in the notation, we keep in mind throughout that the map Φ, the set U, and the vector fields Y_w depend on the base point $x_0 \in V$ as well as δ.

The following lemmas are proved in [22]. Though the authors only claim the results in the case $k = 2$, their proofs extend to the multilinear case with almost no alteration. For what follows, all bounds are uniform in $K > C_\varepsilon$, $0 < \delta_1, \dots, \delta_k < c_{K,\varepsilon}$ satisfying (6.3), and the base point $x_0 \in V$, but may depend on ε and the X_j.

Lemma 1. *If $B_r(0) \subset U$, for some $0 < r \lesssim 1$, then*

$$(6.11) \qquad\qquad Y_{w_i}(t) = \partial_i + O(\tfrac{|t|}{K})$$

and in particular,

$$(6.12) \qquad\qquad |\det(Y_{w_1}, \dots, Y_{w_n})(t)| \sim 1$$

for all $t \in B_r(0)$.

Lemma 2. *If $B_r(0) \subset U$ for some $0 < r \lesssim 1$, then*

(6.13)
$$|\Lambda \circ \Phi(t)| \sim (K\delta)^{\deg I_{x_0}} |\lambda_{I_{x_0}} \circ \Phi(t)|$$

on $B_r(0)$.

Lemma 3. *There exists $C \sim 1$ so that $B := B_C(0) \subset U$.*

Lemma 4. *If $w \in W$, then*

(6.14)
$$\|Y_w\|_{C^M(B)} \leq C_{w,M},$$

provided K is sufficiently large depending on w, M, and ε.

Lemma 5. *If E is a measurable subset of B, then*

(6.15)
$$|\Phi(E)| \sim K^{-n} |\Lambda(x_0)||E|.$$

We will not repeat the proofs of these lemmas, for which we direct the reader to Proposition 4.1 of [22]. To offer some explanation for the parameter K, however, we will sketch the argument for the first three lemmas.

It is easy to compute $Y_{w_j}(0) = \partial_j$, and moreover, at each point $t \in \mathbb{R}^n$, we have

$$r\partial_r := \sum_{j=1}^{n} t_j \partial_j = \sum_{j=1}^{n} t_j Y_{w_j}.$$

With these facts, and after some algebra and differential identities, we can compute radial derivatives using Lie brackets with the Y_{w_j}. The factor K^{-1} in the definitions above helps minimize the contribution coming from higher order Lie brackets once we pull back (which has the effect of replacing δ with 1), since

(6.16) $$[Y_w, Y_{w'}] = K^{-1}(D\Phi(t))^{-1}(K^{-1}(K\delta)^{\deg w + \deg w'}[X_w, X_{w'}](\Phi(t)))$$

for any $w, w' \in W$. Together with Gronwall's inequality, the bounds on the radial derivatives imply the first two lemmas. The third lemma follows from the first two and continuity, since

$$|\det D\Phi(t)| = \frac{K^{-n}(K\delta)^{\deg I_{x_0}} |\lambda_{I_{x_0}}(\Phi(t))|}{|\det(Y_{w_1}, \ldots, Y_{w_n})(t)|}.$$

We note that (6.16) would hold if we replaced each instance of $K\delta$ with δ in the above discussion, but if we did that, unless each of the w_i was actually in $\{1, \ldots, k\}$, $\Phi(B_1(0))$ would be much smaller than $B(x_0; \delta_1, \ldots, \delta_k)$ for large K.

7. Proof of Theorem 2

Here we adapt Tao and Wright's proof that the bound (2.2) can only hold if $b(p)$ (see (2.5)) lies in the Newton polytope P.

We first record some geometric properties of P. By definition, P is convex and

$$(7.1) \qquad b \in P \text{ and } b' \geq b \text{ implies } b' \in P.$$

Moreover, by Theorem 1, we may assume that $P \neq \emptyset$. Since the vertices of P are k-tuples of non-negative integers, by (7.1) P has only finitely many vertices. It is also clear from the definition that $b \in P$ implies that $\sum_{j=1}^{k} b_j \geq n$.

Now suppose that p satisfies (2.1) and that $b(p)$ does not lie in P. Then there exists $a \in \mathbb{R}^k$ and $d \in \mathbb{R}$ so that

$$b(p) \in \left\{ b \in \mathbb{R}^k : a \cdot b < d \right\} =: H_-, \quad P \subset \left\{ b \in \mathbb{R}^k : a \cdot b > d \right\} =: H_+.$$

From (7.1), each entry of a is non-negative, and since $b(p) \in [0, \infty)^k$, $d > 0$. Since P has finitely many vertices, and since each vertex lying in H_+ implies that P lies in H_+, we may assume that each entry of a is positive by continuity. Finally, by scaling, we may assume that $d = 1$.

Let $\delta_0 > 0$. Then since $a \in (0, \infty)^k$,

$$(7.2) \qquad \delta := (\delta_0^{a_1}, \ldots, \delta_0^{a_k})$$

satisfies (6.3) for some $\varepsilon > 0$, independent of δ_0. Shrinking ε if needed, we may assume that the set $\mathbf{I}$ defined in (6.4) contains all of the vertices of P and that

$$(7.3) \qquad a \cdot b(p) < 1 - \varepsilon \qquad a \cdot b > 1 + \varepsilon$$

for each $b \in P$.

We will use the results of Section 6 to show that S is not of restricted weak-type $(p_1, \ldots, p_k)$.

With $x_0 = 0$, choose K large enough that (6.13) and (6.15) hold and (6.14) holds with $w = 1, \ldots, k$ and $M = 0$ on $B = B_C(0)$, whenever δ satisfies (6.2) and (6.3).

Let $0 < c_\varepsilon < C$ be sufficiently small for later purposes, and let

$$\Omega = \Phi(B_{c_\varepsilon}(0)).$$

If $x = \Phi(t) \in \Omega$, $s \ll 1$, and $1 \leq j \leq k$, then

$$e^{s\delta_j X_j}(x) = \Phi(e^{sY_j}(t)) \subset \Phi(B_C(0))$$

by (6.14) and the smallness of c_ε. Hence

$$\frac{|\Omega|}{|\pi_j(\Omega)|} \gtrsim \delta_j.$$

Thus (2.4) implies that

$$\delta^{b(p)} \lesssim |\Omega|.$$

But by (6.15),

$$|\Omega| \lesssim K^{-n}|\Lambda(0)| \lesssim C_K \delta^{b_0}$$

for some $b_0 \in P$. Hence (as K depends on ε), by (7.3) we have

$$\delta_0^{1-\varepsilon} < \delta_0^{b(p)\cdot a} \lesssim C_K \delta_0^{b_0 \cdot a} \lesssim C_K \delta_0^{1+\varepsilon}.$$

Letting $\delta_0 \to 0$, we obtain a contradiction.

8. Multi-parameter Carnot–Carathéodory Balls II

In this section, we return to the multi-parameter Carnot–Carathéodory balls $B(x; \delta_1, \ldots, \delta_k)$, defined in Section 6. We will obtain estimates for the volumes of these balls, and will use these estimates to give alternate statements to Theorems 2 and 3. We will also prove that, under the assumptions (6.2) and (6.3), the balls satisfy the doubling property. As mentioned earlier, most of the needed results can be obtained by translating existing results from the bilinear to the multilinear setting.

As before, we let V be a small neighborhood of 0, and let $X_1, \ldots, X_k$ be smooth vector fields defined on $\mathbb{R}^n$. We assume that there exists $I_0 \in W^n$ so that $|\lambda_{I_0}(x)| \sim 1$ for all $x \in V$. All implicit constants in this section depend on $\varepsilon > 0$ and the vector fields $X_1, \ldots, X_k$.

If $\delta = (\delta_1, \ldots, \delta_k)$ is a k-tuple of positive numbers satisfying (6.3), we let Λ_δ be defined as in (6.5), with $K = 1$.

Proposition 1. *There exists a constant $C_\varepsilon > 1$ such that*

$$|B(x; \delta)| \lesssim C_\varepsilon |\Lambda_\delta(x)|,$$

whenever $x \in V$ and $\delta = (\delta_1, \ldots, \delta_k)$ is a k-tuple of positive numbers satisfying (6.3) and $\delta_i < C_\varepsilon^{-1}$.

Proof. Let $K = K_\varepsilon$ be sufficiently large that the conclusions of Lemmas 1–5 hold (with $M = 0$ in Lemma 4), uniformly in $x \in V$ and δ satisfying (6.2) and (6.3). (Note that $c_{K,\varepsilon}$ is indirectly a function of ε alone.) Henceforth x, δ will be fixed, with Φ and the Y_w defined accordingly.

Define $\tilde{B}(0; c_\varepsilon, \ldots, c_\varepsilon)$ to be the closure of the set of all points

$$e^{t_N Y_{j_N}} \cdots e^{t_1 Y_{j_1}}(0),$$

where N is a non-negative integer, $1 \leq j_i \leq k$ for $1 \leq i \leq N$ and $\sum_{i=1}^{N} |t_i| \leq c_\varepsilon$.

Then, if c_ε is sufficiently small depending on the C^0 norms of $Y_1, \ldots, Y_k$, we have that $\tilde{B}(0; c_\varepsilon, \ldots, c_\varepsilon)$ is contained in B (the domain of Φ). Since Y_i is the pullback by Φ of $\delta_i X_i$,

$$\tilde{B}(0; c_\varepsilon, \ldots, c_\varepsilon) = \Phi^{-1}(B(x; c_\varepsilon \delta_1, \ldots, c_\varepsilon \delta_k)).$$

Thus, by (6.15),

$$|B(x; c_\varepsilon \delta_1, \ldots, c_\varepsilon \delta_k)| \lesssim K^{-n} |\Lambda_{K\delta}(x)| \cdot |B| \sim C_\varepsilon |\Lambda_\delta(x)|. \qquad \blacksquare$$

Proposition 2. *There exists $C_\varepsilon > 1$ such that whenever $x \in V$ and δ obeys (6.3) and $\delta_i \leq C_\varepsilon^{-1}$, we have*

$$|B(x; \delta)| \geq C_\varepsilon^{-1} |\Lambda_\delta(x)|.$$

Furthermore, there exists a sequence $\mathbf{j} = (j_1, \ldots, j_n) \in \{1, \ldots, k\}^n$ so that

(8.1) $$|B_{\mathbf{j}}(x; \delta)| \geq C_\varepsilon^{-1} |\Lambda_\delta(x)|,$$

where $B_{\mathbf{j}}(x; \delta)$ is defined to be the closure of the set

$$\{e^{t_n \delta_{j_n} X_{j_n}} \cdots e^{t_1 \delta_{j_1} X_{j_1}}(x) : |t_i| \leq 1, 1 \leq i \leq n\}.$$

The proof of this proposition, which uses the Arzela–Ascoli theorem and the Nagel–Stein–Wainger theory, is based on the proof of a related lemma in [4]. The proof of the analogous fact in [22] seems more specialized to the bilinear case, as it uses the fact that there are only two possibilities for $\mathbf{j}$, namely $(1, 2, 1, \ldots)$ and $(2, 1, 2, \ldots)$.

Proof. If the statements in the proposition were false, there would exist sequences $\{x^{(\ell)}\}$ of points in V and $\{\delta^{(\ell)}\}$ of k-tuples satisfying the hypotheses of the proposition such that

$$\lim_{\ell \to \infty} \delta^{(\ell)} = (0, \ldots, 0)$$

and if

$$\Phi^{(\ell)} := \Phi_{x^{(\ell)}, K\delta^{(\ell)}},$$

then

(8.2) $$\lim_{\ell \to \infty} |[\Phi^{(\ell)}]^{-1}(B_{\mathbf{j}}(x; \delta^{(\ell)}))| = 0$$

for each $\mathbf{j} \in \{1, \ldots, k\}^n$. (Here $K = K_\varepsilon$ is sufficiently large to allow the applications of Lemmas 1–5 below.)

Letting $Y_i^{(\ell)}$ be the pullback of $\delta_i^{(\ell)} X_i$ by $\Phi^{(\ell)}$, by (6.14) and the Arzela–Ascoli theorem, passing to subsequences if necessary, there exists a vector field Y_i so that $Y_i^{(\ell)} \to Y_i$ in $C^M(B)$, with B as in Lemma 3 and for M arbitrarily large. We take $M \gg \varepsilon^{-1}$ so that each sequence $Y_w^{(\ell)}$ with $w \in \mathbf{I}$ (defined in (6.4)) converges in $C^N(B)$, for N large. By Lemma 1, for each ℓ there exists $(w_1^{(\ell)}, \dots, w_n^{(\ell)}) \in \mathbf{I}$ so that

$$|\det(Y_{w_1^{(\ell)}}^{(\ell)}, \dots, Y_{w_n^{(\ell)}}^{(\ell)})| \sim 1 \qquad \text{in } B.$$

By finiteness of $\mathbf{I}$, after passing to a subsequence we may assume that there is a single such n-tuple, $(w_1, \dots, w_n)$. We then have that

$$|\det(Y_{w_1}, \dots, Y_{w_n})| \sim 1 \qquad \text{in } B.$$

Now we have a contradiction. On the one hand, by the work of Nagel, Stein, and Wainger in [13], there exists a sequence $\mathbf{j}$ and a constant $0 < c_\varepsilon < 1$ so that

(8.3) $$|\tilde{B}_{\mathbf{j}}(0, c_\varepsilon, \dots, c_\varepsilon)| \sim 1,$$

where $\tilde{B}_{\mathbf{j}}(0, c_\varepsilon, \dots, c_\varepsilon)$ is the closure of the set

$$\left\{ e^{t_n c_\varepsilon Y_{j_n}} \cdots e^{t_1 c_\varepsilon Y_{j_1}}(0) : |t_i| \le 1, \, 1 \le i \le n \right\}.$$

Thus, if we let

$$\Phi_{\mathbf{j}}(t) := e^{t_n Y_{j_n}} \cdots e^{t_1 Y_{j_1}}(0),$$

then for some t^0 with $\sum |t_i^0| < c_\varepsilon$, $\det D\Phi_{\mathbf{j}}(t_0) \ne 0$.

On the other hand, because the vector fields $Y_i^{(\ell)}$ converge to Y_i in $C^M(B)$ for large M, the maps $\Phi_{\mathbf{j}}^{(\ell)}$ (defined analogously to the map above) converge to $\Phi_{\mathbf{j}}$ in (say) $C^2(B)$. Thus $|\det D\Phi_{\mathbf{j}}^{(\ell)}(t_0)| > c > 0$ (eventually), and so the $\Phi_{\mathbf{j}}^{(\ell)}$ are injective on a uniform neighborhood of t_0. This gives a lower bound on

$$\left| \tilde{B}_{\mathbf{j}}^{(\ell)}(0; c_\varepsilon) \right| := \Phi_{\mathbf{j}}\left(\{ t \in \mathbb{R}^n : |t_i| \le c_\varepsilon, 1 \le i \le n \} \right).$$

But the above set is just $\Phi_{K\delta^{(\ell)}}^{-1}(B_{\mathbf{j}}(x; \delta^{(\ell)}))$, and we have the promised contradiction. ∎

The two propositions imply the following doubling property.

Corollary. *Whenever $\delta = (\delta_1, \dots, \delta_k)$ satisfies (6.2) and (6.3), we have that*

$$|B(x; 2\delta_1, \dots, 2\delta_k)| \le C_\varepsilon |B(x; \delta_1, \dots, \delta_k)|$$

uniformly in $x \in V$.

We also obtain the following alternative characterization of the Newton polytope P.

Proposition 3.   *The Newton polytope P associated to the vector fields $X_1, \ldots, X_k$ is equal to the set of all points $b = (b_1, \ldots, b_k)$ so that*

$$(8.4) \qquad\qquad |B(0; \delta_1, \ldots, \delta_k)| \gtrsim \delta^b,$$

where the implicit constant depends on ε, but is uniform in $\delta_1, \ldots, \delta_k > 0$ satisfying (6.2) and (6.3).

Proof. If $b \notin P$, then by the propositions of this section and the proof of Theorem 2, for some $\varepsilon > 0$ no uniform bound

$$|B(0; \delta_1, \ldots, \delta_k)| \gtrsim \prod_{j=1}^{k} \delta_j^{b_j}$$

can hold. If $b \in P$ and $\delta_1, \ldots, \delta_k$ are any positive numbers, then

$$\delta^b \leq \sum_{I \in \mathbf{I}_0} \delta^I \sim |\Lambda_\delta(0)|,$$

where $\mathbf{I}_0$ is the set of vertices of P. By the propositions of this section, we thus have (8.4). ∎

By the doubling property, we have

$$|B(x; \delta_1, \ldots, \delta_i, \ldots, \delta_k)| \sim |B(x; \delta_1, \ldots, 2\delta_i, \ldots, \delta_k)|.$$

Since $e^{t\delta_i X_i} B(x; \delta) \subset B(x; \delta_1, \ldots, 2\delta_i, \ldots, \delta_k)$, while $\pi_i \circ e^{tX_j} \equiv \pi_i$, by the coarea formula we must have

$$\frac{|B(x; \delta_1, \ldots, \delta_k)|}{|\pi_i(B(x; \delta_1, \ldots, \delta_k))|} \gtrsim_\varepsilon \delta_i.$$

From this and the propositions, we are able to obtain geometric versions of Theorems 2 and 3.

Theorem 2 is thus equivalent to the following tautology: If S is of restricted weak-type $(p_1, \ldots, p_k)$, then (2.4) (with the implicit constant depending on ε) holds whenever δ satisfies (6.2) and (6.3) and $\Omega = B(0; \delta_1, \ldots, \delta_k)$.

We now give an alternative, more geometric, statement of Theorem 3, analogous to the formulation in [4].

Theorem 4. *Assume that $p = (p_1, \ldots, p_k)$ satisfies (2.1). Suppose that for each $\varepsilon > 0$ there exists $c_\varepsilon > 1$ so that whenever $\delta = (\delta_1, \ldots, \delta_k)$ is a k-tuple of sufficiently small (depending on ε) positive numbers satisfying (6.3), we have*

$$|B(0; \delta)| \leq c_\varepsilon \prod_{j=1}^{k} |\pi_j(B(0; \delta))|^{1/p_j}.$$

Then whenever $\tilde{p} > p$ (i.e. $\tilde{p}_i > p_i$, $1 \leq i \leq k$), we have (1.2).

9. Proof of Theorem 3

In this section we will prove that if V is a sufficiently small neighborhood of 0 and $a \in C_c^\infty(V)$, then S is of restricted weak-type $(p_1, \ldots, p_k)$ whenever p satisfies (2.1) and $b(p)$ lies in the interior of the Newton polytope P. One may use the arguments in this section together with a partition of unity to see the following: If V is bounded and for every vertex b of P and $x \in V$ there exists $I \in W^n$ with $\deg(I) \leq b$ and $\lambda_I(x) \neq 0$, then V is sufficiently small in the above sense. By real interpolation, this proves Theorem 3.

We note that the arguments of this section are largely based on those in [4] and [22], but some new details, such as in the refinement, are needed in the multilinear setting.

We let $\mathbf{I}_0 \subset W^n$ be the (finite) set of all n-tuples of words I such that $\lambda_I(0) \neq 0$ and such that $\deg(I)$ is a vertex of P. By passing to a smaller subset of V if needed, we may assume that if $I \in \mathbf{I}_0$, then $\lambda_I \sim 1$ on V.

Let $\Omega \subset V$ be a Borel set having positive Lebesgue measure, and let $\alpha_1, \ldots, \alpha_k$ be defined as in (2.3). By symmetry, we may assume that

$$\alpha_1 \geq \cdots \geq \alpha_k.$$

Since $\alpha_j \lesssim \mathrm{diam}(V)$, by passing to a smaller subset of V if needed, we may assume that each α_j is as small as we like.

In order to prove that

$$(2.4) \qquad |\Omega| \gtrsim \alpha_1^{b_1} \cdots \alpha_k^{b_k}$$

for b lying in the interior of P, it suffices to show that if $b \in P$, there exists a constant $C > 0$ so that for every $\varepsilon > 0$ we have

$$(9.1) \qquad |\Omega| \gtrsim \alpha_k^{C\varepsilon} \alpha^b,$$

where in the preceding statement and for the remainder of this section the implicit constant is allowed to depend on ε. To see that this suffices, note that if $b \in \mathrm{int}\, P$, then there exists $b' \in \mathrm{int}\, P$ such that $b' < b$. Then $b > b' + (0, \ldots, 0, C\varepsilon)$ if ε is sufficiently small, so (9.1) with $b = b'$ implies (2.4), by smallness of the α_j.

We will assume throughout that ε is small enough that $\mathbf{I}_0 \subset \mathbf{I}$, where $\mathbf{I}$ is the set defined in (6.4).

9.1. Refining Ω

To apply what we learned in previous sections, we must put ourselves in the situation of considering a "large" subset of Ω which "looks like" a Carnot Carathéodory ball with weakly comparable radii.

In this subsection, we will make an initial refinement of Ω which will give us the weakly-comparable "radii".

By boundedness of V, we may decompose Ω as the disjoint union of $\lesssim \alpha_k^{-C\varepsilon}$ Borel sets of diameter $\lesssim \alpha_k^\varepsilon$. Henceforth, we will work with the largest of these, denoted $\tilde{\Omega}$, which has measure

$$|\tilde{\Omega}| \gtrsim \alpha_k^{C\varepsilon}|\Omega|.$$

Next, we refine $\tilde{\Omega}$. Let $\mathbf{J} \in \{1, \ldots, k\}^{nk^n}$ be a sequence which is formed by concatenating all of the elements of $\{1, \ldots, k\}^n$ in some order. Our next goal is to construct a sequence of refinements

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_{nk^n} \subset \tilde{\Omega}$$

of $\tilde{\Omega}$ so that $\Omega_0 \neq \emptyset$ and so that for $1 \leq i \leq nk^n$ and $x \in \Omega_{i-1}$, the set

$$\{t : |t| \ll 1 \text{ and } e^{tX_{J_i}}(x) \in \Omega_i\}$$

has a particular form.

The following definition is due to Tao and Wright.

Definition 1. If $0 < \varepsilon, w \ll 1$, then a *central set of width w* is a subset S of $[-w, w]$ having positive measure and such that for any interval I

$$|I \cap S| \lesssim \left(\frac{|I|}{w}\right)^\varepsilon |S|.$$

There is an analogous definition due to Christ in [4].

Lemma 6. *Let $\Omega' \subset \tilde{\Omega}$ with $|\Omega'| \gtrsim \alpha_k^{C\varepsilon}|\tilde{\Omega}|$. Then if $1 \leq j \leq k$, there exists a subset $\langle\Omega'\rangle_j \subset \Omega'$ so that $|\langle\Omega'\rangle_j| \gtrsim \alpha_k^{C'\varepsilon}|\tilde{\Omega}|$ and so that for each $x \in \langle\Omega'\rangle_j$,*

$$\left\{t : |t| \ll 1 \text{ and } e^{tX_j}(x) \in \Omega'\right\}$$

is a central set of width w,

$$\alpha_k^{C'\varepsilon}\alpha_j \lesssim w \lesssim \alpha_k^\varepsilon$$

and measure $\gtrsim \alpha_k^{C'\varepsilon}\alpha_j$. Here C' is a constant which is larger than C, but independent of ε.

The proof is the same as that of Lemma 8.2 in [22] and will be omitted. We now define the refinements as follows:

$$\Omega_{nk^n} := \langle\tilde{\Omega}\rangle_{J_{nk^n}},$$

given Ω_i, $2 \le i \le nk^n$,

$$\Omega_{i-1} := \langle \Omega_i \rangle_{J_{i-1}},$$

and $\Omega_0 := \Omega_1$. Then $|\Omega_0| \gtrsim \alpha_k^{C\varepsilon}|\Omega| > 0$, so $\Omega_0 \ne \emptyset$. Moreover, since Ω_i satisfies the conclusion of Lemma 6 with $j = J_i$, and since $\Omega_{i-1} \subset \Omega_i$, whenever $x \in \Omega_{i-1}$,

$$\left\{ t : |t| \ll 1 \ \text{ and } \ e^{tX_{J_i}}(x) \in \Omega_i \right\}$$

is a central set of width w_i, $C_\varepsilon^{-1}\alpha_k^{C\varepsilon}\alpha_{J_i} \le w_i \lesssim \alpha_k^\varepsilon$ and measure $\ge C_\varepsilon^{-1}\alpha_k^{C\varepsilon}\alpha_{J_i}$, where C_ε is a constant depending on ε.

9.2. Filling out $\tilde{\Omega}$

In this subsection, we show that $\tilde{\Omega}$ contains a set which looks like a Carnot–Carathéodory ball with radii coming from the measures of the central sets in the previous subsection. We also sketch a heuristic argument for the conclusion of the proof.

Fix a base point $x_0 \in \Omega_0$ and set

$$\delta_j := C_\varepsilon^{-1}\alpha_k^{C\varepsilon}\alpha_j, \quad 1 \le j \le k.$$

Then the δ_j satisfy (6.3), though possibly with a smaller value of ε. Hence we may choose an n-tuple $\mathbf{j} \in \{1, \ldots, k\}^n$ so that

$$|B_{\mathbf{j}}(x_0; \delta_1, \ldots, \delta_k)| \sim |B(x_0; \delta_1, \ldots, \delta_k)|.$$

For $1 \le i \le n$, define on a ball B_i centered at 0 in $\mathbb{R}^i$ of radius ~ 1

$$\Phi_{\mathbf{j}}^i(t_1, \ldots, t_i) := e^{t_i X_{j_i}} \cdots e^{t_1 X_{j_1}}(x_0).$$

Let ℓ be such that $(J_{\ell n+1}, \ldots, J_{\ell n+n}) = (j_1, \ldots, j_n)$. Define

$$T_1 = \left\{ t_1 \in \mathbb{R} : |t_1| \ll 1 \ \text{ and } \ \Phi_{\mathbf{j}}^1(t_1) \in \Omega_{\ell n+1} \right\}.$$

Then since $x_0 \in \Omega_{\ell n+1}$, T_1 is a central set of width w_1 (after reindexing), with

$$\alpha_k^{C\varepsilon}\alpha_{j_1} \lesssim w_1 \lesssim \alpha_k^\varepsilon$$

and measure $\gtrsim \alpha_k^{C\varepsilon}\alpha_{j_1}$. Assuming T_{i-1} has been defined, and $2 \le i \le n$, we define

$$\tau_i(t) := \left\{ t_i \in \mathbb{R} : |t_i| \ll 1 \ \text{ and } \ \Phi_{\mathbf{j}}^i(t, t_i) \in \Omega_{\ell n+i} \right\}$$

whenever $t \in T_{i-1}$, and let

$$T_i := \left\{ (t, t_i) \in \mathbb{R}^i : t \in T_{i-1} \ \text{ and } \ t_i \in \tau_i(t) \right\}.$$

Then each of the $\tau_i(t)$ is a central set of width w_i,

$$\alpha_k^{C\varepsilon}\alpha_{j_i} \lesssim w_i \lesssim \alpha_k^{\varepsilon}$$

and measure $\gtrsim \alpha_k^{C\varepsilon}\alpha_{j_i}$.

Since $\Phi_{\mathbf{j}}^n(T_n) \subset \tilde{\Omega}$, it suffices to prove a lower bound for $|\Phi_{\mathbf{j}}^n(T_n)|$. In fact, since

$$|B_{\mathbf{j}}(x_0; \delta_1, \ldots, \delta_k)| \sim |B(x_0; \delta_1, \ldots, \delta_k)| \gtrsim \sum_{I \in \mathbf{I}_0} \delta^{\deg(I)} \gtrsim \alpha_k^{C\varepsilon}\alpha^b,$$

it suffices to show that

$$|\Phi_{\mathbf{j}}^n(T_n)| \gtrsim \alpha_k^{C\varepsilon}|B_{\mathbf{j}}(x_0; \delta_1, \ldots, \delta_k)|.$$

The rest of this section will be devoted to making the following heuristic argument rigorous:

Let

$$\tilde{T}_n = \big\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : |t_i| \le \delta_{j_i} \big\}.$$

Then

$$|B_{\mathbf{j}}(x_0; \delta_1, \ldots, \delta_k)| = |\Phi_{\mathbf{j}}^n(\tilde{T}_n)| \sim \int_{\tilde{T}_n} |\det \partial_t \Phi_{\mathbf{j}}^n(t)| dt$$

$$\lesssim \frac{|\tilde{T}_n|}{|T_n|} \int_{T_n} |\det \partial_t \Phi_{\mathbf{j}}^n(t)| dt \lesssim \alpha_k^{-C\varepsilon}|\Phi_{\mathbf{j}}^n(T_n)|.$$

In the lines above, we certainly ignored some details, but despite this, the properties of central sets, together with the smoothness of the X_j make this heuristic surprisingly close to the truth.

9.3. Polynomials on T_n

This subsection closely follows the work of Christ in [4].

Lemma 7. *If S is a central set of width $w \lesssim 1$, and P is a polynomial of degree D on $\mathbb{R}$, then*

$$|P| \gtrsim C_D\|P\|_{L^\infty([-w,w])}$$

on a subset $S' \subset S$ of measure $\gtrsim |S|$.

Sketch of proof. This is proved in [4]. Roughly, the values of $x \in [-w, w]$ such that $|P(x)| \ll \|P\|_{L^\infty([-w,w])}$ lie near the $\le D$ complex zeros of P a distance $\lesssim w$ from $[-w, w]$. Thus off the union of $\le D$ intervals I_i of length $\ll D^{-1}w$, $|P| \gtrsim C_D\|P\|_{L^\infty([-w,w])}$. The intersection of S with this union is small by centrality, so we may take $S' = S\setminus\bigcup_i I_i$. ∎

In particular, if we take $S = [-w, w]$, we see that

$$\|P\|_{L^\infty([-w,w])} \sim w^{-1} \int_{[-w,w]} |P|.$$

Lemma 8. *If $T_i \subset \mathbb{R}^i$, $1 \le i \le n$, if $T_{i+1} \subset T_i \times [-1, 1]$, $1 \le i \le n - 1$, and if the sets*

$$\tau_1 := T_1 \qquad \tau_i(t) := \{s \in [-C, C] : (t, s) \in T_i\}$$

are central sets of width w_1 and w_i for each $t \in T_{i-1}$, respectively, and if P is a polynomial of degree D on $\mathbb{R}^n$, then

$$|P| \gtrsim C_D \|P\|_{L^\infty(\prod_{i=1}^n [-w_i, w_i])}$$

on a subset $T_n' \subset T_n$ of measure $\gtrsim |T_n|$.

Proof. We briefly recount the proof of this from [4]. Its proof from the previous lemma is as follows. Considering P^2 if necessary, we may assume that $P \ge 0$.

We define polynomials P_i on $\mathbb{R}^i$, $1 \le i \le n$ of degree $\le D$ as follows:
Let $P_n := P$, and for $1 \le i \le n - 1$ and $t \in \mathbb{R}^i$, define

$$P_i(t) = w_{i+1}^{-1} \int_{[-w_{i+1}, w_{i+1}]} P_{i+1}(t, s) ds.$$

For each i, $2 \le i \le n$ and for each $t \in T_{i-1}$,

$$\tau_i'(t) := \left\{ s \in \tau_i(t) : P_i(t, s) \gtrsim C_D \|P_i(t, \cdot)\|_{L^\infty([-w_i, w_i])} \right\}$$

has measure $\gtrsim |\tau_i(t)|$ (because $s \mapsto P_i(t, s)$ is a polynomial of degree $\le D$). Let

$$T_1' := \left\{ s \in T_1 : |P_1(s)| \gtrsim C_D \|P_1\|_{L^\infty([-w_1, w_1])} \right\}$$

and define the sets

$$T_i' := \left\{ (t, s) : t \in T_{i-1}' \text{ and } s \in \tau_i'(t) \right\}.$$

Then for each i, $|T_i'| \sim |T_i|$, and if $t \in T_n'$,

$$P(t) \sim C_D \|P\|_{L^\infty(\prod_{i=1}^n [-w_i, w_i])}.$$

∎

9.4. Back to $\Phi_{\mathbf{j}}^n$

In this subsection, we use the polynomial lemmas to make the heuristic arguments from an earlier subsection rigorous and thereby complete the proof of Theorem 3. The main step will be to prove that the mapping $\Phi_{\mathbf{j}}^n$ is $O(N)$-to-1 off of a negligible set. This, when combined with the lemmas on generic sets, will give us the lower bound on the volume of $\Phi_{\mathbf{j}}^n(T_n)$ that we need.

Because $\Phi_{\mathbf{j}}^n$ is smooth, from the definition of $B_{\mathbf{j}}(x_0; \delta_1, \ldots, \delta_k)$, we have the upper bound

$$(9.2) \qquad |B_{\mathbf{j}}(x_0; \delta_1, \ldots, \delta_k)| = |\Phi_{\mathbf{j}}^n(\tilde{T}_n)| \leq \int_{\tilde{T}_n} |\det(\partial_t \Phi_{\mathbf{j}}^n(t))| \, dt$$

$$\leq \delta^{\deg(\mathbf{j})} \|\det \partial_t \Phi_{\mathbf{j}}^n\|_{L^\infty(\tilde{T}_n)} =: \delta^{\deg(\mathbf{j})} \mathcal{J}_{\mathbf{j}}(\delta).$$

In addition, we have already seen that

$$(9.3) \qquad |B_{\mathbf{j}}(x_0; \delta_1, \ldots, \delta_k)| \sim |B(x_0; \delta_1, \ldots, \delta_k)| \sim |\Lambda_\delta(x_0)|.$$

Combining (9.2) and (9.3), we thus have the inequality

$$\tfrac{1}{\delta^{\deg(\mathbf{j})}} |\Lambda_\delta(x_0)| \lesssim \mathcal{J}_{\mathbf{j}}(\delta).$$

Because $\Phi_{\mathbf{j}}^n$ is smooth, $\mathcal{J}_{\mathbf{j}}(\delta)$ is bounded above, and because $|\Lambda_\delta(x_0)| \gtrsim \delta^{\deg(I_0)}$, $\mathcal{J}_{\mathbf{j}}(\delta)$ is bounded below. In particular, we have

$$(9.4) \qquad \alpha_k^C \lesssim \mathcal{J}_{\mathbf{j}}(\delta) \lesssim 1$$

(by the definition of the δ_j and monotonicity of the α_j).

We now use this information to establish a lower bound for $|\Phi_{\mathbf{j}}^n(T_n)|$. This will complete the proof of the theorem. Let $Q_w := \prod_{j=1}^n [-w_j, w_j]$ be the rectangle whose side-lengths are given by the widths w_j (so $T_n \subset Q_w$). Then smoothness of $\Phi_{\mathbf{j}}^n$, the lower bound (9.4), and the fact that $w_j \lesssim \alpha_k^\varepsilon$ imply that we can control high order remainder terms in the Taylor series of $\Phi_{\mathbf{j}}^n$ on Q_w. In fact, taking $N \gg \varepsilon^{-1}$ and the α_j sufficiently small, we have

$$(9.5) \qquad \|\Psi_{\mathbf{j}} - \Phi_{\mathbf{j}}^n\|_{C^2(Q_w)} \leq c_0 \alpha_k^C (\mathcal{J}_{\mathbf{j}}(\delta))^2,$$

where $\Psi_{\mathbf{j}}$ is the Taylor polynomial of $\Phi_{\mathbf{j}}^n$ of degree N centered at 0, and we may choose c_0 as small and C as large as we like. (N will also depend on C, c_0.) Here we may choose the α_j as small as needed (how small depends on b, ε, N, and $\|X_j\|_{C^{M_N}(V)}$, $j = 1, \ldots, k$) by making the initial partition of unity sufficiently fine.

We apply a linear transformation A_w, mapping Q_w onto the unit cube Q. The images of the τ_j, denoted $\tilde{\tau}_j$, are then central sets of width 1, and, denoting by $\tilde{\Psi}$ and $\tilde{\Phi}$ the compositions $\Psi_{\mathbf{j}} \circ A_w^{-1}$ and $\Phi_{\mathbf{j}}^n \circ A_w^{-1}$ (respectively), we have that $\tilde{\Psi}$ and $\tilde{\Phi}$ are smooth, that $\tilde{\Psi}$ is the degree N Taylor polynomial of $\tilde{\Phi}$, and that the bound (9.5) is transformed to

$$(9.6) \qquad \|\tilde{\Psi} - \tilde{\Phi}\|_{C^2(Q)} \leq c_0 (\tilde{\mathcal{J}})^2,$$

where $\tilde{\mathcal{J}} = \|\det \partial_t \tilde{\Phi}\|_{L^\infty(Q)}$.

We are now in a position to apply Lemma 7.1 of [4], the upshot of which we state below.

Lemma 9. *The bound* (9.6) *and the centrality of the intervals* $\tilde{\tau}_j$ *imply that*

$$(9.7) \qquad |\tilde{\Phi}(A_w(T_n))| \gtrsim |A_w T_n| \tilde{\mathcal{J}} \gtrsim \frac{|A_w T_n|}{|A_w \tilde{T}_n|} \int_{A_w \tilde{T}_n} |\det \partial_t \tilde{\Phi}(t)| \, dt.$$

Undoing the linear transformation, this implies that

$$|\Phi_{\mathbf{j}}^n(T_n)| \gtrsim \frac{|T_n|}{|\tilde{T}_n|} \int_{\tilde{T}_n} |\det \partial_t \Phi_{\mathbf{j}}^n(t)| \, dt \gtrsim \alpha_k^{C\varepsilon} |B_{\mathbf{j}}(x; \delta_1, \ldots, \delta_n)|,$$

and the theorem is proved.

We briefly sketch the proof of Lemma 9.

Sketch of proof. First, we may use the bound (9.6) and centrality of the $A_w(\tau_j)$ to refine $A_w(T_n)$ to the region where

$$|\det \partial_t \tilde{\Psi}(t)| \sim |\det \partial_t \tilde{\Phi}(t)| \gtrsim \tilde{\mathcal{J}},$$

without significantly reducing the volume. Denote the refined region by ω. The idea is to show that $\tilde{\Phi}$ is $O(N)$-to-1 on ω, which implies that

$$|\tilde{\Phi}(A_w(T_n))| \geq |\tilde{\Phi}(\omega)| \geq \frac{1}{C_N} \int_\omega |\det \partial_t \tilde{\Phi}(t)| \, dt;$$

(9.7) follows.

By quantitative forms of the inverse function theorem, $\tilde{\Phi}$ is injective on balls of diameter less than a small constant times $\tilde{\mathcal{J}}$. We cover ω by a family $\{Q_j\}$ of such balls. The next step is to use the polynomial approximation to complete the argument.

Let Q_j^* denote the dilate of Q_j by a factor C_d. Then these cubes have bounded overlap, and by Bezout's theorem from algebraic geometry, a point $y \in \mathbb{R}^n$ may lie in the image of at most $C_{N,d}$ of these cubes under the (polynomial) mapping $\tilde{\Psi}$. The final step of the proof is to transfer this property to $\tilde{\Phi}$ by showing that $\tilde{\Phi}(Q_n) \subset \tilde{\Psi}(Q_n^*)$.

The argument is topological. Let $y \in \tilde{\Phi}(Q_j)$ and let B be a ball with $Q_j \subset B \subset Q_j^*$. Local injectivity and the bound (9.6) imply that $y \notin \tilde{\Psi}(\partial B) \cup \tilde{\Phi}(\partial B)$ and that

$$\tilde{\Psi}|_{\partial B} : \partial B \to \mathbb{R}^n \backslash \{y\} \qquad \tilde{\Phi}|_{\partial B} : \partial B \to \mathbb{R}^n \backslash \{y\}$$

are homotopic. By local injectivity, their topological degree must then be 1, which implies in particular that $y \in \tilde{\Psi}(B)$. Further details may be found in [4]. ∎

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