Finiteness of endomorphism algebras of CM modular abelian varieties

Josep González

Abstract
Let $A_f$ be the abelian variety attached by Shimura to a normalized newform $f \in S^2(\Gamma_1(N))^{\text{new}}$. We prove that for any integer $n > 1$ the set of pairs of endomorphism algebras $(\text{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}, \text{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q})$ obtained from all normalized newforms $f$ with complex multiplication such that $\dim A_f = n$ is finite. We determine that this set has exactly 83 pairs for the particular case $n = 2$ and show all of them.

We also discuss a conjecture related to the finiteness of the set of number fields $\text{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}$ for the non-CM case.

1. Introduction

For an abelian variety $A$ defined over a field $L$, we denote by $\text{End}_L(A)$ the ring of all its endomorphisms defined over $L$ and $\text{End}_L^0(A) := \text{End}_L(A) \otimes \mathbb{Q}$ denotes is endomorphism algebra over $L$. There is the following conjecture attributed to Robert Coleman.

Conjecture 1.1 Let $n, m \geq 1$ be positive integers. Then, up to isomorphism, there is only finitely many $\mathbb{Q}$-algebras $M$ such that $M \simeq \text{End}_L^0(A)$ for some abelian variety $A$ of dimension $n$ defined over a number field $L$ of degree $m$.

This conjecture is the starting point of this article and, next, we focus our attention on modular abelian varieties. Let us denote by $A_f/\mathbb{Q}$ the abelian variety attached by Shimura to a normalized newform $f \in S_2(\Gamma_1(N))^{\text{new}}$ with Fourier expansion $\sum_{n>0} a_n q^n$, where $q = e^{2\pi i z}$. It is well-known that $A_f$ is a simple quotient over $\mathbb{Q}$ of the jacobian of the modular curve $X_1(N)$, whose endomorphism algebra $\text{End}_\mathbb{Q}(A_f)$ is isomorphic to the number field $E_f = \mathbb{Q}(\{a_n\})$ and, moreover, $[E_f: \mathbb{Q}] = \dim A_f$. Also, we know that any simple

\textbf{2000 Mathematics Subject Classification:} 11G18, 14K22.

\textbf{Keywords:} Modular abelian varieties, complex multiplication.
quotient over $\mathbb{Q}$ of $\text{Jac}(X_1(M))$ is isogenous over $\mathbb{Q}$ to $A_f$ for some $f \in S_2(\Gamma_1(N))_{\text{new}}$ with $N | M$.

The modular abelian varieties $A_f$ can be classified in three types according to their endomorphism algebras over $\overline{\mathbb{Q}}$. Indeed, every $A_f$ is isogenous over $\mathbb{Q}$ to a power of an absolutely simple abelian variety $B_f$. The center of the algebra $\mathcal{A} = \text{End}_0^0(\mathbb{Q})(A_f)$ is either an imaginary quadratic field $K$ or the totally real number field $\mathbb{F} = \mathbb{Q}(\{a_p^2/\varepsilon(p)\}_{p | N})$. In the first case, $\mathcal{A}$ is a matrix algebra over $K$, $B_f$ is an elliptic curve with complex multiplication by $K$ and it is said that $f$ has CM by $K$. Otherwise, either $\mathcal{A}$ is a matrix algebra over $\mathbb{F}$, i.e. $B_f$ has real multiplication by $\mathbb{F}$ (RM), or $\mathcal{A}$ is a matrix algebra over a quaternion algebra $\mathcal{B}$ with center $\mathbb{F}$, i.e. $B_f$ has quaternionic multiplication by $\mathcal{B}$ (QM).

For an integer $n \geq 1$, we consider the set $S_n$ consisting of the pairs of isomorphic classes of $\mathbb{Q}$-algebras $(\text{End}_0^0(\mathbb{Q})(A_f), \text{End}_0^0(\mathbb{Q})(A_f))$, where $f$ runs over the set of normalized newforms with $\dim A_f = n$. As we show in Section 2, the set of degrees of the smallest number fields where the abelian varieties $A_f$ of dimension $n$ have all their endomorphisms defined is bounded. According to Conjecture 1.1, the set $S_n$ should be finite. Of course, we know that for $n = 1$ this is true and, more precisely, $S_1$ has exactly 10 pairs: $(\mathbb{Q}, \mathbb{Q})$ and the pairs $(K, \mathbb{Q})$, where $K$ is an imaginary quadratic field of class number one.

Since the algebra of endomorphisms defined over $\mathbb{Q}$ of a simple abelian variety over $\mathbb{Q}$ of dimension $n$ is a $\mathbb{Q}$-vector space of dimension at most $n$, the modular abelian varieties $A_f$ can be viewed as those with a richer arithmetical structure. Thus, the finiteness of the sets $S_n$ appears as an interesting case to test Coleman’s conjecture. This finiteness has been studied for modular abelian varieties with quaternionic multiplication, and partial results can be found in [17] and, for the particular case of surfaces, in [3]. Here, we center our attention on the CM case.

The plan of this paper is as follows. Section 2 is preliminary and devoted to introduce notation and summarize some known facts concerning modular abelian varieties with CM. In Section 3 we prove the modular conjecture for this class of modular abelian varieties and we also determine the set of pairs $(\text{End}_0^0(\mathbb{Q})(A_f), \text{End}_0^0(\mathbb{Q})(A_f))$ for the particular case $\dim A_f = 2$, which turns out to be the most laborious part of this article. In the last section we discuss the non-CM case. For every integer $n > 1$, we introduce a value $\tilde{B}(n) \in \mathbb{N} \cup \{+\infty\}$ which depends on the Fourier coefficients of the normalized newforms $f$ without CM with $\dim A_f = n$. We present some evidence that show that $\tilde{B}(n)$ could be finite and prove that if this is the case, then the set of number fields $\mathbb{E}_f$ with $\dim A_f = n$ is finite, which implies the finiteness of the set $S_n$ for the RM case. We finish this paper by giving a lower bound for $\tilde{B}(2)$. 
2. On modular abelian varieties with CM

Let us denote by New\(_N\) the set of normalized newforms of \(S_2(\Gamma_1(N))\). For a given \(f \in \text{New}_N\) with Fourier expansion \(\sum_{n>0} a_n q^n\) and a Dirichlet character \(\nu\) of conductor \(M\), we denote by \(f \otimes \nu\) the only normalized newform with \(q\)-expansion \(\sum_{n>0} b_n q^n\) that satisfies \(b_n = \nu(n) a_n\) for all integers \(n\) with \((n, NM) = 1\).

Let \(K\) be an imaginary quadratic field in a fixed algebraic closure \(\overline{\mathbb{Q}}\). We denote by \(O\) its ring of integers and by \(\chi\) the Dirichlet character attached to \(K\). Let \(f = \sum_{n>0} a_n q^n \in \text{New}_N\). By [16], we know that the following three conditions are equivalent:

(i) The newform \(f\) has CM by \(K\).

(ii) The newform \(f\) satisfies \(f = f \otimes \chi\), i.e. \(a_n = \chi(n)a_n\) for all positive integers with \((n, N) = 1\).

(iii) There is a primitive Hecke character \(\psi : I(m) \to \overline{\mathbb{Q}}^\times\) of conductor an integral ideal \(m\) of \(K\) such that

\[
f = \sum_{a \in I(m), a \subset O} \psi(a) q^{N(a)},
\]

where \(I(m)\) denotes the multiplicative group of fractional ideals of \(K\) relatively prime to \(m\) and \(N(a)\) denotes the norm of the ideal \(a\).

Moreover, for a Hecke character \(\psi\) as above, the level \(N\) of \(f\) is \(N(m)\) times \(D\), where \(D\) is the absolute value of the discriminant of \(K\). Attached to \(\psi\) we also have:

- The number fields \(E_f = \mathbb{Q}({a_n})\) and \(E = \mathbb{Q}({\psi(a)})\). One has

\[
E = E_f K.
\]

- The character \(\eta_\psi : (O/m)^* \to \overline{\mathbb{Q}}^\times\) defined by \(\eta_\psi(a) = \psi((a))/a\), which is also primitive of conductor \(m\) and satisfies \(\eta_\psi(u) = 1/u\) for \(u \in O^\times\). For every primitive character \(\eta\) of conductor \(m\) satisfying the last condition we have \(\eta = \eta_\psi\) for some primitive Hecke character \(\psi\) of conductor \(m\). When there is no risk of confusion, we shall write \(\eta\) instead of \(\eta_\psi\).

- The Nebentypus \(\varepsilon\) of \(f\), which is the Dirichlet character mod \(N\) defined by \(\varepsilon(d) = \chi(d)\eta(d)\).

- The totally real number field \(L_\varepsilon = \overline{\mathbb{Q}}^\text{ker}\varepsilon\). Here \(\varepsilon\) is viewed as a character of the absolute Galois group Gal(\(\overline{\mathbb{Q}}/\mathbb{Q}\)).
Let us denote by $\Phi$ the set of $K$-embeddings $E \hookrightarrow \overline{Q}$. In [10] the following is proved:

(i) There is a quotient abelian variety $A$ of $A_f$ defined over $K$ equipped with an isomorphism $\iota : E \cong \text{End}_K(A) \otimes Q$ such that $(A, \iota)$ is of CM-type $\Phi$ over $K$.

(ii) The abelian varieties $A_f$ and $A$ are $K$-isogenous if and only if $K \not\subseteq E_f$. Otherwise, $A_f$ is $K$-isogenous to $A \times \overline{A}$, where the bar $\overline{\cdot}$ stands for complex conjugation.

(iii) The smallest number field $L$ where $A$ has all its endomorphisms defined is a cyclic extension of the Hilbert class field $H$ of $K$ which is contained in the ray class field of $K$ mod $m$ and such that $[L : H] = \text{ord} \eta/|O^*|$. The extension $L/K$ is characterized by the property that a prime ideal $p \in I(m)$ splits completely in $L$ if and only if $\psi(p) \in K$.

(iv) There exists an elliptic curve $E$ defined over $L$ with CM by $O$ such that $E^{\dim A}$ and $A_f$ are isogenous over $L_0$, where $L_0$ is the maximal real subfield of $L$.

We say that a number field is the splitting field of an abelian variety if it is the smallest number field where the abelian variety has all its endomorphisms defined. Thus, the number field $L$ in (iii) is the splitting field of $A_f$.

In the next section we will use the following result.

**Proposition 2.1** Let $\psi$ be a Hecke character of $K$ of conductor $m$ whose attached newform $f = \sum_{n \geq 0} a_n q^n \in \text{New}_N \epsilon$. Assume $K \not\subseteq E_f$. We have that

(i) The ideal $m$ coincides with $\overline{m}$.

(ii) The extension $L/\overline{Q}$ is Galois.

(iii) The number field $L_\epsilon$ is contained in $L$.

(iv) There exists an elliptic curve $E$ with CM by $O$ defined over $L_0$ such that $E^{\dim A_f}$ and $A_f$ are isogenous over $L_0$, where $L_0$ is the maximal real subfield of $L$.

Moreover, the following conditions are equivalent:

(a) The Nebentypus $\epsilon$ is trivial, i.e., $f \in S_2(\Gamma_0(N))$.

(b) The number field $E_f$ is totally real.

(c) The Hecke characters $\psi$ and $\psi_\epsilon$ agree.

(d) For all positive integers $n$ coprime to $N$ we have $\eta(n) = \chi(n)$. 
Proof. Part (i) is proved in Lemma 2.1 in [10]. Since \( \psi_c = \sigma \psi \) for \( \sigma \in \Phi \), \( \ker \eta = \ker \eta_{\psi_c} \) and, thus, \( \ker \eta \) is stable under complex conjugation. This fact implies \( L = \overline{E} \) (cf. Section 3 in [10]) and, therefore, \( L/\mathbb{Q} \) is a Galois extension. Next, we give two proofs for the inclusion \( L_{\xi} \subset L \), because the arguments involved in both of them will be used in the sequel.

Since \( L/\mathbb{Q} \) is a Galois extension, it suffices to prove that every rational prime \( p \nmid N \) which splits completely in \( \mathbb{L} \), also splits completely in \( L_{\xi} \). Let \( p \) and \( \overline{p} \) be the prime ideals of \( \mathbb{K} \) over such a prime \( p \). Due to the fact that \( p \) and \( \overline{p} \) split completely in \( \mathbb{L} \), we have \( \psi(p), \psi(\overline{p}) \in \mathbb{K} \). The condition \( \psi_c = \sigma \psi \) for some \( \sigma \in \Phi \) implies \( \psi(\overline{p}) = \overline{\psi(p)} \) and, thus, \( a_p = \psi(p) + \overline{\psi(p)} \in \mathbb{Z} \). Since all elliptic curves with CM by \( \mathcal{O} \) are ordinary at every prime of good reduction over a rational prime which splits in \( \mathbb{K} \) and \( A_f \) has good reduction at all primes not dividing \( N \), \( a_p \neq 0 \) (mod \( p \)) (see Proposition 5.2 in [2]) and, in particular, \( a_p \neq 0 \). Hence, \( \epsilon(p) = a_p/\overline{a_p} = 1 \) and it follows that \( p \) splits completely in \( L_{\xi} \).

Now, we present an alternative proof of part (iv). The Weil involution \( W_N \) of \( X_1(N) \) is defined over the \( N \)-th cyclotomic field \( \mathbb{Q}(\zeta_N) \) and satisfies \( \tau_d W_N = W_N \langle d \rangle \) for all \( d \in (\mathbb{Z}/N\mathbb{Z})^* \), where \( \tau_d \) is the element of \( \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \) mapping \( \zeta_N \) to \( \zeta_N^d \) and \( \langle d \rangle \) denotes the diamond automorphism of \( X_1(N) \), which is defined over \( \mathbb{Q} \). The Weil involution and the diamonds induce automorphisms on \( A_f \). Since \( \langle d \rangle \) acts trivially on \( A_f \) if and only if \( \epsilon(d) = 1 \), \( L_{\xi} \) is the smallest field of definition for \( W_N \) acting on \( A_f \) and, thus, \( L_{\xi} \subset L \).

Let us prove part (iv). We know that \( A_f \) is isogenous over \( \mathbb{L} \) to \( E^{\dim A_f} \) for some elliptic curve \( E \) with CM by \( \mathcal{O} \). Since \( \mathbb{L} \) is the splitting field of \( A_f \), we can take \( E \) with \( j \)-invariant \( j(\mathcal{O}) \) together with an isomorphism \( \mu : E \to \mathbb{E} \) defined over \( \mathbb{L} \). Due to the action of \( \mu \) and \( \overline{\mu} \) on regular differentials of \( E \) and \( \mathbb{E} \), we obtain that \( \mu \circ \overline{\mu} = \text{id} \). Therefore, by Weil’s descent criterion (cf. [18]) we know that \( E \) admits a descent over \( \mathbb{L}_0 \) and, we can assume that \( E \) is defined over \( \mathbb{L}_0 \). By Faltings’s criterion, to prove the statement it suffices to check, that for every prime \( \mathfrak{P}_0 \) of \( \mathbb{L}_0 \) over a prime \( p \nmid N \) with \( N_{\mathbb{L}_0}/\mathbb{Q}(\mathfrak{P}_0) = p^m \), the reductions of \( A_f \) and \( E^{\dim A_f} \) modulo \( \mathfrak{P}_0 \) are isogenous over \( \mathbb{F}_{p^m} \). When \( \mathfrak{P}_0 \) splits in \( \mathbb{L} \), this fact is obvious. Since \( p \nmid N \), \( \mathfrak{P}_0 \) does not ramify in \( \mathbb{L} \) so we can assume that \( \mathfrak{P}_0 \) is inert in \( \mathbb{L} \). The prime \( p \) is also inert in \( \mathbb{K} \) because \( L = L_0 \mathbb{K} \) and, thus, \( m \) is the residue degree of \( p \mathcal{O} \) over \( \mathbb{L} \). Since \( L_{\xi} \subset \mathbb{L}_0 \), \( \epsilon(p^m) = 1 \) and \( a_{p^m}^\mathfrak{P} = \psi(p\mathcal{O})^m = \eta(p^m) p^m = (-1)^m p^m \). Due to the fact that the residue degree of any prime ideal \( p \in I(\mathfrak{m}) \) in \( \mathbb{L} \) is the smallest positive exponent \( e \) such that \( \psi(p)^e \in \mathbb{K} \), \( m \) is odd. Let us denote by \( \mathfrak{P} \) the prime ideal of \( \mathbb{L} \) over \( \mathfrak{P}_0 \). By the Eichler-Shimura congruence, the characteristic polynomial of the endomorphism \( \text{Frob}_p \) acting on the \( l \)-adic Tate module of the reduction of \( A_f/\mathbb{L} \) modulo \( \mathfrak{P} \) is \( (x + p^m)^{2 \dim A_f} \).
and, consequently, the characteristic polynomial of the endomorphism \( \text{Frob}_P \) corresponding to the reduction of \( E/L \) modulo \( \mathfrak{P} \) is \( (x + p^m)^2 \). It follows that the characteristic polynomials of the endomorphism \( \text{Frob}_P \) acting on the \( l \)-adic Tate modules of the reductions of \( A_f/L_0 \) and \( E/L_0 \) modulo \( \mathfrak{P}_0 \) are \( (x^2 + p^m)^{\dim A_f} \) and \( x^2 + p^m \) respectively, which completes the proof of part (iv).

Finally, note that the equivalence between (a) and (b) holds for any newform in \( \text{New}_N \). Indeed, if \( \varepsilon \) is trivial, then it is obvious that \( E_f \) is totally real. For the converse, see the proof of Lemma 6.17 of [1]. The remaining equivalences of the statement are immediate.

\[ \blacksquare \]

Remark 2.1 Observe that the splitting field of \( A_f \) is \( LL_\varepsilon \). Indeed, by part (iii) of the above proposition, it is immediate for \( K \not\subseteq E_f \). When \( K \subseteq E_f \), it follows from the fact that \( L \) is the splitting field of \( A \) and \( KL_\varepsilon \) is the smallest field of definition of the induced morphism by the Weil involution \( W_N \) between \( A \) and \( \overline{A} \).

Proposition 2.2 Let \( n \) be a positive integer. The set of degrees of the splitting fields of all abelian varieties \( A_f \) of dimension \( n \) (with or without CM) is bounded.

**Proof.** Assume that \( \dim A_f = n \) and let us denote by \( M \) the splitting field of \( A_f \). Let \( k \) be the greatest integer such that \( \varphi(k) \mid 2n \), where \( \varphi \) stands for Euler’s function. It is clear that \( [L_\varepsilon : Q] \leq k \).

First, we consider the CM case. With the above notation, \( M = LL_\varepsilon \) and, moreover, \( \ord \eta \leq k \) since \( Q(\eta) \subseteq E \). Due to the fact that the class number of \( K \) divides \( [E : K] \) (cf. Theorem 3.1 of [19]) and \( [E : K] \) divides \( n \), we obtain

\[
[M : Q] \leq [L : Q][L_\varepsilon : Q] \leq [H : Q] \ord \eta k \leq 2nk^2.
\]

For the non-CM case, the number field \( M \) is described in Proposition 2.1 of [9] and it is easy to check that \( [M : Q] \leq 2^n k \).

\[ \blacksquare \]

3. Finiteness for the CM case

Let us denote by \( \text{New}_{cm}^N \) the subset of \( \text{New}_N \) consisting of the newforms with CM by an imaginary quadratic field. For every integer \( n > 0 \), let us define \( S_{cm}^n \) as the set of pairs of number fields \( (K, M) \) such that \( K \) is an imaginary quadratic field, \( M \simeq E_f \) for some \( f \in \text{New}_{cm}^N \) with CM by \( K \) and \( [M : Q] = n \). It is clear that the map \( S_{cm}^n \to S_n \) sending a pair \( (K, M) \) to \( (M_n(K), M) \) yields a bijection between \( S_{cm}^n \) and the subset of \( S_n \) obtained from newforms with CM.
Theorem 3.1 For any \( n > 0 \), the set \( S_n^{cm} \) is finite.

Proof. Let us prove that the set \( \{(K, M : K) : (K, M) \in S_n^{cm}\} \) is finite, which is equivalent to the statement. Take \( f \) \( \in \text{New}_n^m \) attached to a Hecke character \( \psi \) of an imaginary quadratic field \( K \) such that \( \dim A_f = n \). By Theorem 3.1 of \([19]\), the class number \( h \) of \( K \) divides \( n \) and, moreover, the order \( k \) of \( \eta \) satisfies that \( \varphi(k) | 2n \).

We set \( \mathcal{N} = \{ m \in \mathbb{Z}^+ : \varphi(m) | 2n \} \) and denote by \( \mathcal{K} \) the set of imaginary quadratic fields whose class number divides \( n \). It is clear that both sets are finite and, consequently, it suffices to prove that, for every pair \((K, k) \in \mathcal{K} \times \mathcal{N} \), the set of number fields \( \mathcal{Q}(\psi) = \mathcal{Q}(\{\psi(a)\}) \) obtained when \( \psi \) runs over the set of Hecke characters of \( K \) whose attached character \( \eta \) has order \( k \) is finite (even without fixing the degree of \( \mathcal{Q}(\psi) \)).

We denote by \( \zeta_m \) a primitive \( m \)-th root of unity. Given a pair \((K, k) \in \mathcal{K} \times \mathcal{N} \), take a Hecke character \( \psi \) of \( K \) of conductor \( m \) for which \( \eta \) has order \( k \). Let \( \{a_1, \ldots, a_h\} \) be a set of representative ideals of the class group of \( K \) and let us denote by \( n_i, 1 \leq i \leq h \), the order of \( a_i \) in \( \text{Gal}(\mathcal{H}/K) \). For every positive integer \( i \leq h \), we take \( a_i \in \mathcal{Q} \) such that \( a_i^{n_i} \in K \) and \( a_i^{n_i} \mathcal{O} = a_i^{n_m} \). Choose \( \alpha_i \in K \) such that \( \alpha_i a_i \) is relatively prime to \( m \). Then, \( \mathcal{Q}(\psi) = \mathcal{K}(\zeta_k)(\psi(\alpha_1 a_1), \ldots, \psi(\alpha_h a_h)) \). Since \( \psi(\alpha_i a_i)^{m_i} = a_i^{n_i} \alpha_i^{m_i} \eta(\alpha_i a_i)^{m_i} \), the statement follows from the fact that \( \mathcal{Q}(\psi) \) is a subfield of the number field \( \mathcal{K}(\zeta_{k, h})(a_1, \ldots, a_h) \).

Next, we focus our attention on the two dimensional case. In order to simplify the notation, we represent an element \((K, M) \in S_2^{cm} \) by the pair \((-d, m) \), where \( d \) is the positive square-free integer such that \( K = \mathcal{Q}(\sqrt{-d}) \) and \( m \neq 1 \) is the square-free integer such that \( M = \mathcal{Q}(\sqrt{m}) \).

Theorem 3.2 With the above notation, the set \( S_2^{cm} \) has exactly the following 83 pairs:

(i) For the values of \( d \) such that \( \mathcal{Q}(\sqrt{-d}) \) has class number \( h = 1 \):

<table>
<thead>
<tr>
<th>( d )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( -1, \pm 2, \pm 3 )</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( -1, \pm 3 )</td>
</tr>
<tr>
<td>( 2, 7, 11, 19, 43, 67, 163 )</td>
<td>( -1, -3, \pm d, 3d )</td>
</tr>
</tbody>
</table>

(ii) For the values of \( d \) such that \( \mathcal{Q}(\sqrt{-d}) \) has class number \( h = 2 \):

<table>
<thead>
<tr>
<th>( d )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \cdot 5, 7 \cdot 5, 3 \cdot 17, 7 \cdot 13, 5 \cdot 23, 3 \cdot 41, 11 \cdot 17, 5 \cdot 47, 3 \cdot 89, 13 \cdot 31, 7 \cdot 61 )</td>
<td>( \pm p_1 )</td>
</tr>
</tbody>
</table>

where \( p_1 \) is the unique prime dividing \( d \) such that \( p_1 \equiv 1 \pmod{4} \), and also

<table>
<thead>
<tr>
<th>( d )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5, 13, 37 )</td>
<td>( -1, d )</td>
</tr>
<tr>
<td>( 2 \cdot 3, 2 \cdot 11 )</td>
<td>( -1, \pm 2 )</td>
</tr>
<tr>
<td>( 2 \cdot 5, 2 \cdot 29 )</td>
<td>( -1, \pm d/2 )</td>
</tr>
</tbody>
</table>
Proof. We know that all \( f \in \text{New}_K^\text{cm} \) with \( \dim A_f = 2 \) have CM by an imaginary quadratic field whose class number is either 1 or 2. From now on, \( K \) is an imaginary quadratic field of discriminant \( -D \) and class number \( h \leq 2 \). The square-free part \( d \) of \( D \) is either \( D \) or \( D/4 \) depending on whether \( D \) is odd or not. Let us denote by \( w \) the order of \( \mathcal{O}^* \), namely \( w = 2, 4 \) or 6 according to \( D > 4, D = 4 \) or \( D = 3 \), respectively.

Firstly, we prove that \((-d,-d) \in S_2^\text{cm}\) if and only if \( h = 1 \). Indeed, if \((-d,-d) \in S_2^\text{cm}\) then \( E_f = K \) and \( [E : K] = 1 \), which implies \( h = 1 \). Conversely, assume \( h = 1 \) and let \( p \) be a prime ideal over an odd prime \( p \). We can choose \( p \) such that \( p \equiv 1 + w \pmod{w^2} \) and, thus, there exists a character \( \eta \) of conductor \( p \) and order \( w \) satisfying \( \eta(u) = 1/u \) for all \( u \in \mathcal{O}^* \). A Hecke character \( \psi \) of conductor \( m = p \) and character \( \eta \) provides a newform \( f \) with \( E = K \).

Since \( m \neq \overline{m} \), part 2 of Proposition 2.1 implies that \( K \subseteq E_f \), and then \( E_f = E = K \). Therefore, for \( h = 1 \) all pairs \((-d,-d) \) lie in \( S_2^\text{cm} \).

From now on, we focus our attention on Hecke characters \( \psi \) whose number field \( E \) is a biquadratic field \( \mathbb{Q}(\sqrt{-d}, \sqrt{m}) \) for which \( E_f = \mathbb{Q}(\sqrt{m}) \). We recall that, in this case, Proposition 2.1 applies and \( \mathbb{L} \) is the splitting field of \( A_f \). We split the proof in two cases according to the class number \( h \), and for each of them we examine all possibilities for the values of \( \ord(\eta) \).

1. Case \( h = 1 \). In this case, \( E = K(\eta) \) and \( [K(\eta) : K] = 2 \). Therefore, \( \varphi(k)/\varphi(w) = 2 \), where \( k \) is the order of \( \eta \). So we have the following possibilities for \( k, E \) and \( m \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( d = 1 )</th>
<th>( d = 3 )</th>
<th>( d \neq 1, 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Q}(\sqrt{-1}, \sqrt{-2}) ); ( \mathbb{Q}(\sqrt{-1}, \sqrt{-3}) )</td>
<td>( \mathbb{Q}(\sqrt{-1}, \sqrt{-3}) )</td>
<td>( \mathbb{Q}(\sqrt{-d}, \sqrt{-1}); \mathbb{Q}(\sqrt{-d}, \sqrt{-3}) )</td>
<td></td>
</tr>
<tr>
<td>( -2, 2 )</td>
<td>( -3, 3 )</td>
<td>( -1, d )</td>
<td>( -3, 3d )</td>
</tr>
</tbody>
</table>

Next, we prove that all these possibilities for \((-d, m) \) do occur. Let \( \ell \) be the prime ideal of \( K \) over the unique rational prime \( \ell \) dividing \( D \) and consider the integral ideal \( \mathfrak{m}_0 \) defined by

\[
\mathfrak{m}_0 = \begin{cases} 
\ell & \text{if } D \neq 3 \text{ is odd}, \\
\ell^2 & \text{if } D = 3, \\
\ell^3 & \text{if } D = 4, \\
\ell^5 & \text{if } D = 8. 
\end{cases}
\]

Let \( \eta_0 \) be a character mod \( \mathfrak{m}_0 \) satisfying the following three conditions: the order of \( \eta_0 \) is \( w \), \( \eta_0(u) = 1/u \) for all \( u \in \mathcal{O}^* \) and \( \eta_0(n) \chi(n) \) is the trivial Dirichlet character mod \( D \). Note that \( \eta_0 \) is unique except for \( D = 8 \).
Let \( \psi_0 \) be the unique Hecke character of conductor \( \mathfrak{m}_0 \) with character \( \eta_0 \). Since \( \mathbb{Q}(\psi_0) = \mathbb{K} \) and the newform \( f_0 \) attached to \( \psi_0 \) has trivial Nebentypus, \( \dim A_{f_0} = 1 \). By taking Dirichlet characters \( \chi_1 \) of order 4 if \( d \neq 1 \) and \( \chi_2 \) of order 3 if \( d \neq 3 \), the newforms \( f_0 \otimes \chi_1 \) and \( f_0 \otimes \chi_2 \) have CM by \( \mathbb{K} \) and provide the values \( m = -1 \) and \( m = -3 \), respectively. This yields \((-d, -1), (-d, -3) \in S_2^{cm} \) for all \( d \) when \( h = 1 \).

For \( d \neq 3 \), take an inert prime \( p \) in \( \mathbb{K} \) such that \( p \equiv -1 \) (mod 3). Choose a character \( \eta' \mod p\mathcal{O} \) of order 3. The newform obtained from a Hecke character of conductor \( \mathfrak{m} = \mathfrak{m}_0 \cdot p\mathcal{O} \) and character \( \eta = \eta_0 \times \eta' \) has trivial Nebentypus. Since \( \mathcal{E} = \mathbb{Q}(\sqrt{-d}, \sqrt{-3}) \), it follows \((-d, 3d) \in S_2^{cm} \).

For \( d \neq 1 \), take an inert prime \( p \) in \( \mathbb{K} \) such that \( p \equiv -1 \) (mod 4) and choose a character \( \eta' \mod p\mathcal{O} \) of order 4. Proceeding as before for the character \( \eta_0 \times \eta' \), we obtain \((-d, d) \in S_2^{cm} \).

For \( d = 1 \), take \( p = 7 \) and the character \( \eta' \mod p\mathcal{O} \) of order 8 defined by \( \eta'(2 + i) = (1 + i)\sqrt{2}/2 \). For the character \( \eta = \eta_0 \times \eta' \), we obtain \((-1, 2) \in S_2^{cm} \).

To complete the case \( h = 1 \), we need to prove \((-1, -2) \in S_2^{cm} \). Take the characters \( \eta_2 \) and \( \eta_3 \) mod \( 2\mathcal{O} \) and \( 3\mathcal{O} \), respectively, defined by \( \eta_2(i) = -1 \) and \( \eta_3(1 - i) = (1 + i)\sqrt{2}/2 \). The newform \( f \) obtained from a Hecke character with character \( \eta_2 \times \eta_3 \) satisfies \( \mathcal{E}_f = \mathbb{Q}(\sqrt{-2}) \).

(2) Case \( h = 2 \). If \( [\mathcal{E} : \mathbb{Q}] = 4 \), then the order \( k \) of \( \eta \) can only be 2, 4 or 6 and, moreover, \( k \) must divide \( 2D \) (cf. Theorem 3.5 of [19]). Note that, in this particular setting, \( \mathcal{E}_f \) is a quadratic field if and only if \( \psi_\varepsilon = \varepsilon^2 \psi \) for some \( \sigma \in \Phi \). Therefore, for all \( \alpha \in (\mathcal{O}/\mathfrak{m})^* \), we have \( \eta(\overline{\alpha}) = \overline{\eta(\alpha)} \) if \( \varepsilon \) is trivial and, otherwise, \( \eta(\overline{\alpha}) = \overline{\varepsilon \eta(\alpha)} = \overline{\eta(\alpha)} \) for the non-trivial \( \sigma \in \text{Gal}(\mathcal{E}/\mathbb{K}) \) (for \( k = 2 \), both conditions agree). In particular, if \( \mathfrak{m} = \mathfrak{m}_1 \cdot \mathfrak{m}_2 \) with \( (\mathfrak{m}_1, \mathfrak{m}_2) = 1 \), \( \mathfrak{m}_i = \mathfrak{m}^i \) for \( i \leq 2 \), and \( \eta = \eta_1 \times \eta_2 \), where \( \eta_i \) is a character mod \( \mathfrak{m}_i \), then each \( \eta_i \) has to satisfy the same condition as \( \eta \) for all \( \alpha \in (\mathcal{O}/\mathfrak{m}_i)^* \).

When \( h = 2 \), \( D \) has exactly two prime divisors. Let \( \ell \) be such a prime satisfying that the prime ideal of \( \mathbb{K} \) over \( \ell \) is non-principal. For a given non-principal prime ideal \( \mathfrak{p} \in I(\mathfrak{m}) \), we have \( \mathfrak{p}^2 = \alpha^2 \ell \mathcal{O} \) for some \( \alpha \in \mathbb{K}^* \) and, thus, \( \psi(\mathfrak{p})^2 = \alpha^2 \ell \eta(\alpha^2 \ell) \) and \( \sqrt{\ell \eta(\alpha^2 \ell)} \in \mathcal{E} \). Therefore, we have to consider the only following possibilities for \( k \) and \( \mathcal{E} \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( D )</th>
<th>( 4 \mid D )</th>
<th>( 3 \mid D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \mathbb{Q}(\sqrt{-d}, \sqrt{\ell}) ), ( \mathbb{Q}(\sqrt{-d}, \sqrt{-\ell}) )</td>
<td>( \mathbb{Q}(\sqrt{-d}, \sqrt{-1}) )</td>
<td>( \mathbb{Q}(\sqrt{-d}, \sqrt{-3}) )</td>
</tr>
</tbody>
</table>

Next, we split the proof according to the value of \( k \).

(i) Subcase \( k = 2 \). In this case, \( \mathbb{L} = \mathbb{H} \) and, again by Theorem 3.5 of [19], we know that \( [\mathcal{E} : \mathbb{K}] = 2 \) for any Hecke character of \( \mathbb{K} \) with \( k = 2 \).
Let $p_1$ be the prime that divides $d$ such that the field $\mathbb{K}' = \mathbb{Q}(\sqrt{p_1})$ is the real quadratic subfield of $\mathbb{H}$. Let us now prove that, for $h = 2$ and $k = 2$, the pair $(-d, m)$ lies in $S^m_2$ if and only if $D \not\equiv 4 \pmod{8}$ and $|m| = p_1$.

Let $\psi$ be a Hecke character such that $\text{ord} \eta = 2$ and $E_f = \mathbb{Q}(\sqrt{m})$. Since $\mathbb{H}$ is the splitting field of $A_f$, there exists an elliptic curve defined over $\mathbb{H}$ with CM by $\mathcal{O}$ which has all the isogenies to its Galois conjugates defined over $\mathbb{H}$. The case $D \equiv 4 \pmod{8}$, i.e. $D = 4 \cdot p_1$ with $p_1 \equiv 1 \pmod{4}$, cannot occur because there are no elliptic curves with CM by $\mathcal{O}$ satisfying this property (see 11.3 in [11]). If $D \not\equiv 4 \pmod{8}$, then $d/p_1$ is the other prime $p_2$ that divides $D$ and, moreover, the number field $\mathbb{E}$ is either $\mathbb{Q}(\sqrt{-d, \sqrt{p_1}})$ or $\mathbb{Q}(\sqrt{-d, -\sqrt{p_1}})$, which implies $|m| \in \{p_1, p_2\}$.

By Proposition 2.1, there exists an elliptic curve $E$ with CM by $\mathcal{O}$ defined over $\mathbb{K}'$ such that $A_f$ is isogenous over $\mathbb{Q}$ to the Weil restriction $\text{Res}_{\mathbb{K}'/\mathbb{Q}} E$. Since $E_f = \mathbb{Q}(\sqrt{m})$, for the non-trivial $\sigma \in \text{Gal}(\mathbb{K}'/\mathbb{Q})$ there is $\mu \in \text{Hom}_{\mathbb{K}'}(E, \sigma E) \otimes \mathbb{Q}$ such that $\mu \circ \sigma \mu = [m]$, which implies that $m$ is a norm of $\mathbb{K}'$. Due to the fact that neither $p_2$ nor $-p_2$ are norms of $\mathbb{K}'$, it follows that $|m| = p_1$.

Next, we prove that, for $D \not\equiv 4 \pmod{8}$, the pairs $(d, \pm p_1) \in S^m_2$. Let us denote by $p_1$ and $p_2$ the ideals over $p_1$ and $p_2$, respectively. Take the integral ideals

$$m_0 = \begin{cases} p_1 \cdot p_2 & \text{if } p_1 \text{ and } p_2 \text{ are odd}, \\ p_1 \cdot p_2 & \text{if } p_2 = 2, \\ p_1^2 \cdot p_2 & \text{if } p_1 = 2, \end{cases}$$

and

$$m_1 = \begin{cases} p_2 & \text{if } p_2 \text{ is odd}, \\ p_2^2 & \text{if } p_2 = 2. \end{cases}$$

For a quadratic character $\eta_0$ of conductor $m_0$ such that $\eta_0(n) = \chi(n)$ for all integers $n$ coprime to $D$, we obtain a Hecke character $\psi$ whose newform has trivial Nebentypus and, thus, $E_f$ is a real quadratic field. Therefore $m = p_1$. For an odd quadratic character $\eta_1$ of conductor $m_1$, we obtain a Hecke character $\psi$ whose newform $f$ has non-trivial Nebentypus and its Fourier coefficient $a_{p_1}$ satisfies $a_{p_1}^2 = -p_1$. Therefore, $E = \mathbb{K}(\sqrt{-p_1})$. It can be easily proved that, for a prime $p$ of $\mathbb{K}$, one has $\psi(p) = \psi_c(p) \in \mathbb{K}$ when $p$ is principal and, otherwise, $\psi(p) = -\psi_c(p)$ and $\psi(p)\sqrt{-p_1} \in \mathbb{K}$. Hence, $\psi_c = \sigma \psi$ for the non-trivial $\sigma \in \text{Gal}(\mathbb{E}/\mathbb{K})$ and, thus, $E_f = \mathbb{Q}(\sqrt{-p_1})$.

\[(ii)\ \textbf{Subcase } k = 4.\] For a Hecke character with $k = 4$, we know by [19] that $[\mathbb{E} : \mathbb{K}] = 2$ if and only if $4 \mid D$ and $\eta(2\alpha^2)$ has order 4 for some (and every) $\alpha \in \mathbb{K}^*$ such that $2\alpha^2 \mathcal{O} \in I(m)$. The last condition amounts to saying that, for any generator $\beta$ of the square of some (and every) non-principal prime ideal $p \in I(m)$, $\eta(\beta)$ has order 4. If this is the case, $E = \mathbb{Q}(\sqrt{-d}, \sqrt{d})$ and we only have to consider the cases $m = -1$ and $m = d$. So we can assume $4 \mid D$ and, in this case, $D$ has a unique odd prime divisor $p_1$. We denote by $p_1$ and $p_2$ the prime ideals over $p_1$ and $2$, respectively.
We first prove that, for \( k = 4 \), \((-d,d) \in S_2^m\) if and only if \( d \) is odd, i.e., \( D \equiv 4 \pmod{8} \).

For \( d \) odd, take \( \mathfrak{m} = p_1 \cdot p_2^3 \) and \( \eta = \eta_1 \times \eta_2 \), where \( \eta_1 \) is the quadratic character mod \( p_1 \) and \( \eta_2 \) is a character mod \( p_2^3 \) of order 4 (\( \eta_2(\sqrt{-p_1}) = \pm i \)).

For each possible value of \( d \) (\( d = 5, 13, 37 \)), it is easy to find a generator \( \beta \) of the square of a non-principal prime ideal \( \mathfrak{p} \) and check that its real part is even. So \( \eta(\beta) = \pm \eta_2(\beta) \) has order 4 because \( \eta_2(\beta^2) = -1 \) and \( \mathbb{E} = \mathbb{Q}(\sqrt{-d}, \sqrt{d}) \).

Since \( \varepsilon = 1 \), we obtain \((-d,d) \in S_2^m\).

Assume now that \( d \) is even and \( \mathbb{E}_f = \mathbb{Q}(\sqrt{d}) \). Since \( \mathfrak{m} = \mathfrak{m}' \), \( \eta \) is primitive of conductor \( \mathfrak{m} \) and \( \eta(n) = \chi(n) \) for all integers \( n \) coprime to the level \( N \) of \( f \), it must be that:

- The ideal \( \mathfrak{m} \) is of the form \( p_1 \cdot p_2^5 \cdot \prod_{i=1}^{r} p_i' \mathcal{O} \) for some primes \( p_i' \) with \( (p_i', D) = 1 \),
- The character \( \eta \) is of the form \( \eta_1 \times \eta_2 \times \prod_{i=1}^{r} \eta_i' \), where \( \eta_1 \) is quadratic mod \( p_1 \), \( \eta_2 \) is of order 2 or 4 mod \( p_2^5 \), and each \( \eta_i' \) is of order 2 or 4 and primitive of conductor \( p_i' \mathcal{O} \),
- The following conditions have to be satisfied:

  (i) \( \eta_1(n) \cdot \eta_2(n) = \chi(n) \) for all integers \( n \) coprime to \( D \),

  (ii) for each \( i \leq r \), \( \eta_i'(n) = 1 \) for all integers \( n \) coprime to \( p_i' \) and \( \eta_i'(\alpha) = \eta_i'(\overline{\alpha}) \) for all \( \alpha \in (\mathcal{O}/p_i' \mathcal{O})^* \).

If \( \mathfrak{m} \) were the ideal \( p_1 \times p_2^3 \) and \( \eta_2 \) a character of order 4, then \( \eta_2(1 + \sqrt{-d})^2 = -1 \). In this case, the degree \([\mathbb{E} : \mathbb{Q}]\) would be greater than 4 since for each possible value of \( d \) (\( d = 6, 10, 22, 58 \)) it can be found a generator \( \beta \) of the square of a non-principal prime ideal \( \mathfrak{p} \in I(\mathfrak{m}) \) and checked that \( \eta_2(\beta)^2 = 1 \). So \( \eta_i'(\beta) \) must necessarily have order 4, which leads to a contradiction. Indeed, for an inert prime \( p_i' \) in \( \mathbb{K} \), \( \mathbb{N}_{\mathbb{K}/\mathbb{Q}}(\beta) = p^2 \) implies that \( \beta \) is a square in the finite field \( \mathcal{O}/p_i' \mathcal{O} \) and, thus, \( \eta_i'(\beta)^2 = 1 \). For the split case, the conditions \( \mathbb{N}_{\mathbb{K}/\mathbb{Q}}(\beta) = p^2 \) and \( \eta_i'(\beta) = \eta_i'(\overline{\beta}) \) also implies that \( \eta_i'(\beta)^2 = 1 \). So \((-d,d) \notin S_2^m \) for \( d \) even.

Let us now prove that, for \( k = 4 \), \((-d,-1) \in S_2^m \) for all \( d \).

Assume \( p_1 \equiv 1 \pmod{4} \). We take \( \mathfrak{m} = p_1 \) and let \( \eta \) be a character mod \( p_1 \) of order 4. It is clear that \( \eta(\alpha) = \eta(\overline{\alpha}) \) for all \( \alpha \in (\mathcal{O}/\mathfrak{m})^* \). Moreover, since \( p_1 \not\equiv 1 \pmod{8} \), \( \eta(-1) = -1 \) and \( \eta(2) = \pm i \). Due to the fact that \( \eta(2) \) has order 4, for any Hecke character \( \psi \) with attached character \( \eta \) we have \( \mathbb{E} = \mathbb{Q}(\sqrt{-d}, \sqrt{d}) \) and, moreover, \( \psi(p_2) = \pm (1 \pm i) \). Let \( \mathfrak{p} \) be a prime ideal over a prime \( p \) which splits in \( \mathbb{K} \). By using the fact that \( \mathfrak{p} \) is either principal or \( \mathfrak{p} = \alpha p_2 \) with \( \alpha \in I(\mathfrak{m}) \), we obtain \( a_p \in \mathbb{Q}(i) \) and, thus, \( \mathbb{E}_f = \mathbb{Q}(i) \).
For \( p_1 \not\equiv 1 \pmod{4} \), we take \( m = p_1 \cdot p_2^2 \) and \( \eta = \eta_1 \times \eta_2 \), where \( \eta_1 \) is a quadratic character mod \( p_1 \) and \( \eta_2 \) is an even character of conductor \( p_2^2 \) and order 4 such that \( \eta_2(1 + \sqrt{-d}) = \eta_2(5) = \pm i \). It is easy to check that for any Hecke character \( \psi \) with character \( \eta \) we have \( \ord \varepsilon = 4 \) and \( E_f = \mathbb{Q}(i) \).

(iii) Subcase \( k = 6 \). By [19], we know that \( [E : \mathbb{K}] = 2 \) implies \( 3 \mid D \). Assume \( 3 \mid D \) and set \( p_1 = 3 \), so that \( p_2 = d/p_1 \) is the other prime divisor of \( D \). Now, \( \mathbb{H} = \mathbb{K}(\sqrt{p_2}) \). Let \( p_1 \) and \( p_2 \) be the prime ideals over \( p_1 \) and \( p_2 \), respectively.

Although we have already proved that \( (-d, p_2) \in S_2^{cm} \) when \( k = 2 \), we point out that, for \( k = 6 \), this pair is also attained. Let us now prove that \( (-d, -3) \not\in S_2^{cm} \). Assume that there exists a Hecke character for which \( \eta \) has order 6 and \( E_f = \mathbb{Q}(\sqrt{-3}) \). The Nebentypus \( \varepsilon \) is non-trivial and its order divides 6. Since the newform \( g = f \otimes \varepsilon \) has CM by \( \mathbb{K}, E_g = \mathbb{Q}(\sqrt{-3}) \) and its Nebentypus is \( \varepsilon^3 \), which must have order 2. So, we can assume that \( \varepsilon \) has order 2. Due to the fact that \( \varepsilon \not\equiv 1 \), the Weil involution \( W_N \) acting on \( A_f \) is non-trivial and, thus, provides an elliptic quotient \( E \) of the abelian surface \( A_f \) defined over the real quadratic field \( \mathbb{K}' = \mathbb{L}_e \). The curve \( E \) has CM by an order \( \mathcal{O}' \) of \( \mathbb{K} \) and the ring class field of \( \mathcal{O}' \), which contains \( \mathbb{H} \), is \( \mathbb{K}' \mathbb{K} \). Therefore, \( \mathbb{H} = \mathbb{K}' \mathbb{K} \) and \( \mathbb{K}' \) must be \( \mathbb{Q}(\sqrt{p_2}) \). Since \( A_f \) is isogenous over \( \mathbb{Q} \) to \( \text{Res}_{\mathbb{K}/\mathbb{Q}} E \) and \( E_f = \mathbb{Q}(\sqrt{-3}) \), \( -3 \) should be a norm of \( \mathbb{K}' \) but this condition does not occur when \( 3 \mid D \).

4. On the finiteness for endomorphism algebras over \( \mathbb{Q} \)

This section is devoted to present evidences about a behavior of the Fourier coefficients of normalized newforms. We show that this conjectural behavior implies the finiteness of the set of number fields \( E_f \) with degree \( n \) and, thus, the finiteness of the set \( S_n \) for the RM case.

For \( f \in \text{New}_N \), we denote by \( S_2(A_f) \) the \( \mathbb{C} \)-vector space generated by the Galois conjugates of \( f \), whose dimension is \( \dim A_f \). We consider the positive integer defined by

\[
B(f) := \max \{ \ord_{\infty} h : h \in S_2(A_f) \}.
\]

In other words, \( B(f) \) is the positive integer for which there is a single cuspidal form in \( S_2(A_f) \) whose \( q \)-expansion is

\[
q^{B(f)} + \sum_{m > B(f)} a_m q^m.
\]

Since \( \Omega^1(A_f) \simeq S_2(A_f) dq/q \subseteq \Omega^1(X_1(N)) \), we know that

\[
\dim A_f \leq B(f) \leq 2g_1 - 1,
\]

where \( g_1 \) denotes the genus of \( X_1(N) \). If there exists a curve \( C \) defined over \( \mathbb{C} \) along with a morphism \( \pi : X_1(N) \to C \) such that \( S_2(A_f) dq/q \subseteq \pi^* (\Omega^1(C)) \),
then we can improve the upper bound of $B(f)$ since $B(f) \leq 2g(C) - 1$, where $g(C)$ denotes the genus of $C$. But, we cannot ensure the existence of such a curve with a genus less or equal than a constant depending on $\dim A_f$. In fact there is a conjecture about the finiteness of such curves (see Conjecture 1.1 in [1]).

It is natural to ask about the asymptotic behavior of $B(f)$ when $f$ runs over the set of normalized newforms $f$ whose abelian varieties $A_f$ have a given dimension.

For every integer $n > 0$ and every $x \in \mathbb{R}$ we define

\[
B(n, x) := \max\{B(f) : f \in \text{New}_N, \dim A_f = n, N \leq x\}, \\
B(n) := \lim_{x \to +\infty} B(n, x).
\]

Of course, $B(1) = 1$. After computing $B(f)$ for $n = 2$ and $N \leq 3000$ with $f$ running over the set of all normalized newforms with trivial Nebentypus, we obtained the results displayed in the following table:

<table>
<thead>
<tr>
<th>$B(f)$</th>
<th>$#{A_f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1372</td>
</tr>
<tr>
<td>3</td>
<td>1536</td>
</tr>
<tr>
<td>5</td>
<td>1155</td>
</tr>
<tr>
<td>7</td>
<td>504</td>
</tr>
<tr>
<td>11</td>
<td>139</td>
</tr>
<tr>
<td>13</td>
<td>43</td>
</tr>
<tr>
<td>17</td>
<td>11</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
</tr>
</tbody>
</table>

Note that if $\dim A_f = 2$, then $B(f) = k$ if and only if $a_i \in \mathbb{Z}$ for all $i < k$ and $a_k \not\in \mathbb{Z}$, where $f = \sum_{m > 0} a_m q^m$. By the properties of the Fourier coefficients, if the order of the Nebentypus of $f$ is 1 or 2, then $B(f)$ is a prime.

With regard to the above computational table, one could think that $B(n) < +\infty$ for $n > 1$. Nevertheless, this assertion is not right. In fact, $B(n)$ would be $+\infty$ if we only took into account newforms with CM to define $B(n, x)$. Indeed, take for instance $K = \mathbb{Q}(\sqrt{-7})$ and $n = 2$. For any integer $k > 7$, let $p_1 < \cdots < p_r$ be the primes $\leq k$ and let $p > k$ be a prime such that $p \equiv 1 \pmod{4}$ and splits in $K$. Choose a prime $p_i$ over each $p_i$ and a prime $p$ over $p$. Let $m' = p_i^2 \cdot \prod_{i=2}^r p_i$. We take $m$ to be either $m'$ or $m' \cdot p$. Depending on whether the primitive quadratic character of conductor $m'$ is odd or not, let $\eta$ be the primitive quadratic character of conductor $m$ and let $\psi$ be the corresponding Hecke character. It is clear that $E_f = \mathbb{Q}(\sqrt{-7})$ and $a_m = 0$ for all $1 < m \leq k$. It follows that $\text{ord}_{f - \sqrt{7}} > k$ and, thus, $B(f) > k$.

For newforms without CM, we can use a similar procedure that consists on twisting a newform $f$ by suitable quadratic Dirichlet characters of large conductor to obtain newforms $g$ with $B(g) > k$ and $\dim A_f = \dim A_g$. For this reason, we shall refine the above definitions to avoid the distortion caused by the effect of twists.

In the sequel $\chi$ stands for a Dirichlet character of any conductor and order. For an integer $n \geq 1$, we say that a normalized newform $f$ without CM
is \( n \)-primitive if \( \dim A_f = n \) and \( \dim A_{f \otimes \chi} \geq n \) for all Dirichlet characters \( \chi \). The reason to exclude the CM case in this definition is the following. For two Hecke characters \( \psi \) and \( \psi' \) of \( K \), \( \psi' \) can be viewed as a twist of \( \psi \) by a character of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and it may be that the corresponding newforms \( f \) and \( f' \) satisfy that \( \dim A_f < \dim A_f' = n \) and \( \dim A_{f \otimes \chi} \geq n \) for all Dirichlet characters \( \chi \).

Now, we define

\[
\tilde{B}(f) := \min \{ B(f \otimes \chi) : \dim A_{f \otimes \chi} = \dim A_f \}, \quad \tilde{B}(n, x) := \max \{ \tilde{B}(f) : f \in \text{New}_N \backslash \text{New}^\text{cm}_N, N \leq x, f \text{ is } n\text{-primitive} \}, \quad \tilde{B}(n) := \lim_{x \to +\infty} \tilde{B}(n, x).
\]

The range \( N \leq 3000 \) is too small to detect the effect of twists, but we can see in the above table the quick decrease in the number of abelian surfaces \( A_f \) when \( B(f) \) increases. Now, one suspects that an affirmative answer to the question \( \tilde{B}(n) < +\infty \) should be considered. Next, we show two important consequences about this hypothesis.

**Proposition 4.1** Let \( n > 1 \) be an integer. Assume that \( \tilde{B}(m) < +\infty \) for all positive integers \( m \leq n \). Then,

(i) The set of number fields \( E_f \) of degree \( n \) obtained when \( f \) runs over the set of all normalized newforms is finite.

(ii) If \( n = 2 \), then \( 4 \tilde{B}(2) - 1 \) is an upper bound for all primes \( p \equiv 1 \) (mod 4) such that the modular curve \( X_0^+(p) = X_0(p)/\langle w_p \rangle \) has non cuspidal rational points without CM.

**Proof.** Let us prove (i). By Theorem 3.1 we can restrict our attention to the non-CM case. First, we assume that \( f \in \text{New}_N \backslash \text{New}^\text{cm}_N \) is \( n \)-primitive. Let \( g = f \otimes \chi = \sum_{m > 0} a_m q^m \) be such that \( \dim A_f = \dim A_g \) and \( B(g) = \tilde{B}(f) \). Let us denote by \( \tau_1, \ldots, \tau_n \) the \( \mathbb{Q} \)-embeddings of \( E_g \) into \( \mathbb{Q} \). The matrix \( (\tau_i a_j)_{i \leq n, j \leq B(g)+1} \) has rank \( n \) and, thus, \( \{a_1, \ldots, a_{B(g)+1}\} \) is a system of generators of the \( \mathbb{Q} \)-vector space \( E_g \). Under the assumption \( \tilde{B}(n) < +\infty \), we have that \( E_g \) is the field \( \mathbb{Q}(a_2, \ldots, a_{B(n)+1}) \). For any integer \( m > 1 \), \( a_m \) is an algebraic integer of degree at most \( n \) such that \( |\tau_i a_m| \leq \sigma_0(m) \sqrt{m} \) for all \( i \leq n \), where \( \sigma_0(m) = \sum_{0 \leq d | m} 1 \). Therefore, there are only finitely many possibilities for the values \( a_m \) and, thus, finitely many possibilities for \( E_g \). The condition \( \dim A_{g \otimes \chi^{-1}} = n \) implies that the order \( k \) of \( \chi \) satisfies \( \varphi(k) | n \). Therefore the number field \( E_f \) is a subfield of the compositum of \( E_g \) with the \( k \)-th cyclotomic field, and it follows that the set of number
fields $\mathbb{E}_f$ obtained for $n$-primitive newforms $f$ is finite. If $\dim A_f = n$ and $f$ is not $n$-primitive, let $\chi$ be a Dirichlet character such that $f_0 = f \otimes \chi$ is $m$-primitive for some $m < n$. By using that there are finitely many possibilities for $\mathbb{E}_{f_0}$ and for $\mathbb{Q}(\chi)$, it follows that the set of number fields $\mathbb{E}_f$ for the case that $\dim A_f = n$ and $f$ is not $n$-primitive is also finite.

Let us prove (ii). The existence of a non cuspidal point in $X^+_0(p)(\mathbb{Q})$ without CM implies the existence of an elliptic curve $E$ without CM defined over a quadratic field $K$ along with a $p$-isogeny $\mu : E \rightarrow \sigma E$, where $\sigma$ is the non-trivial element of $\text{Gal}(K/\mathbb{Q})$. If $p \equiv 1 \pmod{4}$, then $p$ is a norm of $K$ (cf. [8]) and, thus, we can choose $E$ such that $\mu$ is defined over $K$ and $\mu \circ \sigma \mu = [p]$. Therefore, the Weil restriction $A = \text{Res}_{K/\mathbb{Q}}(E)$ is an abelian surface such that $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ is $\mathbb{Q}(\sqrt{p})$, i.e., $A$ is of $\text{GL}_2$-type with real multiplication by $\sqrt{p}$. Due to recent results on Serre’s modularity conjecture by Khare-Wintenberger [12], Dieulefait [5] and Kissin [13], the abelian surface $A$ is modular and there exists a normalized newform $f$ with trivial Nebentypus such that $A$ is $\mathbb{Q}$-isogenous to $A_f$. It is clear that $f$ is $2$-primitive. Let $\chi$ be a quadratic Dirichlet character such that the newform $g = f \otimes \chi = q + \sum_{m>1} a_m q^m$ satisfies $\tilde{B}(f) = B(g)$. Due to the fact that $g$ has an inner-twist by the quadratic character attached to $K$ and its Nebentypus is trivial, the Fourier coefficients of $g$ satisfy the next condition: If $a_m \notin \mathbb{Z}$ then $a_m^2/p \notin \mathbb{Z}$. Let $p_0$ be the least prime such that $a_{p_0} \notin \mathbb{Z}$. Hence, $p_0 \leq \tilde{B}(2)$. The statement follows from the inequality $p \leq a_{p_0}^2 \leq (2\sqrt{p_0})^2 \leq 4 \tilde{B}(2)$.

**Remark 4.1** In view of the results in [15], in the QM case we do not see any reason to derive the finiteness of the set $S_n$ from the finiteness of the set of number fields $\mathbb{E}_f$ of degree $n$ and, thus, from the condition $\tilde{B}(n) < +\infty$.

We conclude by giving a better lower bound for $\tilde{B}(2)$ than the one provided by the above computations for newforms of level $\leq 3000$ and trivial Nebentypus.

**Proposition 4.2** There is a 2-primitive normalized newform $f \in S_2(\Gamma_0(2 \cdot 5^2 \cdot 31159^2))$ such that $\tilde{B}(f) = B(f) = 59$. In particular, $\tilde{B}(2) \geq 59$.

**Proof.** Consider the elliptic curve $E : y^2 = x^3 + Ax + B$, where

$$A = 13709960(2643250204357 - 285242082633\sqrt{-D})$$

$$B = 348980800( -1822416766804803284533 + 63802091292233830777\sqrt{-D}),$$

and $D = 31159$. It can be checked that the conductor of $E$ is the integral ideal of $K = \mathbb{Q}(\sqrt{-D})$ generated by $2 \cdot 5^2 \cdot 31159$. The pair $(E, E)$ provides a non-CM rational point on the curve $X^+_0(137)$, i.e. $E$ is a quadratic $\mathbb{Q}$-curve without CM of degree 137 (cf. [6]).
We claim that the isogeny $\mu : E \to \overline{E}$ of degree 137 is defined over $\mathbb{K}$. Indeed, let $x_1, \ldots, x_{136}$ be the $x$-coordinates of the non-trivial points of the kernel of $\mu$. By [7], it suffices to prove that $-137 \cdot N_{\mathbb{K}/\mathbb{Q}}(s_1) \in (\mathbb{K}^*)^2$, where $s_1 = \sum_{i=1}^{136} x_i$. One way to determine $s_1$ is to compute the 137-th division polynomial of $E$, which has degree 9384, and then to factorize it over $\mathbb{K}$.

A better though approximate way is to determine a basis $\{\omega_1, \omega_2\}$ of periods of $E$ such that $\tau = \omega_1/\omega_2$ is in the upper half-plane and satisfies $j(137 \tau) = j(\tau)$. Then, $s_1 = \sum_{i=1}^{136} \wp(i/\omega_2; \omega_1, \omega_2)$, where $\wp(z; \omega_1, \omega_2)$ denotes the Weierstrass function attached to the period lattice of $E$. After computing, we obtain

$$\omega_1 = -0.00000559349200452413239 \ldots - 0.0000134040043086026752 \ldots i,$$

$$\omega_2 = 0.000336023031664207601 \ldots + 0.000808125822643943557 \ldots i,$$

$$s_1 = 103120(1152883 + 56273\sqrt{-D}).$$

It is now immediate to check that $-137 N_{\mathbb{K}/\mathbb{Q}}(s_1)$ is a square in $\mathbb{K}^*$.

Since $\mu$ is defined over $\mathbb{K}$ and $\mu \circ \pi = [137]$, the Weil restriction $A = \text{Res}_{\mathbb{K}/\mathbb{Q}}(E)$ satisfies $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{137})$ and, thus, $A/\mathbb{Q}$ is modular. By Milne’s formula in [14], the conductor of $A$ is

$$N_{\mathbb{K}/\mathbb{Q}}(\text{cond}(E)) \cdot D^2 = 2^2 \cdot 5^4 \cdot 31159^4$$

and, thus, there exists a normalized newform $f = \sum_{n>0} a_n q^n \in S_2(\Gamma_0(N))$ such that $A_f$ is $\mathbb{Q}$-isogenous to $A$, where

$$N = \sqrt{\text{cond}(A)} = 2 \cdot 5^2 \cdot 31159^2$$

(cf. [4]). The newform $f$ has an inner-twist by the quadratic Dirichlet character attached to $\mathbb{K}$. The Fourier coefficients $a_n$ lie in $\mathbb{Q}(\sqrt{137})$ and we know that, if $a_p \notin \mathbb{Z}$ for a prime $p$, then $A$ has good reduction at $p$, $p$ is inert in $\mathbb{K}$ and $a_p/\sqrt{137} \in \mathbb{Z}$. Then, we have to determine the first inert prime $p_0$ such that $a_{p_0} \neq 0$. By the Eichler-Shimura congruence, for a prime $p \nmid N$, the polynomial $(x^2 - a_x x + p)(x - \sigma a_x x + p)$ is the characteristic polynomial of Frobenius acting on the $l$-adic Tate module of the reduction of $A$ mod $p$. Therefore, the characteristic polynomial of Frobenius acting on the $l$-adic Tate module of the reduction of $E$ mod $p$ is $x^2 - (a_p^2 - 2p) x + p^2$. After computing, we obtain $p_0 = 59$ and $a_{59} = \pm \sqrt{137}$. Then, the $q$-expansion of $(f - \sigma f)/(2\sqrt{137})$ is $\pm q^{59} + O(q^{60})$ and, thus, $B(f) = 59$.

Let $g = f \otimes \chi = \sum_{n>0} b_n q^n$, where $\chi$ is any quadratic Dirichlet character. For every inert prime $p < 59$ of $\mathbb{K}$ at which $A_g$ has good reduction, we know that $A_f$ has also good reduction at $p$. Therefore, $b_p = \chi(p) a_p = 0$ and, thus, $\widetilde{B}(f) = B(f)$. 

\[\blacksquare\]
References


*Recibido: 5 de noviembre de 2009*

Josep González
Universitat Politècnica de Catalunya
Escola Politècnica Superior d’Enginyeria de Vilanova i la Geltrú
Av. Victor Balaguer s/n
08800 Vilanova i la Geltrú, Spain
josepg@ma4.upc.edu

The author is partially supported by DGICYT Grant MTM2009-13060-C02-02.