Rev. Mat. Iberoamericana **26** (2010), no. 3, 915–946

Well-posedness in critical spaces for the compressible Navier-Stokes equations with density dependent viscosities

Qionglei Chen, Changxing Miao and Zhifei Zhang

Abstract

In this paper, we prove the local well-posedness in critical Besov spaces for the compressible Navier-Stokes equations with density dependent viscosities under the assumption that the initial density is bounded away from zero.

1. Introduction

In this paper, we consider the compressible Navier-Stokes equations with density dependent viscosities in $\mathbb{R}^+ \times \mathbb{R}^N (N \geq 2)$:

$$(1.1)\quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) D(u)) - \nabla(\lambda(\rho)\operatorname{div} u) + \nabla P(\rho) = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Here $\rho(t,x)$ and $u(t,x)$ are the density and velocity of the fluid. The pressure P is a smooth function of ρ, $D(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ is the strain tensor, the Lamé coefficients μ and λ depend smoothly on ρ and satisfy

$$(1.2)\qquad \mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0,$$

which ensures that the operator $-\operatorname{div}(2\mu(\rho)D\cdot) - \nabla(\lambda(\rho)\operatorname{div}\cdot)$ is elliptic. An important example is included in the system (1.1): the viscous shallow water equations($N = 2, \mu(\rho) = \rho, \lambda(\rho) = 0$ and $P(\rho) = \rho^2$).

2000 Mathematics Subject Classification: 35Q30, 35D10.
Keywords: Compressible Navier-Stokes equations, Besov spaces, Bony's paraproduct, Fourier localization.

The local existence and uniqueness of smooth solutions for the system (1.1) were proved by Nash [23] for smooth initial data without vacuum. Later on, Matsumura and Nishida [20] proved the global well-posedness for smooth data close to equilibrium, see also [18] for one dimension. Concerning the global existence of weak solutions for the large initial data, we refer to [2, 3, 19, 21]. We may refer to [4, 10, 25] and references therein for the viscous shallow water equations.

This paper is devoted to the study of the well-posedness of the system (1.1) in the critical spaces. Recently, Danchin has obtained several important well-posedness results in the critical spaces for the compressible Navier-Stokes equations [11, 12, 14]. To explain the precise meaning of critical spaces, let us consider the incompressible Navier-Stokes equations

$$\text{(NS)} \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \text{div} u = 0. \end{cases}$$

It is easy to find that if (u, p) is a solution of (NS), then

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t,x) \stackrel{\text{def}}{=} \lambda^2 p(\lambda^2 t, \lambda x) \tag{1.3}$$

is also a solution of (NS). For the (NS) equations, a functional space X is critical if the corresponding norm is invariant under the scaling of (1.3). Obviously, $\dot{H}^{\frac{N}{2}-1}$ is a critical space. Fujita and Kato [16] proved the well-posedness of (NS) in $\dot{H}^{\frac{N}{2}-1}$, see also [5, 6, 22] and references therein for the well-posedness in the other critical spaces. For the compressible Navier-Stokes equations, let us introduce the following transformation

$$\rho_\lambda(t,x) \stackrel{\text{def}}{=} \rho(\lambda^2 t, \lambda x), \quad u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x).$$

Then if (ρ, u) solves (1.1), so does $(\rho_\lambda, u_\lambda)$ provided the viscosity coefficients are constants and the pressure law has been changed into $\lambda^2 P$. This motivates the following definition:

Definition 1.1 *We will say that a functional space is critical with respect to the scaling of the equations if the associated norm is invariant under the transformation:*

$$(\rho, u) \longrightarrow (\rho_\lambda, u_\lambda)$$

(up to a constant independent of λ).

A natural candidate is the homogenous Sobolev space $\dot{H}^{N/2} \times \left(\dot{H}^{N/2-1}\right)^N$, but since $\dot{H}^{N/2}$ is not included in L^∞, we can not obtain a L^∞ control of the density when $\rho_0 \in \dot{H}^{N/2}$.

Instead, we choose the initial data (ρ_0, u_0) for some $\bar{\rho}_0$ in a critical homogenous Besov spaces:

$$(\rho_0 - \bar{\rho}_0, u_0) \in \dot{B}^{\frac{N}{p}}_{p,1} \times \big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big)^N,$$

since $\dot{B}^{\frac{N}{p}}_{p,1}$ is continuously embedded in L^∞.

However, working in the critical spaces, if we deal with the elliptic operators of the momentum equations as a constant coefficient second order operator plus a perturbation induced by the density and viscosity coefficients, the perturbation will be a trouble term. In the case when $\rho - \bar{\rho}_0$ is small in $\dot{B}^{\frac{N}{p}}_{p,1}$ or has more regularity, the perturbation can be treated as a harmless source term and the corresponding local well-posedness can be obtained by following the argument of Danchin [12], see [17].

The purpose of the present paper is to obtain a local well-posedness result in the critical Besov spaces under the natural physical assumption that the initial density is bounded away from zero. Our new observation is that if $\rho - \bar{\rho}_0$ is small in the weighted Besov spaces $\dot{B}^{\frac{N}{p}}_{p,1}(\omega)$(see Section 3 for the definition), the perturbation can still be treated as a harmless source term. Similar idea has been used by the authors of this paper to prove the local well-posedness in $\dot{B}^1_{2,1} \times \big(\dot{B}^0_{2,1}\big)^2$ for the viscous shallow water equations [10]. Very rencently, Danchin [15] proved a similar result for the system (1.1) with constant coefficients. The key of his proof is a new and interesting estimate for a class of parabolic systems with the coefficients in $C([0,T]; \dot{B}^{N/2}_{2,1})$. It seems to be possible to adapt his method to the present model. Here we would like to present a general functional framework to deal with the local well-posedness in the critical spaces for the compressible fluids.

Our main result is as follows:

Theorem 1.2 *Let $\bar{\rho}_0$ and c_0 be two positive constants. Assume that the initial data satisfies*

$$(\rho_0 - \bar{\rho}_0, u_0) \in \dot{B}^{\frac{N}{p}}_{p,1} \times \big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big)^N \quad \textit{and} \quad \rho_0 \geq c_0.$$

Then there exists a positive time T such that

(a) **Existence:** *If $p \in (1, N]$, the system (1.1) has a solution $(\rho - \bar{\rho}_0, u) \in E^p_T$ with*

$$E^p_T \stackrel{\text{def}}{=} C([0,T]; \dot{B}^{\frac{N}{p}}_{p,1}) \times \Big(C([0,T]; \dot{B}^{\frac{N}{p}-1}_{p,1}) \cap L^1(0,T; \dot{B}^{\frac{N}{p}+1}_{p,1})\Big)^N, \quad \rho \geq \frac{1}{2}c_0;$$

(b) **Uniqueness:** *If $p \in (1, N]$, then the uniqueness holds in E^p_T.*

Remark 1.3 *If the Lamé coefficients μ and λ are constants satisfying* (1.2)*, then the range of p in the existence result of the system* (1.1) *can be extended to $p \in (1, 2N)$, since we can take $p \in (1, 2N)$ in Proposition* 5.1 *for the case when $\overline{\lambda}$ and $\overline{\mu}$ are constants.*

The structure of this paper is as follows:

In Section 2, we recall some basic facts about the Littlewood-Paley decomposition and the functional spaces. In Section 3, we firstly introduce the weighted Besov spaces, then present some nonlinear estimates. Section 4 is devoted to the estimates in the weighted Besov spaces for the linear transport equation. Section 5 is devoted to the estimates in the weighted Besov spaces for the linearized momentum equation. In Section 6, we prove the existence of the solution. In Section 7, we prove the uniqueness of the solution.

2. Littlewood-Paley theory and the functional spaces

Let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for all } \xi \neq 0.$$

The frequency localization operator Δ_j and S_j are defined by

$$\Delta_j f = \varphi(2^{-j}D)f, \quad S_j f = \sum_{k\le j-1} \Delta_k f \quad \text{for} \quad j \in \mathbb{Z}.$$

With our choice of φ, one can easily verify that

$$\begin{aligned} &\Delta_j\Delta_k f = 0 \quad \text{if} \quad |j-k| \ge 2 \quad \text{and} \\ &\Delta_j(S_{k-1}f\Delta_k f) = 0 \quad \text{if} \quad |j-k| \ge 5. \end{aligned} \tag{2.1}$$

We denote the space $\mathcal{Z}'(\mathbb{R}^N)$ by the dual space of $\mathcal{Z}(\mathbb{R}^N) = \{f \in \mathcal{S}(\mathbb{R}^N); D^\alpha \hat{f}(0) = 0; \forall\alpha \in \mathbb{N}^N \text{ multi-index}\}$, it also can be identified by the quotient space of $\mathcal{S}'(\mathbb{R}^N)/\mathcal{P}$ with the polynomials space $\mathcal{P}$. The formal equality

$$f = \sum_{k\in\mathbb{Z}} \Delta_k f$$

holds true for $f \in \mathcal{Z}'(\mathbb{R}^N)$ and is called the homogeneous Littlewood-Paley decomposition.

The operators Δ_j help us recall the definition of the Besov space (see also [24]).

Definition 2.1 *Let $s \in \mathbb{R}$, $1 \le p, r \le +\infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by*

$$\dot{B}^s_{p,r} = \{f \in \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{\dot{B}^s_{p,r}} < +\infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big\| 2^{ks} \|\Delta_k f(t)\|_p \Big\|_{\ell^r}.$$

We next introduce the Besov-Chemin-Lerner space $\widetilde{L}^q_T(\dot{B}^s_{p,r})$ which is initiated in [9].

Definition 2.2 *Let $s \in \mathbb{R}$, $1 \le p, q, r \le +\infty$, $0 < T \le +\infty$. The space $\widetilde{L}^q_T(\dot{B}^s_{p,r})$ is defined as the set of all the distributions f satisfying*

$$\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} < +\infty,$$

where

$$\|f\|^r_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \Big\| 2^{ks} \|\Delta_k f(t)\|_{L^q(0,T;L^p)} \Big\|_{\ell^r}.$$

Obviously, $\widetilde{L}^1_T(\dot{B}^s_{p,1}) = L^1_T(\dot{B}^s_{p,1})$. In the sequel, we will constantly use the Bony's decomposition from [1] that

$$uv = T_u v + T_v u + R(u,v), \tag{2.2}$$

with

$$T_u v = \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v, \quad R(u,v) = \sum_{j \in \mathbb{Z}} \Delta_j u \widetilde{\Delta}_j v, \quad \widetilde{\Delta}_j v = \sum_{|j'-j| \le 1} \Delta_{j'} v.$$

Let us conclude this section by collecting some useful lemmas.

Lemma 2.3 ([7]) *Let $1 \le p \le q \le +\infty$. Assume that $f \in L^p(\mathbb{R}^N)$, then for any $\gamma \in (\mathbb{N} \cup \{0\})^N$, there exist constants C_1, C_2 independent of f, j such that*

$$\begin{aligned} \text{supp}\hat{f} \subseteq \{|\xi| \le A_0 2^j\} \quad &\Longrightarrow \quad \|\partial^\gamma f\|_q \le C_1 2^{j|\gamma| + jN(\frac{1}{p} - \frac{1}{q})} \|f\|_p\,, \\ \text{supp}\hat{f} \subseteq \{A_1 2^j \le |\xi| \le A_2 2^j\} \quad &\Longrightarrow \quad \|f\|_p \le C_2 2^{-j|\gamma|} \sup_{|\beta| = |\gamma|} \|\partial^\beta f\|_p\,. \end{aligned}$$

Lemma 2.4 ([12]) *Let $1 < p < \infty$, and $a \ge \bar{a} > 0$ be a bounded continuous function. Assume that $u \in L^p(\mathbb{R}^N)$ and $\text{supp}\, \hat{u} \subset \{\xi : R_1 \le |\xi| \le R_2\}$. Then there exists a constant c depending only on N and R_2/R_1 such that*

$$c\, \bar{a}\, R_1^2 \frac{(p-1)}{p^2} \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}\,x \le -\int_{\mathbb{R}^N} \operatorname{div}(a \nabla u) |u|^{p-2} u \,\mathrm{d}\,x.$$

Lemma 2.5 *Let $s>0$, and $1\le p\le\infty$. Assume that $f,g\in \dot B^{s_1}_{p,1}\cap L^\infty$. Then there holds*

$$\|fg\|_{\dot B^s_{p,1}}\le C\big(\|f\|_{\dot B^s_{p,1}}\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{\dot B^s_{p,1}}\big).$$

Lemma 2.6 *Let $s_1,s_2\le\frac{N}{p}$, $s_1+s_2>N\max(0,\frac{2}{p}-1)$, and $1\le p,q,q_1,q_2\le\infty$ with $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Assume that $f\in\widetilde L^{q_1}_T(\dot B^{s_1}_{p,1})$ and $g\in\widetilde L^{q_2}_T(\dot B^{s_2}_{p,1})$. Then there holds*

$$\|fg\|_{\widetilde L^q_T(\dot B^{s_1+s_2-\frac{N}{p}}_{p,1})}\le C\|f\|_{\widetilde L^{q_1}_T(\dot B^{s_1}_{p,1})}\|g\|_{\widetilde L^{q_2}_T(\dot B^{s_2}_{p,1})}.$$

Lemma 2.7 *Let $s_1\le\frac{N}{p}, s_2<\frac{N}{p}$, $s_1+s_2\ge N\max(0,\frac{2}{p}-1)$, and $1\le p,q,q_1,q_2\le\infty$ with $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Assume that $f\in\widetilde L^{q_1}_T(\dot B^{s_1}_{p,1})$ and $g\in\widetilde L^{q_2}_T(\dot B^{s_2}_{p,\infty})$. Then there holds*

$$\|fg\|_{\widetilde L^q_T(\dot B^{s_1+s_2-\frac{N}{p}}_{p,\infty})}\le C\|f\|_{\widetilde L^{q_1}_T(\dot B^{s_1}_{p,1})}\|g\|_{\widetilde L^{q_2}_T(\dot B^{s_2}_{p,\infty})}.$$

Lemma 2.8 *Let $s\in(-N\min\big(\frac{1}{p},\frac{1}{p'}\big),\frac{N}{p}+1]$ and $1\le p,q,q_1,q_2\le\infty$ with $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Assume that $f\in\widetilde L^{q_1}_T(\dot B^{\frac{N}{p}+1}_{p,1})$ and $g\in\widetilde L^{q_2}_T(\dot B^{s}_{p,1})$. Then there holds*

$$\sum_j 2^{j(s-1)}\|\mathrm{div}[\Delta_j,f]\nabla g\|_{L^q_T(L^p)}\le C\|f\|_{\widetilde L^{q_1}_T(\dot B^{\frac{N}{p}+1}_{p,1})}\|g\|_{\widetilde L^{q_2}_T(\dot B^{s}_{p,1})}.$$

Lemma 2.9 *Let $s>0$ and $1\le p,q\le\infty$. Assume that $F\in W^{[s]+3,\infty}_{loc}(\mathbb{R})$ with $F(0)=0$. Then for any $f\in L^\infty_T(L^\infty)\cap\widetilde L^q_T(\dot B^s_{p,1})$, we have*

$$\|F(f)\|_{\widetilde L^q_T(\dot B^s_{p,1})}\le C\big(1+\|f\|_{L^\infty_T(L^\infty)}\big)^{[s]+2}\|f\|_{\widetilde L^q_T(\dot B^s_{p,1})}.$$

Lemma 2.6-Lemma 2.9 can be easily proved by using Bony's decomposition and Lemma 2.3, see also [8, 12] or Section 3 for similar results.

Remark 2.10 *Lemmas* 2.6–2.9 *still remain true for the usual homogenous Besov spaces. For example, the estimate in Lemma* 2.6 *becomes*

$$\|fg\|_{\dot B^{s_1+s_2-\frac{N}{p}}_{p,1}}\le C\|f\|_{\dot B^{s_1}_{p,1}}\|g\|_{\dot B^{s_2}_{p,1}},$$

with p,s_1,s_2 satisfying the conditions as in Lemma 2.6.

3. Nonlinear estimates in the weighted Besov spaces

Let us firstly introduce the weight function. Let $\{e_k(t)\}_{k\in\mathbb{Z}}$ be a sequence defined in $[0,+\infty)$ satisfying the following conditions:

$$e_k(t)\in[0,1],\ e_k(t)\le e_{k'}(t)\ \text{ if } k\le k' \quad\text{and}\quad e_k(t)\sim e_{k'}(t) \text{ if } \ k\sim k'. \tag{3.1}$$

where $k\sim k'$ means that there exists a constant N_0 such that $|k-k'|\le N_0$. Then the weight function $\{\omega_k(t)\}_{k\in\mathbb{Z}}$ is defined by

$$\omega_k(t)=\sum_{\ell\ge k}2^{k-\ell}e_\ell(t),\quad k\in\mathbb{Z}.$$

It is easy to verify that for any $k\in\mathbb{Z}$,

$$\begin{cases}\omega_k(t)\le 2,\quad e_k(t)\le\omega_k(t),\\ \omega_k(t)\le 2^{k-k'}\omega_{k'}(t)\quad\text{if } k\ge k',\quad \omega_k(t)\le 3\omega_{k'}(t)\quad\text{if } k\le k',\\ \omega_k(t)\sim\omega_{k'}(t)\quad\text{if } k\sim k'.\end{cases} \tag{3.2}$$

Definition 3.1 *Let $s\in\mathbb{R}$, $1\le p,r\le+\infty$, $0<T<+\infty$. The weighted Besov space $\dot B^s_{p,r}(\omega)$ is defined by*

$$\dot B^s_{p,r}(\omega)=\{f\in\mathcal{Z}'(\mathbb{R}^N):\ \|f\|_{\dot B^s_{p,r}(\omega)}<+\infty\},$$

where

$$\|f\|_{\dot B^s_{p,r}(\omega)}\stackrel{\text{def}}{=}\big\|2^{ks}\omega_k(T)\|\Delta_k f\|_p\big\|_{\ell^r}.$$

Definition 3.2 *Let $s\in\mathbb{R}$, $1\le p,q\le+\infty$, $0<T<+\infty$. The weighted function space $\widetilde L^q_T(\dot B^s_{p,1}(\omega))$ is defined by*

$$\widetilde L^q_T(\dot B^s_{p,1}(\omega))=\{f\in L^q_T(\dot B^s_{p,1}(\omega)):\ \|f\|_{\widetilde L^q_T(\dot B^s_{p,1}(\omega))}<+\infty\},$$

where

$$\|f\|_{\widetilde L^q_T(\dot B^s_{p,1}(\omega))}\stackrel{\text{def}}{=}\sum_{k\in\mathbb{Z}}2^{ks}\omega_k(T)\Big(\int_0^T\|\Delta_k f(t)\|_p^q dt\Big)^{\frac1q}.$$

Remark 3.3 *If $e_k(t)$ is continuous on $[0,+\infty)$ and $e_k(0)=0$ for $k\in\mathbb{Z}$, $f\in\widetilde L^\infty_T(\dot B^s_{p,1})$, then for any $\varepsilon>0$, there exists a $\widetilde T\in(0,T]$ such that*

$$\|f\|_{\widetilde L^\infty_{\widetilde T}(\dot B^s_{p,1}(\omega))}\le\varepsilon.$$

Indeed, due to $f \in \widetilde{L}^\infty_T(\dot{B}^s_{p,1})$ *and* $\omega_k(T) \le 2$, *there exists* $N_1 \in \mathbb{N}$ *such that*

$$\sum_{|k|\ge N_1+1} 2^{ks}\omega_k(T)\|\Delta_k f\|_{L^\infty_T(L^p)} \le \varepsilon/3,$$
$$\sum_{|k|\le N_1} 2^{ks} \sum_{\ell\ge k+N_1+1} 2^{k-\ell} e_\ell(T)\|\Delta_k f\|_{L^\infty_T(L^p)} \le \varepsilon/3.$$

Thus, we have

$$\begin{aligned}\|f\|_{\widetilde{L}^\infty_{\widetilde{T}}(\dot{B}^s_{p,1}(\omega))} &\le 2\varepsilon/3 + \sum_{|k|\le N_1} 2^{ks} \sum_{k\le\ell\le k+N_1} 2^{k-\ell} e_\ell(\widetilde{T})\|\Delta_k f\|_{L^\infty_{\widetilde{T}}(L^p)}\\ &\le 2\varepsilon/3 + 2e_{2N_1}(\widetilde{T}) \sum_{|k|\le N_1} 2^{ks}\|\Delta_k f\|_{L^\infty_{\widetilde{T}}(L^p)}\\ &\le \varepsilon,\end{aligned}$$

if $\widetilde{T} \in (0, T]$ *is chosen such that*

$$2e_{2N_1}(\widetilde{T}) \sum_{|k|\le N_1} 2^{ks}\|\Delta_k f\|_{L^\infty_{\widetilde{T}}(L^p)} \le \varepsilon/3.$$

Next, we present some estimates in the weighted Besov spaces.

Lemma 3.4 *Let* $1 \le p \le \infty$. *Assume that* $f \in \dot{B}^{s_1}_{p,1}(\omega), g \in \dot{B}^{s_2}_{p,1}$. *Then there hold*
(a) *if* $s_2 \le \frac{N}{p}$, *we have*

$$\|T_g f\|_{\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega)} \le C\|f\|_{\dot{B}^{s_1}_{p,1}(\omega)}\|g\|_{\dot{B}^{s_2}_{p,1}};$$

(b) *if* $s_1 \le \frac{N}{p} - 1$, *we have*

$$\|T_f g\|_{\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega)} \le C\|f\|_{\dot{B}^{s_1}_{p,1}(\omega)}\|g\|_{\dot{B}^{s_2}_{p,1}};$$

(c) *if* $s_1 + s_2 > N\max(0, \frac{2}{p} - 1)$, *we have*

$$\|R(f, g)\|_{\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega)} \le C\|f\|_{\dot{B}^{s_1}_{p,1}(\omega)}\|g\|_{\dot{B}^{s_2}_{p,1}}.$$

Proof. Due to (2.1), we have

$$\Delta_j(T_g f) = \sum_{|j'-j|\le 4} \Delta_j(S_{j'-1}g\Delta_{j'}f),$$

then we get by Lemma 2.3 and (3.2) that

$$\begin{aligned}\|T_g f\|_{\dot B^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega)} &= \sum_j 2^{j(s_1+s_2-\frac{N}{p})}\omega_j(T)\|\Delta_j(T_g f)\|_p \\ &\le C\sum_j 2^{j(s_1+s_2-\frac{N}{p})}\omega_j(T)\|S_{j-1}g\|_\infty\|\Delta_j f\|_p \\ &\le C\|f\|_{\dot B^{s_1}_{p,1}(\omega)}\|g\|_{\dot B^{s_2}_{p,1}},\end{aligned}$$

where we used in the last inequality

$$\|S_{j-1}g\|_\infty \le C\sum_{\ell\le j-2} 2^{\ell\frac{N}{p}}\|\Delta_\ell g\|_p \le C2^{j(-s_2+\frac{N}{p})}\|g\|_{\dot B^{s_2}_{p,1}}.$$

This proves (a). We next prove (b). Similarly, we have

$$\begin{aligned}\|T_f g\|_{\dot B^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega)} &= \sum_j 2^{j(s_1+s_2-\frac{N}{p})}\omega_j(T)\|\Delta_j(T_f g)\|_p \\ &\le C\sum_j 2^{j(s_1+s_2-\frac{N}{p})}\omega_j(T)\|S_{j-1}f\|_\infty\|\Delta_j g\|_p,\end{aligned}$$

and by Lemma 2.3 and (3.2), we have

$$\begin{aligned}\omega_j(T)\|S_{j-1}f\|_\infty &\le C2^j\sum_{\ell\le j-2} 2^{\ell(\frac{N}{p}-1)}\omega_\ell(T)\|\Delta_\ell f\|_p \\ &\le C2^{j(\frac{N}{p}-s_1)}\|f\|_{\dot B^{s_1}_{p,1}(\omega)},\end{aligned}$$

which lead to (b). Now we prove (c). Notice that

$$\Delta_j(R(f,g)) = \sum_{j'\ge j-3}\Delta_j(\Delta_{j'}f\widetilde\Delta_{j'}g),$$

then we get by Lemma 2.3 that if $p \ge 2$

$$\begin{aligned}\|R(f,g)\|_{\dot B^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega)} &\le C\sum_j\sum_{j'\ge j-3} 2^{j(s_1+s_2)}\omega_j(T)\|\Delta_{j'}f\|_p\|\widetilde\Delta_{j'}g\|_p \\ &\le C\sum_j\sum_{j'\ge j-3}\sum_{\ell\ge j} 2^{j-\ell}e_\ell(T)2^{j(s_1+s_2)}\|\Delta_{j'}f\|_p\|\widetilde\Delta_{j'}g\|_p \\ &= \sum_j\sum_{j'\ge j-3}\sum_{\ell\ge j,j'}\square + \sum_j\sum_{j'\ge j-3}\sum_{j'\ge\ell\ge j}\square \\ &\triangleq I + II.\end{aligned}$$

For II, using the fact that $e_\ell(T) \le e_{j'}(T) \le \omega_{j'}(T)$ if $\ell \le j'$, we get

$$\begin{aligned} II &\le C\sum_j \sum_{j'\ge j-3} \omega_{j'}(T)2^{j(s_1+s_2)}\|\Delta_{j'}f\|_p\|\widetilde{\Delta}_{j'}g\|_p \\ &\le C\sum_j \sum_{j'\ge j-3} \omega_{j'}(T)2^{(j-j')(s_1+s_2)}2^{j's_1}\|\Delta_{j'}f\|_p\|g\|_{\dot B^{s_2}_{p,1}} \\ &\le C\|f\|_{\dot B^{s_1}_{p,1}(\omega)}\|g\|_{\dot B^{s_2}_{p,1}}, \end{aligned}$$

and for I, using the fact that

$$\sum_{\ell\ge j,j'} 2^{j-\ell}e_\ell(T) \le 2^{j-j'}\sum_{\ell\ge j'} 2^{j'-\ell}e_\ell(T) = 2^{j-j'}w_{j'}(T),$$

we obtain

$$\begin{aligned} I &\le C\sum_j \sum_{j'\ge j-3} \omega_{j'}(T)2^{j(s_1+s_2)}2^{j-j'}\|\Delta_{j'}f\|_p\|\widetilde{\Delta}_{j'}g\|_p \\ &\le C\sum_j \sum_{j'\ge j-3} \omega_{j'}(T)2^{(j-j')(s_1+s_2+1)}2^{j's_1}\|\Delta_{j'}f\|_p\|g\|_{\dot B^{s_2}_{p,1}} \\ &\le C\|f\|_{\dot B^{s_1}_{p,1}(\omega)}\|g\|_{\dot B^{s_2}_{p,1}}. \end{aligned}$$

If $p < 2$, we get by Lemma 2.3 that

$$\begin{aligned} &\|R(f,g)\|_{\dot B^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega)} \le \\ &\le C\sum_j \sum_{j'\ge j-3} 2^{j(s_1+s_2-N(\frac{2}{p}-1))}\omega_j(T)\|\Delta_{j'}f\|_p\|\widetilde{\Delta}_{j'}g\|_{p'} \\ &\le C\sum_j \sum_{j'\ge j-3}\sum_{\ell\ge j} 2^{j-\ell}e_\ell(T)2^{j(s_1+s_2-N(\frac{2}{p}-1))}\|\Delta_{j'}f\|_p\|\widetilde{\Delta}_{j'}g\|_p 2^{N(\frac{2}{p}-1)j'}. \end{aligned}$$

Then treating it as in the case of $p \ge 2$, we obtain the same inequality for $s_1 + s_2 > N(\frac{2}{p} - 1)$. This proves (c). ■

We have a similar result in the weighted Besov spaces with the time.

Lemma 3.5 *Let* $1 \le p, q, q_1, q_2 \le \infty$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. *Assume that* $f \in \widetilde{L}^{q_1}_T(\dot B^{s_1}_{p,1}(\omega)), g \in \widetilde{L}^{q_2}_T(\dot B^{s_2}_{p,1})$. *Then there hold*

(a) *if* $s_2 \le \frac{N}{p}$, *we have*

$$\|T_g f\|_{\widetilde{L}^q_T(\dot B^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega))} \le C\|f\|_{\widetilde{L}^{q_1}_T(\dot B^{s_1}_{p,1}(\omega))}\|g\|_{\widetilde{L}^{q_2}_T(\dot B^{s_2}_{p,1})};$$

(b) *if* $s_1 \le \frac{N}{p} - 1$, *we have*

$$\|T_f g\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega))} \le C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1}(\omega))}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})};$$

(c) *if* $s_1 + s_2 > N\max(0, \frac{2}{p} - 1)$, *we have*

$$\|R(f,g)\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega))} \le C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1}(\omega))}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})}.$$

The following proposition is a direct consequence of Lemma 3.5.

Proposition 3.6 *Let* $s_1 \le \frac{N}{p} - 1, s_2 \le \frac{N}{p}, s_1 + s_2 > N\max(0, \frac{2}{p} - 1)$, *and* $1 \le p, q, q_1, q_2 \le \infty$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. *Assume that* $f \in \widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1}(\omega))$ *and* $g \in \widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})$. *Then there holds*

$$\|fg\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}(\omega))} \le C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1}(\omega))}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})}.$$

From the proof of Lemma 3.4, we can also obtain

Proposition 3.7 *Let* $s_1 \le \frac{N}{p} - 1, s_2 < \frac{N}{p}, s_1 + s_2 \ge N\max(0, \frac{2}{p} - 1)$, *and* $1 \le p, q, q_1, q_2 \le \infty$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. *Assume that* $f \in \widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1}(\omega))$ *and* $g \in \widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,\infty})$. *Then there holds*

$$\|fg\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,\infty}(\omega))} \le C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1}(\omega))}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,\infty})}.$$

Proposition 3.8 *Let* $s > 0$ *and* $1 \le p, q \le \infty$. *Assume that* $F \in W^{[s]+3,\infty}_{loc}(\mathbb{R})$ *with* $F(0) = 0$. *Then for any* $f \in L^\infty_T(L^\infty) \cap \widetilde{L}^q_T(\dot{B}^s_{p,1}(\omega))$, *we have*

$$\|F(f)\|_{\widetilde{L}^q_T(\dot{B}^s_{p,1}(\omega))} \le C(1 + \|f\|_{L^\infty_T(L^\infty)})^{[s]+2}\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,1}(\omega))}.$$

Proof. We decompose $F(f)$ as

$$\begin{aligned} F(f) = \sum_{j'\in\mathbb{Z}} F(S_{j'+1}f) - F(S_{j'}f) &= \sum_{j'\in\mathbb{Z}} \Delta_{j'}f \int_0^1 F'(S_{j'}f + \tau\Delta_{j'}f)d\tau \\ &\triangleq \sum_{j'\in\mathbb{Z}} \Delta_{j'}f\, m_{j'}(f), \end{aligned}$$

where $m_{j'}(f) = \int_0^1 F'(S_{j'}f + \tau\Delta_{j'}f)d\tau$. Furthermore, we write

$$\Delta_j F(f) = \sum_{j'<j} \Delta_j(\Delta_{j'}f\, m_{j'}(f)) + \sum_{j'\ge j} \Delta_j(\Delta_{j'}f\, m_{j'}(f)) \triangleq I + II.$$

By Lemma 2.3, we have

$$\begin{aligned}\|I\|_{L^q_T(L^p)} &\leq \sum_{j'<j} \|\Delta_j(\Delta_{j'} f\, m_{j'}(f))\|_{L^q_T(L^p)} \\ &\leq \sum_{j'<j} 2^{-j|\alpha|} \sup_{|\gamma|=|\alpha|} \|D^\gamma \Delta_j(\Delta_{j'} f\, m_{j'}(f))\|_{L^q_T(L^p)},\end{aligned} \tag{3.3}$$

with α to be determined later. Notice that for $|\gamma| \geq 0$, we have

$$\|D^\gamma m_{j'}(f)\|_\infty \leq C 2^{j'|\gamma|}(1+\|f\|_\infty)^{|\gamma|}\|F'\|_{W^{|\gamma|,\infty}},$$

from which and (3.3), it follows that

$$2^{js}\|I\|_{L^q_T(L^p)} \leq C 2^{j(s-|\alpha|)} \sum_{j'<j} 2^{j'|\alpha|}\|\Delta_{j'} f\|_{L^q_T(L^p)}(1+\|f\|_{L^\infty_T(L^\infty)})^{|\alpha|}\|F'\|_{W^{|\alpha|,\infty}},$$

thus, if we take $|\alpha| = [s]+2$, we get by (3.2) that

$$\begin{aligned}\sum_j \omega_j(T) 2^{js}\|I\|_{L^q_T(L^p)} &\leq \\ &\leq C \sum_{j'} 2^{j's}\omega_{j'}(T)\|\Delta_{j'} f\|_{L^q_T(L^p)} \times \\ &\quad \times \sum_{j>j'} 2^{(j-j')(s-|\alpha|+1)}(1+\|f\|_{L^\infty_T(L^\infty)})^{|\alpha|}\|F'\|_{W^{|\alpha|,\infty}} \\ &\leq C\,(1+\|f\|_{L^\infty_T(L^\infty)})^{[s]+2}\|F'\|_{W^{[s]+2,\infty}}\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,1}(\omega))}.\end{aligned} \tag{3.4}$$

Now, let us turn to the estimate of II. We get by Lemma 2.3 that

$$\|II\|_{L^q_T(L^p)} \leq C \sum_{j'\geq j} \|\Delta_{j'} f\|_{L^q_T(L^p)}.$$

Then we write

$$\begin{aligned}\sum_j \omega_j(T) 2^{js}\|II\|_{L^q_T(L^p)} &\leq C \sum_j 2^{js} \sum_{j'\geq j} \|\Delta_{j'} f\|_{L^q_T(L^p)} \sum_{j'\geq \ell\geq j} 2^{j-\ell} e_\ell(T) \\ &\quad + C \sum_j 2^{js} \sum_{j'\geq j} \|\Delta_{j'} f\|_{L^q_T(L^p)} \sum_{\ell\geq j,j'} 2^{j-\ell} e_\ell(T),\end{aligned}$$

from which and a similar argument of (c) in Lemma 3.4, we infer that

$$\sum_j \omega_j(T) 2^{js}\|II\|_{L^q_T(L^p)} \leq C\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,1}(\omega))},$$

from which and (3.4), we conclude the proof of Proposition 3.8. ■

4. Estimates of the linear transport equation

In this section, we study the linear transport equation

$$\begin{cases} \partial_t f + v\cdot\nabla f = g, \\ f(0,x) = f_0. \end{cases} \tag{4.1}$$

Proposition 4.1 [14] *Let* $s \in (-N\min(\frac{1}{p},\frac{1}{p'}), 1+\frac{N}{p})$, $1 \le p, r \le +\infty$, *and* $s = 1+\frac{N}{p}$, *if* $r = 1$. *Let* v *be a vector field such that* $\nabla v \in L^1_T(\dot B^{\frac{N}{p}}_{p,r}\cap L^\infty)$. *Assume that* $f_0 \in \dot B^s_{p,r}$, $g \in L^1_T(\dot B^s_{p,r})$ *and* f *is the solution of* (4.1). *Then there holds for* $t\in[0,T]$,

$$\|f\|_{\widetilde L^\infty_t(\dot B^s_{p,r})} \le e^{CV(t)}\bigg(\|f_0\|_{\dot B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|g(\tau)\|_{\dot B^s_{p,r}}d\tau\bigg),$$

where $V(t) \overset{\text{def}}{=} \int_0^t \|\nabla v(\tau)\|_{\dot B^{\frac{N}{p}}_{p,r}\cap L^\infty}d\tau$. *If* $r<+\infty$, *then* f *belongs to* $C([0,T];\dot B^s_{p,r})$.

Proposition 4.2 *Let* $p\in[1,+\infty]$ *and* $s \in (-N\min(\frac{1}{p},\frac{1}{p'}), \frac{N}{p}]$. *Let* v *be a vector field such that* $\nabla v \in L^1_T(\dot B^{\frac{N}{p}}_{p,1})$. *Assume that* $f_0\in \dot B^s_{p,1}$, $g \in L^1_T(\dot B^s_{p,1})$ *and* f *is the solution of* (4.1). *Then there holds for* $t\in[0,T]$,

$$\|f\|_{\widetilde L^\infty_t(\dot B^s_{p,1}(\omega))} \le e^{CV(t)}\bigg(\|f_0\|_{\dot B^s_{p,1}(\omega)} + \int_0^t e^{-CV(\tau)}\|g(\tau)\|_{\dot B^s_{p,1}(\omega)}d\tau\bigg),$$

where $V(t) \overset{\text{def}}{=} \int_0^t \|\nabla v(\tau)\|_{\dot B^{\frac{N}{p}}_{p,1}}d\tau$.

Proof. Applying the operator Δ_j to the transport equation, we obtain

$$\partial_t\Delta_j f + v\cdot\nabla\Delta_j f = \Delta_j g + [v,\Delta_j]\cdot\nabla f. \tag{4.2}$$

Assume that $p<+\infty$. Multiplying both sides of (4.2) by $|\Delta_j f|^{p-2}\Delta_j f$, we get by integrating by parts over $\mathbb{R}^N$ for the resulting equation that

$$\frac{1}{p}\frac{d}{dt}\|\Delta_j f\|_p^p - \frac{1}{p}\int_{\mathbb{R}^N}|\Delta_j f|^p \mathrm{div} v dx \le \big(\|\Delta_j g\|_p + \|[v,\Delta_j]\cdot\nabla f\|_p\big)\|\Delta_j f\|_p^{p-1},$$

then we have

$$\|\Delta_j f(t)\|_p \le \|\Delta_j f_0\|_p + \int_0^t \big(\|\Delta_j g\|_p + \|[v,\Delta_j]\cdot\nabla f\|_p + \frac{1}{p}\|\mathrm{div} v\|_\infty\|\Delta_j f\|_p\big)d\tau,$$

from which, it follows that

$$\|f\|_{\widetilde{L}^\infty_t(\dot{B}^s_{p,1}(\omega))} \le \|f_0\|_{\dot{B}^s_{p,1}(\omega)} + C\int_0^t \|\mathrm{div} v(\tau)\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \|f(\tau)\|_{\widetilde{L}^\infty_\tau(\dot{B}^s_{p,1}(\omega))} d\tau$$
$$+\int_0^t \|g(\tau)\|_{\dot{B}^s_{p,1}(\omega)} d\tau + \int_0^t \sum_j \omega_j(T) 2^{js} \|[v,\Delta_j]\cdot\nabla f(\tau)\|_p \, d\tau,$$

from which and Lemma 4.3, we infer that

$$\|f\|_{\widetilde{L}^\infty_t(\dot{B}^s_{p,1}(\omega))} \le \|f_0\|_{\dot{B}^s_{p,1}(\omega)} + C\int_0^t \|v(\tau)\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|f(\tau)\|_{\widetilde{L}^\infty_\tau(\dot{B}^s_{p,1}(\omega))} d\tau$$
$$+\int_0^t \|g(\tau)\|_{\dot{B}^s_{p,1}(\omega)} d\tau.$$

Then Gronwall's lemma applied implies the desired inequality. ∎

Lemma 4.3 *Let* $p\in[1,\infty]$, $s\in(-N\min(\frac1p,\frac1{p'}),\frac Np]$. *Assume that* $v\in\dot{B}^{\frac Np+1}_{p,1}$ *and* $f\in\dot{B}^s_{p,1}(\omega)$. *Then there holds*

$$\sum_j \omega_j(T)2^{js}\|[v,\Delta_j]\cdot\nabla f\|_p \le C\|v\|_{\dot{B}^{\frac Np+1}_{p,1}}\|f\|_{\dot{B}^s_{p,1}(\omega)}.$$

Proof. Using the Bony's decomposition, we write

$$\begin{aligned}[v,\Delta_j]\cdot\nabla f &= [T_{v^k},\Delta_j]\partial_k f + T_{\partial_k\Delta_j f}v^k + R(v^k,\partial_k\Delta_j f)\\ &\quad -\Delta_j(T_{\partial_k f}v^k) - \Delta_j R(v^k,\partial_k f).\end{aligned}$$

Using Lemma 3.4 with $s_1 = s-1$ and $s_2=\frac Np+1$, we get

$$\sum_j \omega_j(T)2^{js}\|\Delta_j(T_{\partial_k f}v^k)\|_p \le C\|v\|_{\dot{B}^{\frac Np+1}_{p,1}}\|f\|_{\dot{B}^s_{p,1}(\omega)},$$
$$\sum_j \omega_j(T)2^{js}\|\Delta_j R(v^k,\partial_k f)\|_p \le C\|v\|_{\dot{B}^{\frac Np+1}_{p,1}}\|f\|_{\dot{B}^s_{p,1}(\omega)}.$$

Notice that

$$T'_{\partial_k\Delta_j f}v^k \triangleq T_{\partial_k\Delta_j f}v^k + R(v^k,\partial_k\Delta_j f) = \sum_{j'\ge j-2} S_{j'+2}\Delta_j\partial_k f\,\Delta_{j'}v^k,$$

then we get by Lemma 2.3 that

$$\begin{aligned}\sum_j \omega_j(T)2^{js}\|T'_{\partial_k\Delta_j f}v^k\|_p &\le C\sum_j \omega_j(T)2^{js}\|\Delta_j\nabla f\|_\infty \sum_{j'\ge j-2}\|\Delta_{j'}v^k\|_p\\ &\le C\sum_j \omega_j(T)2^{j(s+1+\frac Np)}\|\Delta_j f\|_p\sum_{j'\ge j}\|\Delta_{j'}v^k\|_p\\ &\le C\|v\|_{\dot{B}^{\frac Np+1}_{p,1}}\|f\|_{\dot{B}^s_{p,1}(\omega)}.\end{aligned}$$

Now, we turn to estimate $[T_{v^k},\Delta_j]\partial_k f$. Set $h(x)=(\mathcal{F}^{-1}\varphi)(x)$, we get by using Taylor's formula that

$$\begin{aligned}
[T_{v^k},\Delta_j]\partial_k f &= \sum_{|j'-j|\le 4}[S_{j'-1}v^k,\Delta_j]\partial_k\Delta_{j'}f\\
&=\sum_{|j'-j|\le 4}2^{Nj}\int_{\mathbb{R}^N}h(2^j(x-y))(S_{j'-1}v^k(x)-S_{j'-1}v^k(y))\partial_k\Delta_{j'}f(y)dy\\
&=\sum_{|j'-j|\le 4}2^{(N+1)j}\int_{\mathbb{R}^N}\int_0^1 y\cdot\nabla S_{j'-1}v^k(x-\tau y)d\tau\partial_k h(2^j y)\Delta_{j'}f(x-y)dy\\
&\quad+2^{Nj}\int_{\mathbb{R}^N}h(2^j(x-y))\partial_k S_{j'-1}v^k(y)\Delta_{j'}f(y)dy,
\end{aligned}$$

from which and the Minkowski inequality, we infer that

$$\begin{aligned}
\sum_j\omega_j(T)2^{js}\|[T_{v^k},\Delta_j]\partial_k f\|_p &\le C\sum_j\omega_j(T)2^{js}\sum_{|j'-j|\le 4}\|\nabla S_{j'-1}v\|_\infty\|\Delta_{j'}f\|_p\\
&\le C\|v\|_{\dot B^{\frac{N}{p}+1}_{p,1}}\|f\|_{\dot B^s_{p,1}(\omega)}.
\end{aligned}$$

Summing up the above estimates, we conclude the proof of Lemma 4.3. ■

5. Estimates of the linearized momentum equation

In this section, we study the linearized momentum equation

$$\left\{\begin{array}{l}\partial_t u-\mathrm{div}(\overline{\mu}\nabla u)-\nabla((\overline{\lambda}+\overline{\mu})\mathrm{div}\,u)=G,\\ u|_{t=0}=u_0.\end{array}\right. \tag{5.1}$$

In what follows, we assume that the viscosity coefficients $\overline{\lambda}(\rho)$ and $\overline{\mu}(\rho)$ depend smoothly on the function ρ and there exists a positive constant c_1 such that

$$\overline{\mu}\ge c_1,\quad \overline{\lambda}+2\,\overline{\mu}\ge c_1.$$

Fix a positive constant c to be chosen later. In this section, the weighted function $\omega_k(t)$ is given by

$$\omega_k(t)=\sum_{\ell\ge k}2^{k-\ell}e_\ell(t),$$

with $e_\ell(t)=(1-e^{-c2^{2\ell}t})^{\frac12}$. It is easy to verify that the function $e_\ell(t)$ satisfies (3.1).

Proposition 5.1 *Let $q \in [1,\infty]$. Assume that $G \in L^1_T(\dot{B}^{s-1}_{p,1}), u_0 \in \dot{B}^{s-1}_{p,1}$, and $\rho - \underline{\rho} \in L^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})$. Let u be a solution of (5.1). Then there hold*

(a) *If $p \in (1,N]$, $s \in (-N\min(\frac{1}{p},\frac{1}{p'})+1,\frac{N}{p}]$, we have*

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^{s-1+2/q}_{p,1})} \leq \leq C\Big(\|u_0\|_{\dot{B}^{s-1}_{p,1}} + \|G(\tau)\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1})} + A(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}\Big);$$

In addition, if $\rho-\underline{\rho} \in L^\infty_T(\dot{B}^{\frac{N}{p}+1}_{p,1})$, $p \in (1,\infty)$, $s \in (-N\min(\frac{1}{p},\frac{1}{p'})+1,\frac{N}{p}+1]$, then

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^{s-1+2/q}_{p,1})} \leq \leq C\Big(\|u_0\|_{\dot{B}^{s-1}_{p,1}} + \|G(\tau)\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1})} + A(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s}_{p,1})}\Big);$$

(b) *If $p \in (1,N]$, $s \in (-N\min(\frac{1}{p},\frac{1}{p'})+1,\frac{N}{p}]$, we have*

$$\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})} + \|u\|_{\widetilde{L}^2_T(\dot{B}^{s}_{p,1})} \leq \leq C\Big(\|u_0\|_{\dot{B}^{s-1}_{p,1}(\omega))} + \|G(\tau)\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1}(\omega))} + A(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}\Big).$$

Here $A(T) \stackrel{\text{def}}{=} \big(1+\|\rho\|_{L^\infty_T(L^\infty)}\big)^{[\frac{N}{p}]+2}$.

Proof. Set $d = \mathrm{div}u$ and $w = \mathrm{curl}u$. From (5.1), we find that (d,w) satisfies

$$\begin{cases} \partial_t d - \mathrm{div}(\overline{\nu}\nabla d) = \mathrm{div}G + F_1, \\ \partial_t w - \mathrm{div}(\overline{\mu}\nabla w) = \mathrm{curl}G + F_2, \\ (d,w)|_{t=0} = (\mathrm{div}u_0,\ \mathrm{curl}u_0) \triangleq (d_0,w_0), \end{cases} \tag{5.2}$$

where $\overline{\nu} = \overline{\lambda} + 2\overline{\mu}$ and

$$F_1 = \mathrm{div}(\nabla\overline{\mu}\cdot\nabla u) + \mathrm{div}(\nabla(\overline{\lambda}+\overline{\mu})d),$$
$$F_2^{i,j} = \mathrm{div}(\partial_j\overline{\mu}\nabla u^i - \partial_i\overline{\mu}\nabla u^j), \quad i,j = 1,\dots,N.$$

Applying the operator Δ_j to (5.2), we obtain

$$\begin{cases} \partial_t\Delta_j d - \mathrm{div}(\overline{\nu}\nabla\Delta_j d) = \mathrm{div}\Delta_j G + \Delta_j F_1 + \mathrm{div}[\Delta_j,\overline{\nu}]\nabla d, \\ \partial_t\Delta_j w - \mathrm{div}(\overline{\mu}\nabla\Delta_j w) = \mathrm{curl}\Delta_j G + \Delta_j F_2 + \mathrm{div}[\Delta_j,\overline{\mu}]\nabla w. \end{cases}$$

Multiplying the first equation by $|\Delta_j d|^{p-2}\Delta_j d$, we get by integrating over $\mathbb{R}^N$ that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\Delta_j d\|_p^p - \int_{\mathbb{R}^N} \mathrm{div}(\overline{\nu}\nabla\Delta_j d)|\Delta_j d|^{p-2}\Delta_j d \,\mathrm{d}\,x = \\ = \int_{\mathbb{R}^N}\big(\mathrm{div}\Delta_j G + \Delta_j F_1 + \mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\big)|\Delta_j d|^{p-2}\Delta_j d\,\mathrm{d}\,x$$

Lemma 2.4 ensures there exists a positive constant c_p depending on c_0, p, N such that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\Delta_j d\|_p^p + c_p 2^{2j}\|\Delta_j d\|_p^p \leq \\ \leq \|\Delta_j d\|_p^{p-1}\big(\|\mathrm{div}\Delta_j G\|_p + \|\Delta_j F_1\|_p + \|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_p\big).$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j d\|_p + c_p 2^{2j}\|\Delta_j d\|_p \leq \|\mathrm{div}\Delta_j G\|_p + \|\Delta_j F_1\|_p + \|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_p\,,$$

which implies that

$$\|\Delta_j d(t)\|_p \leq e^{-c_p 2^{2j}t}\|\Delta_j d_0\|_p + \int_0^t e^{-c_p 2^{2j}(t-\tau)}\big(\|\mathrm{div}\Delta_j G\|_p + \|\Delta_j F_1\|_p \\ + \|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_p\big)\,\mathrm{d}\,\tau.$$

Similarly, we can obtain

$$\|\Delta_j w(t)\|_p \leq e^{-c_p 2^{2j}t}\|\Delta_j w_0\|_p + \int_0^t e^{-c_p 2^{2j}(t-\tau)}\Big(\|\,\mathrm{curl}\Delta_j G\|_p \\ + \|\Delta_j F_2\|_p + \|\mathrm{div}[\Delta_j,\overline{\mu}]\nabla w\|_p\Big)\,\mathrm{d}\,\tau.$$

From the above two inequalities, we infer that for any $q \in [1,\infty]$ and $t \in [0,T]$,

(5.3)

$$\|\Delta_j d(t)\|_{L_t^q(L^p)} + \|\Delta_j w(t)\|_{L_t^q(L^p)} \leq \\ \leq C2^{-2j/q}c_j(T)^{\frac{1}{q}}(\|\Delta_j d_0\|_p + \|\Delta_j w_0\|_p) \\ + C2^{-2j/q}c_j(T)^{\frac{1}{q}}\big(\|\mathrm{div}\Delta_j G\|_{L_t^1(L^p)} + \|\Delta_j F_1\|_{L_t^1(L^p)} + \|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_{L_t^1(L^p)}\big) \\ + C2^{-2j/q}c_j(T)^{\frac{1}{q}}\big(\|\mathrm{curl}\Delta_j G\|_{L_t^1(L^p)} + \|\Delta_j F_2\|_{L_t^1(L^p)} + \|\mathrm{div}[\Delta_j,\overline{\mu}]\nabla w\|_{L_t^1(L^p)}\big),$$

with $c_j(T) = 1 - e^{c_p 2^{2j}T}$. Notice that

$$2^j\|\Delta_j u\|_p \sim \|\Delta_j d\|_p + \|\Delta_j w\|_p, \quad e_j(T) \leq \omega_j(T),$$

which together with (5.3) implies that

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^{s-1+2/q}_{p,1})} \le C\Big(\|u_0\|_{\dot{B}^{s-1}_{p,1}} + \|G\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1})} + \|(F_1,F_2)\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1})}\Big)$$
$$+ C\sum_j 2^{j(s-2)}\big(\|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_{L^1_T(L^p)} + \|\mathrm{div}[\Delta_j,\overline{\mu}]\nabla w\|_{L^1_T(L^p)}\big), \tag{5.4}$$

and with $c = c_p$ in the definition of $e_k(t)$,

$$\|u\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1})} + \|u\|_{\widetilde{L}^2_T(\dot{B}^{s}_{p,1})} \le$$
$$\le C\big(\|u_0\|_{\dot{B}^{s-1}_{p,1}(\omega)} + \|G\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1}(\omega))} + \|(F_1,F_2)\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1}(\omega))}\big)$$
$$+ C\sum_j 2^{j(s-2)}\omega_j(T)\big(\|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_{L^1_T(L^p)} + \|\mathrm{div}[\Delta_j,\overline{\mu}]\nabla w\|_{L^1_T(L^p)}\big). \tag{5.5}$$

First of all, we deal with the right hand side of (5.4). From Lemma 2.6 and 2.9, we infer that

$$\begin{aligned}\|F_1\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1})} &\le C\big(\|\nabla\overline{\mu}\cdot\nabla u\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1})} + \|\nabla(\overline{\lambda}+\overline{\mu})d\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1})}\big)\\ &\le C\big(\|\overline{\mu}-\overline{\mu}(\underline{\rho})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} + \|\overline{\lambda}-\overline{\lambda}(\underline{\rho})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\big)\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}\\ &\le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.\end{aligned} \tag{5.6}$$

Similarly, we have

$$\|F_2\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1})} \le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}. \tag{5.7}$$

While, we write

$$[\Delta_j,\overline{\nu}]\nabla d = [\Delta_j,\overline{\nu}-\overline{\nu}(\underline{\rho})]\nabla d = \Delta_j((\overline{\nu}-\overline{\nu}(\underline{\rho}))\nabla d) - (\overline{\nu}-\overline{\nu}(\underline{\rho}))\Delta_j\nabla d,$$

then by Lemma 2.6, Lemma 2.9 we get for $p\in[1,N]$

$$\sum_j 2^{j(s-2)}\|\mathrm{div}\Delta_j((\overline{\nu}-\overline{\nu}(\underline{\rho}))\nabla d)\|_{L^1_T(L^p)} \le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})},$$

$$\sum_j 2^{j(s-2)}\Big(\|\overline{\nu}-\overline{\nu}(\underline{\rho})\|_{L^\infty_T(L^\infty)}\|\Delta_j\mathrm{div}\nabla d\|_{L^1_T(L^p)}$$
$$+ \|\mathrm{div}(\overline{\nu}-\overline{\nu}(\underline{\rho}))\|_{L^\infty_T(L^N)}\|\Delta_j\nabla d\|_{L^1_T(L^{\frac{pN}{N-p}})}\Big) \le$$
$$\le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})},$$

which imply that

$$\sum_j 2^{j(s-2)}\|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_{L^1_T(L^p)} \le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}. \tag{5.8}$$

Similarly, we have

$$\sum_j 2^{j(s-2)}\|\mathrm{div}[\Delta_j,\overline{\mu}]\nabla w\|_{L^1_T(L^p)} \le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}. \tag{5.9}$$

Then the first inequality of Proposition 5.1 (a) can be deduced from (5.4) and (5.6)–(5.9). On the other hand, using Lemma 2.6 and 2.9, we also have

$$\|F_1\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1})}+\|F_2\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1})}\le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s}_{p,1})},$$

and by Lemma 2.8,

$$\sum_j 2^{j(s-2)}\big(\|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_{L^1_T(L^p)}+\|\mathrm{div}[\Delta_j,\overline{\mu}]\nabla w\|_{L^1_T(L^p)}\big)\le$$
$$\le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s}_{p,1})},$$

which together with (5.4) lead to the second inequality of Proposition 5.1 (a).

Next, we deal with the right hand side of (5.5). From Proposition 3.6 and 3.8, it follows that

$$\begin{aligned}\|F_1\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1}(\omega))}&\le C\big(\|\nabla\overline{\mu}\cdot\nabla u\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1}(\omega))}+\|\nabla(\overline{\lambda}+\overline{\mu})d\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{p,1}(\omega))}\big)\\&\le C\big(\|\overline{\mu}-\overline{\mu}(\underline{\rho})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}+\|\overline{\lambda}-\overline{\lambda}(\underline{\rho})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\big)\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}\\&\le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.\end{aligned}\tag{5.10}$$

Similarly, we have

$$\|F_2\|_{\widetilde{L}^1_T(\dot{B}^{s-2}_{p,1}(\omega))}\le CA(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.\tag{5.11}$$

Notice that

$$[\Delta_j,\overline{\nu}]\nabla d=[\Delta_j,\overline{\nu}-\overline{\nu}(\underline{\rho})]\nabla d,\quad [\Delta_j,\overline{\mu}]\nabla w=[\Delta_j,\overline{\mu}-\overline{\mu}(\underline{\rho})]\nabla w,$$

which together with Lemma 5.2 and Proposition 3.8 ensures that

$$\begin{aligned}\sum_j 2^{j(s-2)}\omega_j(T)\big(\|\mathrm{div}[\Delta_j,\overline{\nu}]\nabla d\|_{L^1_T(L^p)}+\|\mathrm{div}[\Delta_j,\overline{\mu}]\nabla w\|_{L^1_T(L^p)}\big)\le&\\ \le C\big(\|\overline{\nu}-\overline{\nu}(\underline{\rho})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}+\|\overline{\mu}-\overline{\mu}(\underline{\rho})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\big)\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}&\\ \le A(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|u\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.&\end{aligned}\tag{5.12}$$

Summing up (5.5) and (5.10)-(5.12), we obtain the inequality of Proposition 5.1 (b). ■

Lemma 5.2 *Let $p \in [1, N]$ and $s \in (-N\min(\frac{1}{p}, \frac{1}{p'}), \frac{N}{p}]$. Assume that $f \in \widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))$ and $g \in \widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})$. Then there holds*

$$\sum_j 2^{j(s-1)}\omega_j(T)\|\mathrm{div}[\Delta_j, f]\nabla g\|_{L^1_T(L^p)} \le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|g\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.$$

Proof. Using the Bony's decomposition, we write

$$\begin{aligned}[f, \Delta_j]\partial_k g &= [T_f, \Delta_j]\partial_k g + T_{\partial_k\Delta_j g} f + R(f, \partial_k\Delta_j g)\\ &\quad -\Delta_j(T_{\partial_k g} f) - \Delta_j R(f, \partial_k g).\end{aligned}$$

Using Lemma 3.5 (a) and (c) with $s_1 = \frac{N}{p}$ and $s_2 = s$, we get

$$\sum_j \omega_j(T)2^{j(s-1)}\|\mathrm{div}\Delta_j(T_{\partial_k g} f)\|_{L^1_T(L^p)} \le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|g\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})},$$
$$\sum_j \omega_j(T)2^{j(s-1)}\|\mathrm{div}\Delta_j R(f, \partial_k g)\|_{L^1_T(L^p)} \le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|g\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.$$

Thanks to the proof of Lemma 4.3, we have

$$T'_{\partial_k\Delta_j g} f \triangleq T_{\partial_k\Delta_j g} f + R(f, \partial_k\Delta_j g) = \sum_{j' \ge j-2} S_{j'+2}\Delta_j\partial_k g\Delta_{j'} f,$$

then we get by Lemma 2.3 and (3.2) that

$$\begin{aligned}&\sum_j \omega_j(T)2^{j(s-1)}\|\mathrm{div}T'_{\partial_k\Delta_j g} f\|_{L^1_T(L^p)}\\ &\quad\le C\sum_j \omega_j(T)2^{js}\Big(\|\Delta_j\nabla g\|_{L^1_T(L^\infty)}\sum_{j'\ge j-2}\|\Delta_{j'} f\|_{L^1_T(L^p)}\\ &\qquad\qquad + \|\Delta_j g\|_{L^1_T(L^\infty)}\sum_{j'\ge j-2} 2^{j'}\|\Delta_{j'} f\|_{L^1_T(L^p)}\Big)\\ &\quad\le C\sum_j 2^{j(s+\frac{N}{p})}\|\Delta_j g\|_{L^1_T(L^p)}\sum_{j'\ge j-2}(2^j + 2^{j'})\omega_{j'}(T)\|\Delta_{j'} f\|_{L^1_T(L^p)}\\ &\quad\le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|g\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.\end{aligned}$$

Set $h(x) = (\mathcal{F}^{-1}\varphi)(x)$. Thanks to the proof of Lemma 4.3, we have

$$\begin{aligned}[T_f, \Delta_j]\partial_k g = \sum_{|j'-j|\le 4} 2^{(N+1)j}\int_{\mathbb{R}^N}\int_0^1 y\cdot\nabla S_{j'-1} f(x-\tau y)d\tau\partial_k h(2^j y)\Delta_{j'} g(x-y)dy\\ + 2^{Nj}\int_{\mathbb{R}^N} h(2^j(x-y))\partial_k S_{j'-1} f(y)\Delta_{j'} g(y)dy,\end{aligned}$$

from which and a similar argument of Lemma 3.5 (b), we infer that

$$\sum_j \omega_j(T) 2^{j(s-1)} \|\mathrm{div}[T_f, \Delta_j]\partial_k g\|_{L^1_T(L^p)} \le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \|g\|_{\widetilde{L}^1_T(\dot{B}^{s+1}_{p,1})}.$$

Summing up the above estimates, we conclude the proof of Lemma 5.2. ■

To prove the uniqueness of the solution, we also need the following proposition.

Proposition 5.3 *Let $p \in [2, N]$. Assume that $G \in L^1_T(\dot{B}^{-\frac{N}{p}}_{p,\infty})$, $u_0 \in \dot{B}^{-\frac{N}{p}}_{p,\infty}$, and $\rho - \underline{\rho} \in L^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})$. Let u be a solution of* (5.1). *Then there holds*

$$\|u\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}+2}_{p,\infty})} + \|u\|_{\widetilde{L}^2_T(\dot{B}^{-\frac{N}{p}+1}_{p,\infty})} \le$$
$$\le C\Big(\|u_0\|_{\dot{B}^{-\frac{N}{p}}_{p,\infty}} + \|G(\tau)\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}}_{p,\infty}(\omega))} + A(T)\|\rho-\underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \|u\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}+2}_{p,\infty})}\Big).$$

Proof. We closely follow the proof of Proposition 5.1. From (5.3), we infer that

(5.13)
$$\|u\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}+2}_{p,\infty})} + \|u\|_{\widetilde{L}^2_T(\dot{B}^{-\frac{N}{p}+1}_{p,\infty})} \le$$
$$\le C\Big(\|u_0\|_{\dot{B}^{-\frac{N}{p}}_{p,\infty}(\omega)} + \|G\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}}_{p,\infty}(\omega))} + \|(F_1, F_2)\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}-1}_{p,\infty}(\omega))}\Big)$$
$$+ C\sup_{j\in\mathbb{Z}} 2^{j(-\frac{N}{p}-1)}\omega_j(T)\big(\|\mathrm{div}[\Delta_j, \overline{\nu}]\nabla d\|_{L^1_T(L^p)} + \|\mathrm{div}[\Delta_j, \overline{\mu}]\nabla w\|_{L^1_T(L^p)}\big).$$

We use Proposition 3.7 to get

$$\|(F_1, F_2)\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}-1}_{p,\infty}(\omega))} \le CA(T)\|\rho - \underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \|u\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}+2}_{p,\infty})}.$$

From Lemma 5.4, the second term on the right hand side of (5.13) is bounded by

$$CA(T)\|\rho - \underline{\rho}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \|u\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}+2}_{p,\infty})}.$$

This completes the proof of Proposition 5.3. ■

Lemma 5.4 *Let $p \in [1,N]$. Assume that $f \in \widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))$ and $g \in \widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}+1}_{p,\infty})$. Then there holds*

$$\sup_{j\in\mathbb{Z}} 2^{j(-\frac{N}{p}-1)}\omega_j(T)\|\mathrm{div}[\Delta_j, f]\nabla g\|_{L^1_T(L^p)} \le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \|g\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{N}{p}+1}_{p,\infty})}.$$

The proof of Lemma 5.4 is very similar to that of Lemma 5.2. Here we omit its proof.

6. The proof of existence

We set

$$a(t,x)=\frac{\rho(t,x)-\overline{\rho}_0}{\overline{\rho}_0},\quad \overline{\mu}(\rho)=\frac{\mu(\rho)}{\rho},\quad \overline{\lambda}(\rho)=\frac{\lambda(\rho)}{\rho}.$$

Then the system (1.1) reads

$$\begin{cases}\partial_t a+u\cdot\nabla a=F,\\ \partial_t u-\mathrm{div}(\overline{\mu}\nabla u)-\nabla((\overline{\lambda}+\overline{\mu})\mathrm{div}\,u)=G,\\ (a,u)|_{t=0}=(a_0,u_0),\end{cases}\tag{6.1}$$

with $a_0=\frac{\rho_0(x)-\overline{\rho}_0}{\overline{\rho}_0}$ and

$$F(a,u)=-(1+a)\mathrm{div}\,u,$$
$$G(a,u)=-u\cdot\nabla u+\frac{\overline{\rho}_0P'(\rho)}{\rho}\nabla a+\frac{\mu(\rho)}{\rho^2}\nabla\rho\cdot\nabla u+\frac{\mu(\rho)+\lambda(\rho)}{\rho^2}\nabla\rho\mathrm{div}\,u.$$

Step 1. *The approximate solution sequence.*

We smooth out the data as follows:

$$a_0^n=S_{n+K}\,a_0,\quad u_0^n=S_n\,u_0,$$

where $K\in\mathbb{Z}$ is chosen such that

$$\overline{\rho}_0(1+a_0^n(x))\geq\frac{3}{4}\,c_0.\tag{6.2}$$

A standard linearized argument (as in the proof of Theorem 4.2 in [12]) will ensure that the system (6.1) with the smooth data (a_0^n,u_0^n) has a solution (a^n,u^n) on a time interval $[0,T_n]$ for some $T_n>0$ such that

$$\begin{cases}a^n\in C([0,T_n];\dot{B}^{\frac{N}{p}}_{p,1}\cap\dot{B}^{\frac{N}{p}+1}_{p,1})\quad\text{and}\\ u^n\in C([0,T_n];\dot{B}^{\frac{N}{p}-1}_{p,1}\cap\dot{B}^{\frac{N}{p}}_{p,1})\cap L^1([0,T_n];\dot{B}^{\frac{N}{p}+1}_{p,1}\cap\dot{B}^{\frac{N}{p}+2}_{p,1}).\end{cases}\tag{6.3}$$

In what follows, we also denote by T_n the maximal lifespan of the solution (a^n,u^n).

Step 2. *Uniform estimates.*

Let

$$E_0:=\|a_0\|_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}$$

and $T\in(0,T_n)$. We assume that the solutions (a^n,u^n) satisfy the following inequalities for some positive constants c_1,C_0,A_0 and η (to be determined later):

(H1) $\overline{\rho}_0(1+a_0^n(t,x)) \geq \frac{c_0}{2}$ for any $(t,x) \in [0,T] \times \mathbb{R}^N$;

(H2) $\overline{\mu}^n(t,x) \geq c_1, \overline{\lambda}^n(t,x) + 2\overline{\mu}^n(t,x) \geq c_1$ for any $(t,x) \in [0,T] \times \mathbb{R}^N$;

(H3) $\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} + \|u^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \leq C_0 E_0$;

(H4) $\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \leq A_0\eta$, $\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})} + \|u^n\|_{\widetilde{L}^2_T(\dot{B}^{\frac{N}{p}}_{p,1})} \leq \eta$.

In what follows, we will show that if the conditions (H1) to (H4) are satisfied for some $T > 0$, then they are actually satisfied with strict inequalities. Since all those conditions depend continuously on the time variable and are satisfied initially, a standard bootstrap argument will ensure that (H1) to (H4) are indeed satisfied for T.

First of all, we get by Proposition 4.1 that

$$\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \leq e^{CV^n(T)}\big(\|a_0\|_{\dot{B}^{\frac{N}{p}}_{p,1}} + \|F^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}}_{p,1})}\big), \tag{6.4}$$

and by Proposition 5.1, we have

$$\begin{aligned}\|u^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} &\leq C\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + C\|G^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \\ &\quad + CA^n(T)\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})},\end{aligned} \tag{6.5}$$

where

$$V^n(t) = \int_0^t \|\nabla u^n(\tau)\|_{\dot{B}^{\frac{N}{p}}_{p,1}} d\tau \quad \text{and} \quad A^n(T) = \big(1 + \|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\big)^{[\frac{N}{p}]+3}.$$

For F^n, we apply Lemma 2.6 to get

$$\|F^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}}_{p,1})} \leq \|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})} + C\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})},$$

and for G^n, we use Lemma 2.6 and 2.9 to get

$$\begin{aligned}\|G^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} &\leq C\|u^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})} \\ &\quad + CA^n(T)\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\big(T + \|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}\big).\end{aligned}$$

Plugging the above two estimates into (6.4) and (6.5), we obtain

$$\begin{aligned}&\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} + \|u^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \leq \\ &\qquad \leq C_1 e^{C_1V^n(T)}(E_0 + (C_0E_0+1)\eta) + C_1A^n(T)C_0E_0(T+\eta).\end{aligned} \tag{6.6}$$

Next, we get by Proposition 4.2 that

$$\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \le e^{CV^n(T)}\big(\|a_0\|_{\dot{B}^{\frac{N}{p}}_{p,1}(\omega)} + \|F^n\|_{\widetilde{L}^1_T\dot{B}^{\frac{N}{p}}_{p,1}(\omega)}\big), \tag{6.7}$$

and by Proposition 5.1, we have

$$\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})} + \|u^n\|_{\widetilde{L}^2_T(\dot{B}^{\frac{N}{p}}_{p,1})} \le \tag{6.8}$$
$$\le C\Big(\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}(\omega))} + \|G^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}-1}_{p,1}(\omega))} + A^n(T)\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}\Big).$$

For F^n, we have

$$\|F^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \le 2\|F^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}}_{p,1})} \le C(1+\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})})\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})},$$

and for G^n, we use Proposition 3.6 and 3.8 to get

$$\|G^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}-1}_{p,1}(\omega))} \le C\|u^n\|^2_{\widetilde{L}^2_T(\dot{B}^{\frac{N}{p}}_{p,1})} + CA^n(T)\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))}\big(T+\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}\big).$$

Plugging the above two estimates into (6.7) and (6.8), we obtain

$$\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \le e^{C_1V^n(T)}\big(\|a_0\|_{\dot{B}^{\frac{N}{p}}_{p,1}(\omega)} + C_2(1+C_0E_0)\eta\big), \tag{6.9}$$

$$\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})} + \|u^n\|_{\widetilde{L}^2_T(\dot{B}^{\frac{N}{p}}_{p,1})} \le \tag{6.10}$$
$$\le C_3\Big(\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}(\omega))} + \eta^2 + A_0A^n(T)\eta(T+\eta)\Big).$$

According to the definition of V^n and A^n, we have

$$V^n(T) \le \eta, \quad A^n(T) \le (1+C_0E_0)^{[\frac{N}{p}]+3}.$$

Let $C_0 = 4C_1$ and $A_0 = 2C_2(1+C_0E_0)$. Then we take η small enough such that

$$\begin{cases} e^{C_1\eta} < \frac{3}{2},\ (C_0E_0+1)\eta \le E_0,\ C_1(C_0E_0+1)^{[\frac{N}{p}]+3}\eta \le \frac{1}{16}, \\ C_3\eta \le \frac{1}{6},\ C_3A_0(1+C_0E_0)^{[\frac{N}{p}]+3}\eta \le \frac{1}{6}. \end{cases} \tag{6.11}$$

Next, we take T small enough such that

$$C_1(C_0E_0+1)^{[\frac{N}{p}]+3}T \le \frac{1}{16},\ C_3A_0(1+C_0E_0)^{[\frac{N}{p}]+3}T \le \frac{1}{6}. \tag{6.12}$$

and note that $\omega_k(0) = 0$ and $(a_0,u_0) \in \dot{B}^{\frac{N}{p}}_{p,1} \times \dot{B}^{\frac{N}{p}-1}_{p,1}$, we can also take T small enough such that

$$\|a_0\|_{\dot{B}^{\frac{N}{p}}_{p,1}(\omega)} \le \frac{A_0}{12}\eta, \quad \|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}(\omega)} \le \frac{\eta}{6C_3}. \tag{6.13}$$

Then it follows from (6.6) that

$$\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} + \|u^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \leq \frac{7}{8} C_0 E_0,$$

and from (6.9) and (6.10), we infer that

$$\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1}(\omega))} \leq \frac{7}{8} A_0 \eta,\ \|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})} + \|u^n\|_{\widetilde{L}^2_T(\dot{B}^{\frac{N}{p}}_{p,1})} \leq \frac{2}{3}\eta,$$

which ensure that (H3) and (H4) are satisfied with strict inequalities for T and η satisfying (6.11)-(6.13).

Let $X_n(t,x)$ be a solution of

$$\frac{d}{dt} X_n(t,x) = u^n(t, X_n(t,x)), \quad X_n(0,x) = x,$$

and we denote by $X_n^{-1}(t,x)$ the inverse of $X_n(t,x)$. Then $a^n(t,x)$ can be solved as

$$a^n(t,x) = a_0^n(X_n^{-1}(t,x)) + \int_0^t F^n(\tau, X_n(\tau, X_n^{-1}(t,x)))d\tau,$$

thus, we have

$$\overline{\rho}_0(1 + a^n(t,x)) = \rho_0^n(X_n^{-1}(t,x)) + \bar{\rho}_0 \int_0^t F^n(\tau, X_n(\tau, X_n^{-1}(t,x)))d\tau. \tag{6.14}$$

On the other hand, we have

$$\|F^n\|_{L^1_T(L^\infty)} \leq \|\nabla u\|_{L^1_T(L^\infty)}(1 + \|a\|_{L^\infty_T(L^\infty)}) \leq C_4(1 + C_0 E_0)\eta.$$

We take η such that

$$C_4(1 + C_0 E_0)\eta < \frac{1}{8} c_0.$$

Then from (6.14) and (6.2), it follows that

$$\overline{\rho}_0(1 + a^n(t,x)) \geq \frac{3}{4}\, c_0 - \frac{1}{8}\, c_0 \geq \frac{5}{8}\, c_0,$$

that is, (H1) is satisfied with the strict inequality. Finally, take

$$c_1 = \frac{1}{2} \min\Big(\inf_{|\rho| \leq \overline{\rho}_0(1 + C_0 E_0)} \overline{\mu}(\rho), \inf_{|\rho| \leq \overline{\rho}_0(1 + C_0 E_0)} (\overline{\lambda}(\rho) + 2\overline{\mu}(\rho))\Big),$$

which ensures that (H2) is satisfied with strict inequality.

Let T^* be the supremum of all time T such that (6.12) and (6.13) are satisfied. We need to prove that $T_n \geq T^*$. If $T_n < T^*$, then we can prove that

$$\begin{cases} a^n \in \widetilde{L}^\infty(0,T_n;\dot{B}^{\frac{N}{p}}_{p,1} \cap \dot{B}^{\frac{N}{p}+1}_{p,1}) \quad \text{and} \\ u^n \in \widetilde{L}^\infty(0,T_n;\dot{B}^{\frac{N}{p}-1}_{p,1} \cap \dot{B}^{\frac{N}{p}}_{p,1}) \cap L^1([0,T_n];\dot{B}^{\frac{N}{p}+1}_{p,1} \cap \dot{B}^{\frac{N}{p}+2}_{p,1}), \end{cases} \tag{6.15}$$

thus, the solution (a^n, u^n) can be continued beyond T^*. Indeed, from Proposition 4.1, we have

$$\|a^n\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}+1}_{p,1})} \leq e^{CV^n(t)}\big(\|a^n_0\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} + \int_0^t \|F^n(\tau)\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} d\tau\big), \tag{6.16}$$

and by Proposition 5.1 (a), we have

$$\begin{aligned} \|u^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} + \|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+2}_{p,1})} &\leq C\|u^n_0\|_{\dot{B}^{\frac{N}{p}}_{p,1}} + C\|G^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}}_{p,1})} \\ &\quad + CA^n(T)\|a^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}\|u^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1})}, \end{aligned} \tag{6.17}$$

On the other hand, we use Lemma 2.5 and the embedding $\dot{B}^{\frac{N}{p}}_{p,1} \hookrightarrow L^\infty$ to get

$$\|F^n\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \leq \|u^n\|_{\dot{B}^{\frac{N}{p}+2}_{p,1}} + C\|a^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|u^n\|_{\dot{B}^{\frac{N}{p}+2}_{p,1}} + C\|a^n\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}\|u^n\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}},$$

and by Lemma 2.6 and 2.9, we have

$$\begin{aligned} \|G^n\|_{\widetilde{L}^1_T(\dot{B}^{\frac{N}{p}}_{p,1})} &\leq C\|u^n\|_{\widetilde{L}^\infty_T\dot{B}^{\frac{N}{p}}_{p,1}}\|u^n\|_{\widetilde{L}^1_T\dot{B}^{\frac{N}{p}+1}_{p,1}} \\ &\quad + CA^n(T)\|a^n\|_{\widetilde{L}^\infty_T\dot{B}^{\frac{N}{p}+1}_{p,1}}\big(T + \|u^n\|_{\widetilde{L}^1_T\dot{B}^{\frac{N}{p}+1}_{p,1}}\big), \end{aligned}$$

which together with (6.16), (6.17) and (H3-H4) implies (6.15).

Step 3. *Existence of a solution.*

We will use a compact argument to prove that the approximate sequence $\{a^n, u^n\}_{n\in\mathbb{N}}$ tends to some function (a, u) which satisfies the system (6.1) in the sense of distribution.

Since $\{u^n\}$ is uniformly bounded in $L^1_T(\dot{B}^{\frac{N}{p}+1}_{p,1}) \cap L^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})$, we get by the interpolation that $\{u^n\}_{n\in\mathbb{N}}$ is also uniformly bounded in $L^q_T(\dot{B}^{\frac{N}{p}-1+2/q}_{p,1})$ for any $q \in [1,\infty]$. By Lemma 2.6, we have

$$\begin{aligned} \|a^n \mathrm{div} u^n\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} &\leq C\|a^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|u^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}, \\ \|u^n \cdot \nabla a^n\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} &\leq C\|a^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|u^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}, \end{aligned}$$

from which and the first equation of the system (6.1), we infer that $\{\partial_t a^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2_T(\dot{B}^{\frac{N}{p}-1}_{p,1})$. On the other hand, by Lemma 2.6 and Lemma 2.9, we have

$$\|u^n\cdot\nabla u^n\|_{\dot{B}^{\frac{N}{p}-3/2}_{p,1}}\le C\|u^n\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\|u^n\|_{\dot{B}^{\frac{N}{p}+1/2}_{p,1}},$$
$$\|\frac{\overline{\rho}_0 P'(\rho^n)}{\rho^n}\nabla a^n\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\le C(\|a^n\|_\infty)\|a^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}(1+\|a^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}),$$

and

$$\|\mathrm{div}(\overline{\mu}^n\nabla u^n)\|_{\dot{B}^{\frac{N}{p}-3/2}_{p,1}}+\|\nabla((\overline{\lambda}^n+\overline{\mu}^n)\mathrm{div}u^n)\|_{\dot{B}^{\frac{N}{p}-3/2}_{p,1}}+\|G_1^n\|_{\dot{B}^{\frac{N}{p}-3/2}_{p,1}}\le$$
$$\le C(\|a^n\|_\infty)\|a^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}}(1+\|a^n\|_{\dot{B}^{\frac{N}{p}}_{p,1}})\|u^n\|_{\dot{B}^{\frac{N}{p}+1/2}_{p,1}},$$

where

$$G_1^n\overset{\mathrm{def}}{=}\frac{\mu(\rho^n)}{(\rho^n)^2}\nabla\rho^n\cdot\nabla u^n+\frac{\mu(\rho^n)+\lambda(\rho^n)}{(\rho^n)^2}\nabla\rho^n\mathrm{div}\,u^n.$$

Then, from the second equation of the system (6.1), we infer that $\{\partial_t u^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^{\frac{4}{3}}_T(\dot{B}^{\frac{N}{p}-\frac{3}{2}}_{p,1}+\dot{B}^{\frac{N}{p}-1}_{p,1})$.

Let $\{\chi_j\}_{j\in\mathbb{N}}$ be a sequence of smooth functions supported in the ball $B(0,j+1)$ and equal to 1 on $B(0,j)$. The above proof ensures that for any $j\in\mathbb{N}$, $\{\chi_j a^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}([0,T];\dot{B}^{\frac{N}{p}-1}_{p,1})$, and $\{\chi_j u^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{4}}([0,T];\dot{B}^{\frac{N}{p}-\frac{3}{2}}_{p,1}+\dot{B}^{\frac{N}{p}-1}_{p,1})$. Since the embeddings

$$\dot{B}^{\frac{N}{p}-1}_{p,1}\cap\dot{B}^{\frac{N}{p}}_{p,1}\hookrightarrow\dot{B}^{\frac{N}{p}-1}_{p,1}\quad\text{and}\quad\dot{B}^{\frac{N}{p}-3/2}_{p,1}\cap\dot{B}^{\frac{N}{p}-1}_{p,1}\hookrightarrow\dot{B}^{\frac{N}{p}-3/2}_{p,1}$$

are locally compact, by applying Ascoli's theorem and Cantor's diagonal process, there exists some function (a,u) such that for any $j\in\mathbb{N}$,

$$\text{(6.18)}\qquad\begin{cases}\chi_j a^n\longrightarrow\chi_j a & \text{in}\quad C([0,T];\dot{B}^{\frac{N}{p}-1}_{p,1}),\\ \chi_j u^n\longrightarrow\chi_j u & \text{in}\quad C([0,T];\dot{B}^{\frac{N}{p}-\frac{3}{2}}_{p,1}),\end{cases}$$

as n tends to ∞(up to a subsequence).

By interpolation, we also have

$$\text{(6.19)}\qquad\begin{cases}\chi_j a^n\longrightarrow\chi_j a & \text{in}\quad C([0,T];\dot{B}^{\frac{N}{p}-s}_{p,1}),\quad\forall\,0<s\le 1,\\ \chi_j u^n\longrightarrow\chi_j u & \text{in}\quad L^1([0,T];\dot{B}^{\frac{N}{p}+s}_{p,1}),\quad\forall\,-\frac{3}{2}\le s<1.\end{cases}$$

Furthermore, we actually have

$$
\begin{cases} (a,u)\in \widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})\otimes\Big(\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})\cap L^1(0,T;\dot{B}^{\frac{N}{p}+1}_{p,1})\Big),\\ \overline{\rho}_0(1+a(t,x))\geq \frac{c_0}{2}. \end{cases} \tag{6.20}
$$

With (6.18)-(6.20), it is a routine process to verify that (a,u) satisfies the system (6.1) in the sense of distribution(see also [11]). Finally, following the argument in [11], we can show that $(a,u)\in C([0,T];\dot{B}^{\frac{N}{p}}_{p,1})\otimes C([0,T];\dot{B}^{\frac{N}{p}-1}_{p,1})$.

7. The proof of uniqueness

In this section, we prove the uniqueness of the solution. Assume that $(a^1,u^1)\in E^p_T$ and $(a^2,u^2)\in E^p_T$ are two solutions of the system (6.1) with the same initial data. Without loss of generality, we may assume that a^1 satisfies

$$
\rho^1(t,x)=\overline{\rho}_0(1+a^1(t,x))\geq\frac{c_0}{2}.
$$

for any $(t,x)\in[0,T]\times\mathbb{R}^N$. Since $a^2\in C([0,T];\dot{B}^{\frac{N}{p}}_{p,1})$ and $\rho^2(0,x)\geq c_0$, there exists a positive time $\widetilde{T}\in(0,T]$ such that

$$
\rho^2(t,x)=\overline{\rho}_0(1+a^2(t,x))\geq\frac{c_0}{2}.
$$

for any $(t,x)\in[0,\widetilde{T}]\times\mathbb{R}^N$. Set $\delta a=a^1-a^2$ and $\delta u=u^1-u^2$. Then $(\delta a,\delta u)$ satisfies

$$
\begin{cases} \partial_t\delta a+u^2\cdot\nabla\delta a=\delta F-\delta u\cdot\nabla a^1,\\ \partial_t\delta u-\mathrm{div}(\overline{\mu}^1\nabla\delta u)-\nabla((\overline{\lambda}^1+\overline{\mu}^1)\mathrm{div}\,\delta u)=\delta G+\delta H,\\ (\delta a,\delta u)|_{t=0}=(0,0), \end{cases} \tag{7.1}
$$

where

$$
\begin{aligned}
&\delta F=F(a^1,u^1)-F(a^2,u^2),\quad \delta G=G(a^1,u^1)-G(a^2,u^2),\\
&\delta H=\mathrm{div}\big((\overline{\mu}^1-\overline{\mu}^2)\nabla u^2\big)+\nabla\big((\overline{\lambda}^1-\overline{\lambda}^2+\overline{\mu}^1-\overline{\mu}^2)\mathrm{div}\,u^2\big),
\end{aligned}
$$

with $\overline{\lambda}^i=\overline{\lambda}(a^i),\overline{\mu}^i=\overline{\mu}(a^i)$ for $i=1,2$.

In what follows, we set

$$
U^i(t)=\int_0^t\|u^i(\tau)\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}d\tau\qquad\text{for } i=1,2,
$$

and denote by A_T a constant depending on $\|a^1\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}$ and $\|a^2\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}$.

Due to the inclusion relation $E_T^p \subseteq E_T^N$, it suffices to prove the uniqueness of the solution in E_T^N. So, we take $p = N$ in the sequel.

We apply Proposition 4.1 to get for any $t \in [0, T]$,

$$\|\delta a(t)\|_{\dot{B}^0_{p,\infty}} \le e^{CU^2(t)} \int_0^t \big(\|\delta F(\tau)\|_{\dot{B}^0_{p,\infty}} + \|\delta u \cdot \nabla a^1(\tau)\|_{\dot{B}^0_{p,\infty}}\big) d\tau. \tag{7.2}$$

By Lemma 2.7, we have

$$\begin{aligned}\|\delta F(\tau)\|_{\dot{B}^0_{p,\infty}} + \|\delta u \cdot \nabla a^1(\tau)\|_{\dot{B}^0_{p,\infty}} &\le \\ \le C\|u^2\|_{\dot{B}^2_{p,1}} \|\delta a\|_{\dot{B}^0_{p,\infty}} &+ C(1 + \|a^1\|_{\dot{B}^1_{p,1}}) \|\delta u\|_{\dot{B}^1_{p,1}}.\end{aligned}$$

Plugging it into (7.2), we get by Gronwall's inequality that

$$\|\delta a(t)\|_{\dot{B}^0_{p,\infty}} \le e^{CU^2(t)} \int_0^t (1 + \|a^1\|_{\dot{B}^1_{p,1}}) \|\delta u\|_{\dot{B}^1_{p,1}} d\tau. \tag{7.3}$$

We use Proposition 5.3 to get for any $t \in [0, T]$,

$$\begin{aligned}\|\delta u(t)\|_{\widetilde{L}^1_t(\dot{B}^1_{p,\infty})} + \|\delta u(t)\|_{\widetilde{L}^2_t(\dot{B}^0_{p,\infty})} &\le \\ \le C \int_0^t \big(\|\delta G(\tau)\|_{\dot{B}^{-1}_{p,\infty}(\omega)} &+ \|\delta H(\tau)\|_{\dot{B}^{-1}_{p,\infty}}\big) d\tau \\ &+ CA_T \|a^1\|_{\widetilde{L}^\infty_t(\dot{B}^1_{p,1}(\omega))} \|\delta u\|_{\widetilde{L}^1_t(\dot{B}^1_{p,\infty})}.\end{aligned} \tag{7.4}$$

From Lemma 2.7, Proposition 3.7 and 3.8, we infer that for any $t \in [0, \widetilde{T}]$,

$$\|\delta H\|_{\dot{B}^{-1}_{p,\infty}} \le A_T \|u^2\|_{\dot{B}^2_{p,1}} \|\delta a\|_{\dot{B}^0_{p,\infty}}, \tag{7.5}$$

$$\begin{aligned}\|\delta G\|_{\dot{B}^{-1}_{p,\infty}(\omega)} \le C\|(u^1, u^2)\|_{\dot{B}^1_{p,1}} \|\delta u\|_{\dot{B}^0_{p,\infty}} &+ A_T \|a^1\|_{\dot{B}^1_{p,1}(\omega)} \|\delta u\|_{\dot{B}^1_{p,\infty}} \\ &+ A_T (1 + \|u^2\|_{\dot{B}^2_{p,1}}) \|\delta a\|_{\dot{B}^0_{p,\infty}}.\end{aligned} \tag{7.6}$$

We take $\widetilde{T}$ small enough such that

$$\|(u^1, u^2)\|_{\widetilde{L}^1_t(\dot{B}^2_{p,1}) \cap \widetilde{L}^2_t(\dot{B}^1_{p,1})} + \|(a^1, a^2)\|_{\widetilde{L}^\infty_t(\dot{B}^1_{p,1}(\omega))} \ll 1.$$

Thus, plugging (7.5) and (7.6) into (7.4), we infer that for any $t \in [0, \widetilde{T}]$,

$$\|\delta u\|_{\widetilde{L}^1_t(\dot{B}^1_{p,\infty})} \le A_T \int_0^t \big(1 + \|(u^1, u^2)\|_{\dot{B}^2_{p,1}}\big) \|\delta a\|_{\dot{B}^0_{p,\infty}} d\tau. \tag{7.7}$$

Lemma 7.1 ([13]) *Let $s \in \mathbb{R}$. Then for any $1 \le p, \rho \le +\infty$ and $0 < \epsilon \le 1$, we have*

$$\|f\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,1})} \le C \frac{\|f\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,\infty})}}{\epsilon} \log\Big(e + \frac{\|f\|_{\widetilde{L}^\rho_T(\dot{B}^{s-\epsilon}_{p,\infty})} + \|f\|_{\widetilde{L}^\rho_T(\dot{B}^{s+\epsilon}_{p,\infty})}}{\|f\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,\infty})}}\Big).$$

From Lemma 7.1, it follows that

$$\|\delta u\|_{L^1_t(\dot B^1_{p,1})} \le C\|\delta u\|_{\widetilde L^1_t(\dot B^1_{p,\infty})} \log\Big(e + \frac{\|\delta u\|_{\widetilde L^1_t(\dot B^0_{p,\infty})} + \|\delta u\|_{\widetilde L^1_t(\dot B^2_{p,\infty})}}{\|\delta u\|_{\widetilde L^1_t(\dot B^1_{p,\infty})}}\Big),$$

which together with (7.3) and (7.7) yields that for any $t \in [0, \widetilde T]$,

$$\|\delta u\|_{\widetilde L^1_t(\dot B^1_{p,\infty})} \le A_T \int_0^t \big(1 + \|(u^1,u^2)\|_{\dot B^2_{p,1}}\big) \|\delta u\|_{\widetilde L^1_\tau(\dot B^1_{p,\infty})} \log\big(e + C_T \|\delta u\|^{-1}_{\widetilde L^1_\tau(\dot B^1_{p,\infty})}\big) d\tau,$$

where $C_T = \|\delta u\|_{\widetilde L^1_T(\dot B^0_{p,\infty})} + \|\delta u\|_{\widetilde L^1_T(\dot B^2_{p,\infty})}$. Notice that $1 + \|(u^1, u^2)(t)\|_{\dot B^2_{p,1}}$ is integrable on $[0, T]$, and

$$\int_0^1 \frac{dr}{r \log(e + C_T r^{-1})}\, dr = +\infty,$$

Osgood lemma applied concludes that $(\delta a, \delta u) = 0$ on $[0, \widetilde T]$, and a continuity argument ensures that $(a^1, u^1) = (a^2, u^2)$ on $[0, T]$.

References

[1] Bony, J.-M.: Calcul symbolique et propagation des singularitiés pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)* **14** (1981), no. 2, 209–246.

[2] Bresch, D. and Desjardins, B.: Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.* **238** (2003), no. 1-2, 211–223.

[3] Bresch, D., Desjardins, B. and Chi-Kun, L.: On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations* **28** (2003), no. 3-4, 843–868.

[4] Bresch, D., Desjardins, B. and Métivier, G.: Recent mathematical results and open problems about shallow water equations. In *Analysis and simulation of fluid dynamics*, 15–31. Adv. Math. Fluid Mech. Birkhäuser, Basel, 2007.

[5] Cannone, M.: *Ondelettes, paraproduits et Navier-Stokes*. Nouveaux essais. Diderot Editeur, Paris, 1995.

[6] Cannone, M.: Harmonic analysis tools for solving the incompressible Navier-Stokes equations. In *Handbook of mathematical fluid dynamics. Vol. III*, 161–244. North-Holland, Amsterdam, 2004.

[7] Chemin, J.-Y.: *Perfect incompressible fluids*. Oxford Lecture Series in Mathematics and its Applications **14**. The Clarendon Press. Oxford University Press, New York, 1998.

[8] Chemin, J.-Y.: Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. *J. Anal. Math.* **77** (1999), 27–50.

[9] Chemin, J.-Y. and Lerner, N.: Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations* **121** (1995), no. 2, 314–328.

[10] Chen, Q., Miao, C. and Zhang, Z.: On the well-posedness for the viscous shallow water equations. *SIAM J. Math. Anal.* **40** (2008), no. 2, 443–474.

[11] Danchin, R.: Global existence in critical spaces for compressible Navier-Stokes equations. *Invent. Math.* **141** (2000), no. 3, 579–614.

[12] Danchin, R.: Local theory in critical spaces for compressible viscous and heat-conductive gases. *Comm. Partial Differential Equations* **26** (2001), no. 7-8, 1183–1233.

[13] Danchin, R.: Density-dependent incompressible viscous fluids in critical spaces. *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003), no. 6, 1311–1334.

[14] Danchin, R.: On the uniqueness in critical spaces for compressible Navier-Stokes equations. *Nonlinear Differential Equations Appl.* **12** (2005), no. 1, 111–128.

[15] Danchin, R.: Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density. *Comm. Partial Differential Equations* **32** (2007), no. 7-9, 1373–1397.

[16] Fujita, H. and Kato, T.: On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.* **16** (1964), 269–315.

[17] Haspot, B.: Cauchy problem for viscous shallow water equations with a term of capillarity. To appear in *Math. Models Methods Appl. Sci.* (2010). Doi 10.1142/S0218202510004532.

[18] Kazhikhov, A. V. and Shelukhin, V. V.: Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *J. Appl. Math. Mech.* **41** (1977), no. 2, 273–282.

[19] Lions, P.-L.: *Mathematical Topics in Fluid Mechanics. Vol. 2, Compressible models*. Oxford Lecture Series in Mathematics and its Applications **10**. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[20] Matsumura, A. and Nishida, T.: The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. *Proc. Japan Acad. Ser. A Math. Sci.* **55** (1979), no. 9, 337–342.

[21] Mellet, A. and Vasseur, A.: On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations* **32** (2007), no. 1-3, 431–452.

[22] Meyer, Y.: Wavelets, paraproducts and Navier-Stokes equations. In *Current developments in Mathematics, 1996 (Cambridge, MA)*, 105–212. Int. Press, Boston, MA, 1997.

[23] Nash, J.: Le problème de Cauchy pour les équation différentielles d'un fluide général. *Bull. Soc. Math. France* **90** (1962), 487–497.

[24] Triebel, H.: *Theory of function spaces*. Monographs in Mathematics **78**. Birkhäuser Verlag, Basel, 1983.

[25] Wang, W.-K. and Xu, C.-J.: The Cauchy problem for viscous shallow water equations. *Rev. Mat. Iberoamericana* **21** (2005), no. 1, 1–24.

Recibido: 3 de diciembre de 2008

Qionglei Chen
Institute of Applied Physics and Computational Mathematics
Beijing 100088, China
chen_qionglei@iapcm.ac.cn

Changxing Miao
Institute of Applied Physics and Computational Mathematics
Beijing 100088, China
miao_changxing@iapcm.ac.cn

Zhifei Zhang
School of Mathematical Science
Peking University
Beijing 100871, China
zfzhang@math.pku.edu.cn

Q. Chen and C. Miao were partially supported by the NSF of China under grant No. 10931001, No. 10725102. Z. Zhang was partially supported by the NSF of China under grant No. 10990013.