Real analytic parameter dependence of solutions of differential equations

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Abstract

We consider the problem of real analytic parameter dependence of solutions of the linear partial differential equation $P(D)u = f$, i.e., the question if for every family $(f_\lambda) \subseteq \mathscr{D}'(\Omega)$ of distributions depending in a real analytic way on $\lambda \in U$, U a real analytic manifold, there is a family of solutions $(u_\lambda) \subseteq \mathscr{D}'(\Omega)$ also depending analytically on λ such that

$$P(D)u_\lambda = f_\lambda \qquad \text{for every } \lambda \in U,$$

where $\Omega \subseteq \mathbb{R}^d$ an open set. For general surjective variable coefficients operators or operators acting on currents over a smooth manifold we give a solution in terms of an abstract "Hadamard three circle property" for the kernel of the operator. The obtained condition is evaluated giving the full solution (usually in terms of the symbol) for operators with constant coefficients and open (convex) $\Omega \subseteq \mathbb{R}^d$ if $P(D)$ is of one of the following types: 1) elliptic, 2) hypoelliptic, 3) homogeneous, 4) of two variables, 5) of order two or 6) if $P(D)$ is the system of Cauchy-Riemann equations. An analogous problem is solved for convolution operators of one variable. In all enumerated cases, it follows that the solution is in the affirmative if and only if $P(D)$ has a linear continuous right inverse which shows a striking difference comparing with analogous smooth or holomorphic parameter dependence problems.

The paper contains the whole theory working also for operators on Beurling ultradistributions $\mathscr{D}'_{(\omega)}$. We prove a characterization of surjectivity of tensor products of general surjective linear operators on a wide class of spaces containing most of the natural spaces of classical analysis.

1. Introduction

The problem of parameter dependence of solutions of linear partial differential equations is a classical question. As examples of important contributions to that topic we should mention papers of Tréves [72, 73] and Browder [13]. Later the problem of Hörmander on holomorphic parameter dependence of fundamental solutions was solved in [44], [45] and [46]. Tréves studied smooth and holomorphic dependence. Browder considered real analytic dependence – their results form the first source of motivation for our paper.

Browder looked at a differential operator as a densely defined map on a Hilbert space $L_2(\Omega)$. In that case there is no ambiguity in saying "analytically dependent" since all reasonable definitions of Banach valued real analytic functions coincide [1].

We will consider differential operators acting on the space of distributions. In that case we have to explain what does it precisely mean "depending in an analytic way" capturing the intuitive idea of a distribution given by a formula depending analytically on parameters λ. That idea becomes apparent through the following example. The formula

$$(1.1) \qquad f_\lambda(x) := \frac{1}{1 + \lambda^2 \left(\sum_{j=1}^d x_j^2 \right)} \qquad \lambda \in \mathbb{R}$$

gives a distribution for each fixed λ and it is natural to say that it depends analytically on λ. Unfortunately, as follows from [40, p. 107] or [41, p. 97] there is no reasonable topology on the space of distributions (or smooth functions) such that $\lambda \mapsto f_\lambda$ is locally given by a convergent vector valued Taylor series.

The Acta Mathematica paper of Kriegl and Michor [40] (and also their subsequent book [41]) contains a detailed discussion on "real analytic dependence" in the context of their study of real analytic maps between manifolds modelled on locally convex spaces and respective vector bundles. Their research is motivated, for instance, by the search of a real analytic structure of the group of real analytic diffeomorphisms on a real analytic manifold. In order to cover examples like (1.1) we have to follow the choice of Kriegl and Michor: for U a real analytic manifold and E a *complex* locally convex space

we call a vector valued function $g : U \to E$ *real analytic* if for every linear continuous functional $x \in E'$ the scalar function $x \circ g : U \to \mathbb{C}$ is analytic. In the context of distributions our definition means that

$$U \ni \lambda \mapsto f_\lambda \in \mathscr{D}'(\Omega)$$

is real analytic if for every test function $\varphi \in \mathscr{D}(\Omega)$ the scalar function

$$\lambda \mapsto \langle f_\lambda, \varphi \rangle$$

is real analytic. The example (1.1) satisfies that definition but it is not covered by Browder's approach because his analytic dependent objects must be locally represented by the vector valued Taylor series convergent in $L_2(\Omega)$, hence also in $\mathscr{D}'(\Omega)$. Moreover, not every real analytic vector valued function extends to a holomorphic function on some complex neighbourhood of its domain (again the example (1.1)!) and therefore the real analytic dependence differs essentially from the holomorphic one. A characterization of Fréchet spaces X such that all X-valued real analytic functions are locally representable via convergent vector valued Taylor series is given in [4] and [5] but neither the space of smooth functions on a manifold nor the space of distributions satisfies this condition.

The research of Kriegl and Michor in [40] and [41] gives the second motivation of our problem. In their terminology, we ask if every real analytic curve lifts with respect to a surjective linear partial differential operator – in [41, Sec. 21] such a lifting problem is considered for arbitrary quotient maps and the smooth analogue of the question is explicitly formulated as an open problem [41, p. 226].

The author finds the outcome of the present paper as very surprising. We know (see, for instance, discussion in [8]) that the analogous smooth or holomorphic dependence problems for $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ have always positive solutions over any convex open set $\Omega \subseteq \mathbb{R}^d$ and for any linear partial differential equation with constant coefficients. Contrary to that, we will prove:

Theorem A. *The real analytic parameter dependence problem for a partial differential operator $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ with constant coefficients:*

(a) *has a negative answer for all elliptic operators for arbitrary open set $\Omega \subseteq \mathbb{R}^d$, $d > 1$, or for hypoelliptic operators for arbitrary open convex set $\Omega \subseteq \mathbb{R}^d$, $d > 1$ as well as for a system of Cauchy-Riemann equations for arbitrary pseudoconvex domain $\Omega \subseteq \mathbb{C}^d$, $d \geq 1$;*

(b) *has an affirmative answer for operators of two variables on convex open $\Omega \subseteq \mathbb{R}^d$ if and only if the operator is hyperbolic;*

(c) has an affirmative answer for operators of order two on $\mathbb{R}^d$ if and only if after some real linear invertible change of variables the operator $P(D)$ is of the form

$$\mu(\partial_1 - a_1)^2 + c \quad or \quad \mu\left[\sum_{j=1}^{r}(\partial_j - a_j)^2 - \sum_{j=r+1}^{s-1}(\partial_j - a_j)^2 + \lambda i\partial_s\right] + c$$

for μ, c, $a_j \in \mathbb{C}$, $\lambda \in \mathbb{R}$, $1 \le r < s - 1$;

(d) has an affirmative answer for operators with homogeneous symbol on convex open $\Omega \subseteq \mathbb{R}^d$ if and only if the operator has a linear continuous right inverse on $C^\infty(\Omega)$.

In fact, in all the cases above, the solution of our problem is positive if and only if $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ or, equivalently, $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ has a linear continuous right inverse.

As a consequence, the problem of real analytic parameter dependence of solutions has a negative answer for the Laplace, heat and Schrödinger operators on any open convex set Ω as well as for the wave operator with more than one spatial variable on any bounded convex set with C^1-boundary. On the other hand, the answer is affirmative for the wave operator with arbitrarily many variables for $\Omega = \mathbb{R}^d$ as well as for the wave operator with one spatial variable for any open convex set $\Omega \subseteq \mathbb{R}^2$. Clearly, if a linear partial differential operator $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ (or, equivalently, $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ [48]) has a linear continuous right inverse then it has a real analytic parameter dependence. All the above results support a surprising conjecture that the converse holds as well (and give a partial solution to [6, Problem 9.5]). A similar theory for operators on spaces of ultradifferentiable functions of Roumieu type, in particular, on spaces of real analytic functions, will be developed in a forthcoming paper.

Various characterizations of the existence of a linear continuous right inverse for a linear partial differential operator with constant coefficients (solving a problem of L. Schwartz) were given by Meise, Taylor and Vogt [48]. Their conditions were better understood and expressed in many cases in terms of the symbol thanks to efforts of Braun, Meise, Taylor and Vogt (see [48, 52, 53, 12] and many further papers). For homogeneous symbols P and bounded convex Ω with C^1-boundary this holds if and only if P is proportional to a product of linear forms with real coefficients [48, Th. 3.8]. For $\Omega = \mathbb{R}^d$ the condition is well understood up to $d = 3$ [12]. Our research both shed a new light on those results and profits from the deep ideas in the papers listed above (especially, from [55]) which are truly impressive and inspiring for us.

We will prove also some unexpected relations, in particular, that operators with real analytic parameter dependence are just perturbations of homogeneous operators with the same property:

Theorem B. *If a linear partial differential operator $P(D): \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ with constant coefficients, $\Omega \subseteq \mathbb{R}^d$ convex open, has a real analytic parameter dependence then both $P(D): \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ and $P_p(D): \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ satisfy the same condition, where*

$$P_p(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$$

is the principal part of the polynomial $P(z) = \sum_{|\alpha| \leq k} a_\alpha z^\alpha$.

By the Köthe-Grothendieck duality [43, Prop. 6.12], the dual space $H'(\mathbb{D})$ of the space of holomorphic functions on the unit disc $H(\mathbb{D}) = \ker \bar{\partial} \subseteq C^\infty(\mathbb{D})$ can be identified as follows:

$$H'(\mathbb{D}) = \operatorname{ind}_{N \in \mathbb{N}} \left(H_0^\infty(K_N), \|\cdot\|_N^* \right),$$
$$K_N := \mathbb{C}^* \setminus (1 - 1/N)\overline{\mathbb{D}}, \quad \|f\|_N^* := \sup_{|z| > 1-1/N} |f(z)|,$$

where the subscript 0 means that we consider functions vanishing at ∞, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. The Hadamard three circle theorem implies that

$$(1.2) \quad \forall N \; \exists M \; \forall K \; \exists \theta_0 \; \forall \theta \in]\theta_0, 1[\; \exists C \quad \|f\|_M^* \leq C \left(\|f\|_N^* \right)^\theta \left(\|f\|_K^* \right)^{1-\theta}.$$

In fact, this condition is nothing more but a qualitative version of the Hadamard's theorem later generalized by Petsche [66, Cor. 4.5]. In [6], [7] and [8] the problem of splitting of short exact sequences

$$0 \longrightarrow X \longrightarrow Y \longrightarrow F \longrightarrow 0,$$

was studied for F either Fréchet or dual Fréchet and X, Y the so-called PLS-spaces (like spaces of distributions or their closed subspaces, see Section 2). The conditions analogous to (1.2) but for non-metrizable spaces were discovered and proved to be of crucial importance for splitting. In particular, in [6, Cor. 8.4], [8, Prop. 5.4] it was shown that essentially the same condition as (1.2), the so-called dual interpolation estimate for θ close to 1 (see (2.1) below in Section 2) holds for kernels of *arbitrary* partial differential operators with constant coefficients (not only $\bar{\partial}$) on $\mathscr{D}'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ convex. The basis of the whole paper is the following surprising fact that the strong version of the Hadamard's estimate has dramatic consequences (Corollary 6.4):

Theorem C. *An arbitrary surjective linear partial differential operator* $P(D, x)$ *with variable coefficients on* $\mathscr{D}'(\Omega)$ *has real analytic parameter dependence of solutions if and only if its kernel has the dual interpolation estimate for all sufficiently small* $\theta \in]0, 1[$ *(see (2.1)). In fact, this result holds for an arbitrary surjective linear continuous operator* $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ *as well, where* $\mathscr{D}'(\Omega)$ *is either the space of distributions on open* $\Omega \subseteq \mathbb{R}^d$ *or the space of currents on a* C^∞*-manifold* Ω.

It is worth noting that since $\mathscr{D}'(\Omega)^m \simeq \mathscr{D}'(\Omega)$ the result holds for matrices (systems) of differential operators as well.

The whole paper is just the proof of the results above. It is based on a combination of methods from various branches of mathematics and it also gives several interesting by-products. In fact, roughly half of the paper consists of estimates on plurisubharmonic functions on algebraic varieties. First, we use some functional analysis to reformulate the problem to the question on surjectivity of tensor products of surjective operators. The second step is based on homological algebra and gives a reformulation in terms of vanishing of some derived functor, or more precisely, some special cohomology groups Proj^1. The third part based on the operator theory on Hilbert spaces (spectral theory of unbounded operators) and on the modern theory of locally convex spaces evaluates when $\mathrm{Proj}^1 = 0$. The fourth step comes from the function theory: it is a specification of the calculated condition for the space of real analytic functions via a complex analytic construction to obtain the mentioned above dual interpolation estimate for small θ. The fifth part is a translation of the estimate for kernels of differential operators through the fundamental principle of Ehrenpreis and Palamodov. The sixth part, the longest one, is a mixture of algebraic geometry and the theory of plurisubharmonic functions. We obtain estimates for plurisubharmonic functions on the zero variety of the polynomial P which give main results.

Let us emphasize that several results in the paper are formulated and proved for $\mathscr{D}'_{(\omega)}(\Omega)$ (including Denjoy-Carleman style ultradistributions $\mathscr{D}'_{(M_p)}$ –see [39]) but they cover also the most interesting case of the space of classical distributions $\mathscr{D}'(\Omega)$ by taking $\omega(t) = \log(2 + |t|)$. All the results up to Section 6 are valid also for operators with variable coefficients and Ω being a smooth manifold (that means for spaces of currents instead of distributions).

Now, we summarize the content of the paper. In Section 2 we explain the framework of the category of PLS-spaces which contains most of the non-Banach natural spaces of analysis. Among them the spaces of (ultra-)distributions, currents, real analytic functions, smooth and holomorphic functions as well as the spaces of tempered distributions. We recall topological invariants extensively used in the paper: dual interpolation estimates and the so-called P-conditions.

In Section 3 we formulate precisely the problem of real analytic parameter dependence of solutions of the given distributional equation $Tu = f$, where T, f are given and u is an unknown distribution or current. We recall that this is equivalent with surjectivity of the operator $T \otimes \mathrm{id}_{\mathscr{A}(U)}$ defined on the completed tensor product of the domain of T and the space of real analytic functions $\mathscr{A}(U)$. Grothendieck showed that for nuclear Fréchet spaces tensor product of two surjective operators is surjective as well but we consider highly non-metrizable spaces like $\mathscr{D}'(\Omega)$ or $\mathscr{A}(U)$ and therefore our problem of surjectivity of tensor products is much more subtle and so far it has been unsolved – the results of the present paper (see Section 5) are the strongest known up to now.

In Section 4 we explain the relation of the problem of surjectivity of tensor products of surjective operators acting on PLS-spaces to the homological algebra problem of vanishing of Proj^1 functor (the first derived functor of the functor of projective limit) on the tensor product of kernels, see Corollary 4.6. This section is the last part in the introductory part containing essentially known results with only minor original contribution.

Section 5 is the first technical core of the paper. The main result (Theorem 5.2) characterizes when Proj^1 vanishes for a completed tensor product of two PLN-spaces. Via the results of Section 4 it solves completely the problem of surjectivity of tensor products of two surjective endomorphisms on PLN-spaces (therefore on most of the natural spaces of analysis)!

In Section 6 we draw consequences from the previous section for tensor products $X \hat{\otimes}_\varepsilon E$ when one space is fixed. The crucial result is Corollary 6.4 = Theorem C which characterizes when $T \otimes \mathrm{id}_{\mathscr{A}(U)}$ is surjective for surjective $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ or, equivalently, characterizes when the equation $Tu = f$ has a real analytic parameter dependence of solutions. As a consequence we characterize in terms of μ when a convolution operator $T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}) \to \mathscr{D}'_{(\omega)}(\mathbb{R})$ has a real analytic parameter dependence of solutions. The section contains only a selection of the consequences of the constructed theory – an attentive reader will easily produce much more.

Since in Section 6 we show that surjectivity of tensorized maps/real analytic parameter dependence is equivalent with the so-called dual interpolation estimates for the kernel of the map T, it is of crucial importance to decide for a fixed operator T (especially differential operator) when its kernel has this property. In Section 7 we give a characterization for linear partial differential operators with constant coefficients, see Theorem 7.1. We use the fundamental principle of Ehrenpreis and Palamodov and theory of plurisubharmonic functions to translate our conditions into a Phragmén-Lindelöf type estimate for plurisubharmonic functions on a suitable algebraic variety.

In Section 8 we use our theory to the problem of real analytic parameter dependence of solutions of linear partial differential equations with constant coefficients on the space of distributions $\mathscr{D}'(\Omega)$ or the space of ultradistributions of Beurling type $\mathscr{D}'_{(\omega)}(\Omega)$, Ω convex. Using plurisubharmonic functions on algebraic varieties we show the main results of the paper (Theorems A and B). This section is the second technical core of the paper – we apply subtle methods of the theory of plurisubharmonic functions.

The last Section 9 collects open problems which naturally appear in our theory.

To conclude the introductory section let us recall that Fréchet space means a complete metrizable locally convex space. An operator denotes a linear continuous map and by $L(X, Y)$ we denote the space of all such operators $T : X \to Y$. The notation $X \overset{comp}{\subseteq} Y$ means that X is a complemented locally convex subspace of Y. By Ω we always denote either an open subset of $\mathbb{R}^d$ or a smooth manifold both with a fixed but arbitrary compact exhaustion

$$K_1 \Subset K_2 \Subset \cdots \Subset K_N \Subset \cdots \Subset \Omega, \qquad \Omega = \bigcup_{N \in \mathbb{N}} K_N.$$

From Section 7 on, Ω is always an open subset of $\mathbb{R}^d$. If Ω is convex then K_N will be chosen convex as well. By $\mathscr{A}(U)$ we denote the space of real analytic functions with its natural topology (see [47, 18, 24]) on a real analytic manifold U. Throughout the paper we always assume that real analytic manifolds are Hausdorff, paracompact and have countable bases of topology.

A short exact sequence of locally convex spaces

$$0 \longrightarrow X \overset{j}{\longrightarrow} Y \overset{q}{\longrightarrow} Z \longrightarrow 0$$

is called topologically exact if j is a linear topological embedding and q is an open continuous linear map. By $B(x, r)$ we denote a ball of radius r and center x. $H(V)$ denotes the space of holomorphic functions on V. If $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ then $|z| := \left(\sum_{j=1}^d |z_j|^2 \right)^{1/2}$. The symbols $o(\cdot)$ and $O(\cdot)$ are always understood as variable tends to ∞.

For non-explained notions from functional analysis (theory of locally convex spaces, spectral theory etc.) see [57] or/and [37]. For the theory of tensor products see [37]. For applications of homological algebra and derived functors to partial differential equations and functional analysis see [83]. For the theory of plurisubharmonic functions (pluripotential theory) see [38]. For linear partial differential equations see [36, 62]. For the theory of currents see [25].

2. PLS-spaces – preliminaries

The whole paper is written in a very convenient framework of the so-called PLS-spaces. It turns out that typical non-Banach spaces of analysis are either PLS-spaces or their duals. For a review on PLS-spaces see [18].

An *LS-space* (called also a DFS-space) is a locally convex inductive limit of a sequence of Banach spaces with compact linking maps or, equivalently, a strong dual of a Fréchet-Schwartz space. It is known that LS-spaces are reflexive, Schwartz, separable and that every closed subspace and separated quotient of an LS-space is LS as well. Examples of LS-spaces are spaces of distributions of compact support $\mathscr{E}'(\Omega)$, spaces of tempered distributions $\mathscr{S}'$, spaces of germs of holomorphic functions over a compact set $H(K)$, etc.

A projective limit of a sequence of LS-spaces is called a *PLS-space*. It is known that all Fréchet-Schwartz spaces, LS-spaces and their countable products are PLS-spaces. Every closed subspace and complete separated quotient of a PLS-space is PLS as well. The class of PLS-spaces is the smallest class of locally convex spaces containing Fréchet-Schwartz spaces, their duals and closed with respect of taking countable products and closed subspaces.

PLS-spaces are separable, webbed and complete. They are reflexive if and only if they are barrelled or (ultra)bornological. Examples of PLS-spaces are spaces of holomorphic functions $H(V)$, smooth functions $C^\infty(\Omega)$, distributions or currents $\mathscr{D}'(\Omega)$, real analytic functions $\mathscr{A}(U)$, ultradistributions in the sense of Beurling $\mathscr{D}'_{(\omega)}(\Omega)$, ultradifferentiable functions in the sense of Roumieu $\mathscr{E}_{\{\omega\}}(\Omega)$ and Köthe type PLS-spaces $\Lambda(A)$ (see [18]).

If instead of LS-spaces we take LN-spaces (i.e., locally convex inductive limits of sequences of Banach spaces with nuclear linking maps, called also DFN-spaces since they are strong duals of nuclear Fréchet spaces), we get PLN-spaces. All the above mentioned PLS-spaces except some Köthe spaces $\Lambda(A)$ are PLN-spaces. The category of PLS-spaces is more convenient since not every nuclear Fréchet space is a PLN-space (only so-called strongly nuclear Fréchet spaces [37] are PLN-spaces).

If X is a PLS-space we write it throughout the paper as

$$X = \operatorname{proj}_{N\in\mathbb{N}} X_N, \qquad \text{where } X_N = \operatorname{ind}_{n\in\mathbb{N}} X_{N,n} \text{ is an LS-space,}$$

$(X_{N,n}, \|\cdot\|_{N,n})$ are Banach spaces, $i^n_{N,n+1} : X_{N,n} \to X_{N,n+1}$ are compact linking maps, $i^{N+1}_N : X_{N+1} \to X_N$, $i_N : X \to X_N$ are linking maps. Moreover, we denote by $B_{N,n}$ the unit ball of $X_{N,n}$ and $U_{N,n}$ its polar, i.e., the unit ball in $X'_{N,n} \hookrightarrow X'_N$. Without loss of generality we assume that $i^n_{N,n+1}$ has norm ≤ 1, i.e., $B_{N,n} \subseteq B_{N,n+1}$ for $N, n \in \mathbb{N}$. Moreover,

$$i^{N+1}_N(B_{N+1,n}) \subseteq B_{N,n} \qquad \text{for } N, n \in \mathbb{N}.$$

We denote

$$\|f\|^*_{N,n} := \sup_{\|x\|_{N,n} \le 1} |f(x)|$$

the dual norm. The same conventions will be used for other PLS-spaces Y, E, etc. By $W(U_{N,n}, B_{N,n})$ we denote the set of all linear continuous operators

$$T : X'_{N,n} \to Y_{N,n}, \qquad T(U_{N,n}) \subseteq B_{N,n}.$$

The spaces mentioned above are described, for instance, in [6, pp. 333–336]. For the sake of completeness we recall the definition of $\mathscr{D}'_{(\omega)}(\Omega)$ from [11] since these spaces are extensively used in the paper.

We will use the classical multiindex notation for $\alpha = (\alpha_1, \dots, \alpha_d)$,

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \quad f^{(\alpha)}(x) = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d}.$$

We introduce a *weight* $\omega : [0, \infty[\to [0, \infty[$ to be a continuous increasing function satisfying the following conditions:

(α) $\omega(2t) = O(\omega(t))$;

(β) $\omega(t) = O(t)$;

(γ) $\log t = o(\omega(t))$ or $\omega(t) = \log(2 + |t|)$;

(δ) φ is a convex function, $\varphi(t) := \omega(e^t)$.

We extend ω to $\mathbb{C}^d$ by $\omega(z) := \omega(|z|)$. If

$$\int_0^\infty \frac{\omega(t)}{1 + t^2} dt < \infty$$

then the weight is *non-quasianalytic* and we will use later on only such weights.

If $\Omega \subseteq \mathbb{R}^d$ open then the class of *Beurling ultradistributions* $\mathscr{D}'_{(\omega)}(\Omega)$ is defined to be the strong dual of

$$\mathscr{D}_{(\omega)}(\Omega) := \Big\{ f \in \mathscr{D}(\Omega) : \ \forall \, k \in \mathbb{N}$$
$$\|f\|_k := \sup_{x \in \Omega} \sup_{\alpha \in \mathbb{N}^d} |f^{(\alpha)}(x)| \exp\Big(-k\varphi^*\Big(\frac{|\alpha|}{k}\Big)\Big) < \infty \Big\},$$

where

$$\varphi^*(t) := \sup_{x \ge 0}(xt - \varphi(t)) \quad \text{the Young conjugate of } \varphi,$$

equipped with the inductive limit topology (steps are Fréchet spaces with $\|\cdot\|_k$ seminorms):

$$\mathscr{D}_{(\omega)}(\Omega) = \mathrm{ind}_{N \in \mathbb{N}} \, \mathscr{D}_{(\omega)}(K_N), \quad \mathscr{D}_{(\omega)}(K_N) := \{f \in \mathscr{D}_{(\omega)}(\Omega) : \mathrm{supp}\, f \subseteq K_N\}.$$

Thus $\mathscr{D}'_{(\omega)}(\Omega)$ is a PLN-space. We consider the weight $\omega(t) = \log(2 + |t|)$ since it gives $\mathscr{D}'_{(\omega)}(\Omega)$ to be the classical space of distributions $\mathscr{D}'(\Omega)$. If Ω is a smooth manifold then we use the same notation $\mathscr{D}'(\Omega)$ to denote the space of currents over Ω with the analogous topology (we hope that which space is denoted will be clear from the context).

The family of classes $\mathscr{D}'_{(\omega)}$ is well adapted to Fourier transform but it contains also most of the Denjoy-Carleman style classes defined via growth conditions of Taylor coefficients controlled by sequences (M_p), see [9] and [39]. The best known among them are Gevrey classes (see, for instance, [69]): $\Gamma_{(p)} = \mathscr{D}_{(\omega)}$ for $\omega(t) = t^{1/p}$ for $p > 1$.

The space of real analytic functions $\mathscr{A}(\Omega)$ has the topology given as

$$\mathscr{A}(\Omega) = \mathrm{proj}_{N \in \mathbb{N}} \ H(K_N),$$

where $H(K_N)$ is the space of germs of holomorphic functions over K_N,

$$H(K_N) := \mathrm{ind}_{n \in \mathbb{N}} \ H^\infty(U_{N,n}),$$

where $(U_{N,n})_{n \in \mathbb{N}}$ is a basis of complex neighbourhoods of K_N. Analogously, we define topology on $\mathscr{A}(\Omega)$ where Ω is a real analytic manifold. Clearly $\mathscr{A}(\Omega)$ is a PLN-space (for more details see [18]). The space $\mathscr{A}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open, is neither metrizable nor dual metrizable and it has a complicated structure: for instance, it has no Schauder basis [24].

We call a sequence α *regular* if

$$\Lambda_r(\beta) \overset{comp}{\subseteq} \Lambda_r(\alpha) \overset{comp}{\subseteq} \Lambda_r(\gamma), \quad \beta, \gamma \ \text{stable}.$$

Here $\Lambda_0(\alpha)$, $\Lambda_\infty(\alpha)$ denote power series spaces of finite and infinite type, respectively, see [57, Ch. 29].

We will need the following simple observation:

Proposition 2.1 *Let U be a real analytic manifold.*

(a) *If U is compact then $\mathscr{A}(U) \simeq \Lambda'_0(\alpha)$, α regular.*

(b) *If U has infinitely many connected components, all of them compact, then*

$$\mathscr{A}(U) \simeq \prod_{n \in \mathbb{N}} \Lambda'_0(\alpha^{(n)}), \qquad \text{sequences } \alpha^{(n)} \text{ are regular for every } n \in \mathbb{N}.$$

(c) *If U has at least one non-compact connected component then $\mathscr{A}(U)$ has a quotient isomorphic to $\mathscr{A}(\mathbb{R})$ and $\mathscr{A}(U)$ is isomorphic to a quotient of the space $\mathscr{A}(\mathbb{R}^m)$ for suitable big m.*

Proof. (a): Clearly U contains a copy of the unit circle $\mathbb{T}$ as a real analytic submanifold. By [81], the restriction map $r : \mathscr{A}(U) \to \mathscr{A}(\mathbb{T})$ has a continuous linear right inverse, so $\mathscr{A}(\mathbb{T}) \simeq \Lambda_0'(j)$ is isomorphic to a complemented subspace of $\mathscr{A}(U)$. By [60, §2.15], we may embed U into $\mathbb{T}^m$ for m big enough. Again by [81], the restriction map $R : \mathscr{A}(\mathbb{T}^m) \to \mathscr{A}(U)$ has a continuous linear right inverse, so $\mathscr{A}(U)$ is complemented in $\mathscr{A}(\mathbb{T}^m) \simeq \Lambda_0'(j^{1/m})$. This completes the proof by [59, Cor. 1.1], comp. [75, Satz 1.7].

(b): Immediate consequence of (a).

(c): Obviously, we may C^1-embed $\mathbb{R}$ into U. By [34, Th. 2.1.4, 2.5.1], we can also embed $\mathbb{R}$ into U as a real analytic submanifold. The restriction map $r : \mathscr{A}(U) \to \mathscr{A}(\mathbb{R})$ is a linear continuous surjection. Analogously, by [60, §2.15], U embeds into $\mathbb{R}^m$ for sufficiently big m as a real analytic submanifold and the restriction map $R : \mathscr{A}(\mathbb{R}^m) \to \mathscr{A}(U)$ is a linear continuous surjection. By [47, Th. 1.2] and [57, 24.30] the maps r and R are open. ∎

We will use extensively in the paper the following Lemma [11, Lemma 1.7], [10, Prop. 5], [50, Lemma 3.7]:

Lemma 2.2 *For every non-quasianalytic weight ω, $\varepsilon > 0$, and a function $h : [0, \infty[\to [0, \infty[$ such that $\omega = o(h)$ there exists a non-quasianalytic weight $\sigma \geq \omega$, a plurisubharmonic function v on $\mathbb{C}^d$ and a constant $D > 0$ such that*

$$\omega = o(\sigma), \quad \sigma = o(h), \quad -D\omega(z) \leq v(z) \leq \varepsilon |\operatorname{Im} z| - \omega(z) \quad \text{for } z \in \mathbb{C}^d.$$

We will use only PLS-type Köthe sequence spaces with ℓ_∞-norms as defined, for instance, in [6]. Among PLS-type Köthe sequence spaces the most interesting are PLS-type power series spaces $\Lambda_{r,s}(\alpha, \beta)$ see for instance [6, Th. 2.2]. We define for sequences (α_j) and (β_j) of positive numbers such that $\alpha_j + \beta_j \to \infty$ as $j \to \infty$ the following PLS-space

$$\Lambda_{r,s}(\alpha, \beta) := \{x = (x_j) : \ \forall N \ \exists n : \|x\|_{N,n} < \infty\},$$

where

$$\|x\|_{N,n} := \sup_j |x_j| \exp(r_N \alpha_j - s_n \beta_j)$$

and (r_N), (s_n) are arbitrary fixed sequences such that $r_N \nearrow r$, $s_n \nearrow s$. It suffices to consider $r, s = 0, \infty$ and the space does not depend on the choice of (r_N) and (s_n).

We will classify PLS-spaces by means of the so-called dual interpolation estimates and P-properties which are generalizations of (Ω)-type and dual (DN)-type properties introduced by Vogt and Zahariuta (see MSC 2000 number 46A63) and were defined for the first time in [6, 7, 8]. For Fréchet spaces of holomorphic functions of one variable the estimates are closely connected (via Köthe-Grothendieck duality) with the classical Hadamard three

circle theorem and operator theory specialists will recognize similarity to the so-called J-type in the interpolation theory of operators [2]. All these reminiscences are rooted in the proofs later on. A lot of information below is taken from [7]. A PLS-space is *strict* if and only if it is a projective limit of a sequence of LS-spaces where linking maps in the projective spectrum are surjective.

A PLS-space X has a *dual interpolation estimate* if

(2.1)
$$\forall \ N \ \exists \ M \geq N \ \forall \ K \geq M \ \exists \ n \ \forall \ m \geq n \ \forall \ \theta \in]0,1[\ \exists \ k \geq m, C$$
$$\forall \ x \in X'_N \quad \|x \circ i_N^M\|_{M,m}^* \leq C \left(\|x \circ i_N^K\|_{K,k}^*\right)^{1-\theta} \left(\|x\|_{N,n}^*\right)^{\theta}.$$

If we replace the quantifier $\forall \ \theta \in]0,1[$ by

$$\exists \ \theta_0 \in]0,1[\ \forall \ \theta \in]0,\theta_0[\quad (\text{or} \ \exists \ \theta_0 \in]0,1[\ \forall \ \theta \in]\theta_0,1[)$$

we get the definition of the *dual interpolation estimate for small (big) theta*.

Using the proof of [6, Lemma 5.1] we get the following reformulation of the dual interpolation estimate:

$$\forall \ N \ \exists \ M \geq N \ \forall \ K \geq M \ \exists \ n \ \forall \ m \geq n \ \forall \ \theta \in]0,1[\ \exists \ k \geq m, C \ \forall \ r > 0$$
$$\forall \ x \in X'_N : \|x \circ i_N^M\|_{M,m}^* \leq C\left(r^\theta \|x \circ i_N^K\|_{K,k}^* + \frac{1}{r^{1-\theta}}\|x\|_{N,n}^*\right).$$

Clearly taking θ close to 0 or 1 we get the dual interpolation estimate for small or big θ. If we consider only $r \geq 1$ then we get the condition $(P\overline{\overline{\Omega}})$ (for all θ or, equivalently, θ close to 0) or $(P\Omega)$ (for θ close to 1 or, equivalently, for some $\theta \in]0,1[$). Analogously, if we consider only $0 < r \leq 1$ then we get the condition (PA) (for all θ or, equivalently, θ close to 1) or $(P\underline{A})$ (for θ close to 0 or, equivalently, for some $\theta \in]0,1[$).

The dual interpolation estimate for small θ is equivalent to the combination of $(P\overline{\overline{\Omega}})$ and $(P\underline{A})$, for big θ is equivalent to $(P\Omega)$ and (PA). Moreover, $(P\overline{\overline{\Omega}})$ implies $(P\Omega)$ and (PA) implies $(P\underline{A})$. Thus the dual interpolation estimate for all θ is equivalent to the combination of $(P\overline{\overline{\Omega}})$ and (PA).

There are also reformulations of these P-conditions in the similar spirit as in the definition of the dual interpolation estimate. For instance, $(P\Omega)$ is equivalent to

$$\forall \ N \ \exists \ M \geq N \ \forall \ K \geq M \ \exists \ n \ \forall \ m \geq n \ \exists \ \theta \in]0,1[\ \exists \ k \geq m, C$$
$$\forall x \in X'_N : \|x \circ i_N^M\|_{M,m}^* \leq C \left(\|x \circ i_N^K\|_{K,k}^*\right)^{1-\theta} \cdot \max\left(\|x\|_{N,n}^*, \|x \circ i_N^K\|_{K,k}^*\right)^{\theta}.$$

If we take $\forall \theta \in]0,1[$ then we get $(P\overline{\overline{\Omega}})$. Analogously, $(P\underline{A})$ is equivalent to

$$\forall \ N \ \exists \ M \geq N \ \forall \ K \geq M \ \exists \ n \ \forall \ m \geq n \ \exists \ \theta \in]0,1[\ \exists \ k \geq m, C$$
$$\forall \ x \in X'_N : \ \|x \circ i_N^M\|_{M,m}^* \leq C \max\left(\|x \circ i_N^K\|_{K,k}^*, \|x\|_{N,n}^*\right)^{1-\theta} \left(\|x\|_{N,n}^*\right)^{\theta}$$

and if we take $\forall \theta \in]0,1[$ we get (PA).

All these conditions are inherited by countable products and complete quotients. In papers [6, 8, 7] many examples of spaces with P-properties are collected. We summarize them for the sake of convenience (use Prop. 2.1).

Corollary 2.3   *(a) A Fréchet space has a dual interpolation estimate for big θ iff it has (Ω). It has a dual interpolation estimate for small (all) θ iff it has $(\overline{\overline{\Omega}})$.*

   (b) An LS-space has a dual interpolation estimate for small θ iff its dual has $(\underline{DN})$. It has a dual interpolation estimate for big (all) θ iff its dual has (DN).

   (c) The space of distributions $\mathscr{D}'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ arbitrary open, or the space of currents $\mathscr{D}'(\Omega)$, Ω an arbitrary C^∞-manifold, or the space of Beurling ultradistributions $\mathscr{D}'_{(\omega)}(\Omega)$ has the dual interpolation estimate for all θ.

   (d) The space of real analytic functions $\mathscr{A}(\Omega)$, Ω a real analytic manifold has the dual interpolation estimate for small θ.

   (e) The PLS-type power series space $\Lambda_{r,s}(\alpha, \beta)$ has the dual interpolation estimate for big θ iff $s = \infty$ or it is a Fréchet space. It has the dual interpolation estimate for small θ (or $(P\overline{\overline{\Omega}})$) iff it is an LS-space.

Remark. Let us recall that the space of currents $\mathscr{D}'(\Omega)$ is the dual space of the space of differential forms on Ω with compact support. Since Ω can be embedded into some $\mathbb{R}^m$ in such a way that there is a tubular neighbourhood Ω_1 of Ω in $\mathbb{R}^m$ and using C^∞-resolution of identity we can define a linear continuous map extending differential forms with compact support on Ω to such forms on Ω_1. By duality the space of currents $\mathscr{D}'(\Omega)$ is a complemented subspace of the space of currents $\mathscr{D}'(\Omega_1)$. Since Ω_1 is "flat" the latter is a product of finitely many spaces of distributions. That is why the space of currents also has the dual interpolation estimate for all θ.

3. Parameter dependence of solutions versus surjectivity of tensorized maps

Let us assume that
$$T : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$$
is a surjective operator and U is a real analytic manifold. We say that the equation
$$Tu = f,$$
where u is an unknown distribution has the *real analytic parameter dependence (of solutions)* if for every real analytic function
$$U \ni \lambda \mapsto f_\lambda \in \mathscr{D}'_{(\omega)}(\Omega)$$

there is a real analytic function

$$U \ni \lambda \mapsto u_\lambda \in \mathscr{D}'_{(\omega)}(\Omega)$$

such that

$$Tu_\lambda = f_\lambda \qquad \text{for every } \lambda \in U.$$

Recall that for complete locally convex space X:

$$\mathscr{A}(U,X) := \{f : U \to X : \quad \forall\, g \in X' \quad g \circ f \in \mathscr{A}(U)\}.$$

As shown in [40], $\mathscr{A}(U,X)$ contains functions for which vector valued Taylor series is not convergent. In [4, 5] a characterization is given for which Fréchet spaces X every function $F \in \mathscr{A}(U,X)$ has a Taylor series local representation.

Let us define ε-product of Schwartz for complete X, E:

$$X \varepsilon E := L_e(X'_{co}, E)$$

the space of linear continuous operators from the dual X' equipped with the compact-open topology. The subscript e means that we equip $L(X'_{co}, E)$ with the topology of uniform convergence on equicontinuous sets. It is important that ε-product is commutative. Clearly $X \otimes E \subseteq X\varepsilon E$ and if one of the spaces has the approximation property then $X \otimes E$ is dense in $X\varepsilon E$, the latter is always complete if E and X are complete. If one of the spaces is nuclear (thus has the approximation property) then on $X \otimes E$ there is only one natural topology so then $X\varepsilon E$ is the unique natural completion of $X \otimes E$!

It is well known that for complete X (for $\mathscr{A}$ see [4, Th. 16]) spaces of vector valued functions have the following tensor representations:

$$C^\infty(U,X) = C^\infty(U)\varepsilon X, \quad H(U,X) = H(U)\varepsilon X, \quad \mathscr{A}(U,X) = \mathscr{A}(U)\varepsilon X.$$

For vector valued distributions ε-product is just the definition [71]:

$$\mathscr{D}'(U,X) := \mathscr{D}'(U)\varepsilon X, \quad \mathscr{D}'_{(\omega)}(U,X) := \mathscr{D}'_{(\omega)}(U)\varepsilon X.$$

As $T \otimes \mathrm{id}_{\mathscr{A}(U)}$ we define the unique extension of the tensor map

$$T \otimes \mathrm{id}_{\mathscr{A}(U)} : \mathscr{D}'_{(\omega)}(\Omega) \otimes \mathscr{A}(U) \to \mathscr{D}'_{(\omega)}(\Omega) \otimes \mathscr{A}(U)$$

via density to the map: $\mathscr{D}'_{(\omega)}(\Omega)\varepsilon\mathscr{A}(U) \to \mathscr{D}'_{(\omega)}(\Omega)\varepsilon\mathscr{A}(U)$.

Conclusion: Let $T : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ be an operator. The equation $Tu = f$ has the real analytic parameter dependence of solutions if and only if

$$T \otimes \mathrm{id} : \mathscr{D}'_{(\omega)}(\Omega)\varepsilon\mathscr{A}(U) \to \mathscr{D}'_{(\omega)}(\Omega)\varepsilon\mathscr{A}(U)$$

is surjective. An analogous conclusion holds for smooth, holomorphic, ultradifferentiable, (ultra)distributional dependence for operators on spaces of real analytic, smooth, ultradifferentiable, holomorphic functions or (ultra)distributions.

4. Surjectivity of tensorized maps versus Proj^1 functor

The aim of this section is to use homological approach to the problem of surjectivity of tensorized maps. We will translate the problem into the question of vanishing of the so-called Proj^1 functor. Of course, the main idea of this section is known to specialists (see [83]) – we complete it with some minor new results.

Let $V = \mathrm{proj}_{N\in\mathbb{N}} \, V_N$ be a projective limit of linear spaces. We construct the following exact sequence

$$0 \xrightarrow{\quad} V \xrightarrow{\;\;i\;\;} \prod_{N\in\mathbb{N}} V_N \xrightarrow{\;\;\sigma\;\;} \prod_{N\in\mathbb{N}} V_N,$$

where i is the canonical embedding and $\sigma((x_n)_{N\in\mathbb{N}}) := (i_N^{N+1} x_{N+1} - x_N)_{N\in\mathbb{N}}$ and $i_N^{N+1} : V_{N+1} \to V_N$ as usual denote linking maps. We define

$$\mathrm{Proj}^1 \left((V_N, i_N^{N+1})_{N\in\mathbb{N}} \right) := \prod_{N\in\mathbb{N}} V_N / \operatorname{im} \sigma.$$

That functor was introduced to functional analysis by Palamodov [63, 64], and analyzed in depth by Vogt [78, 79], Frerick [28], Frerick and Wengenroth [30] and others, see a survey in [83].

If we assume that (V_N) are complete and the spectrum is *strongly reduced* (i.e., canonical images of V in V_N are dense) then all such spectra are equivalent and we can write $\mathrm{Proj}^1 \, V$ since it does not depend on such a spectrum representing V. If V is a PLS-space then it is a strongly reduced projective limit of LS-spaces and we have the following result:

Theorem 4.1 [83, Cor. 3.3.10] *Let V be a PLS-space. The following assertions are equivalent:*

1. *$\mathrm{Proj}^1 \, V = 0$;*

2. *V is ultrabornological;*

3. *V is barrelled.*

Moreover, the following essentially known result will be used, see [20, Prop. 4.3] and its proof based on [3, 4.3], where the last statement follows from [17, Prop. 2.7].

Theorem 4.2 *Let $V = \mathrm{proj}_{N\in\mathbb{N}} \, V_N$, $W = \mathrm{proj}_{N\in\mathbb{N}} W_N$ are PLS-spaces (represented by strongly reduced spectra of LS-spaces) then*

$$V\varepsilon W = \mathrm{proj}_{N\in\mathbb{N}} \, V_N \varepsilon W_N = \mathrm{proj}_{N\in\mathbb{N}} \, L_b(V_N', W_N)$$

is a PLS-space, spectrum $(V_N \varepsilon W_N)$ is strongly reduced and

$$V_N \varepsilon W_N = L_b(V_N', W_N)$$

is an LS-space with the fundamental sequence of bounded sets (Banach discs)

$$(W(U_{N,n}, B_{N,n}))_{n \in \mathbb{N}},$$

where $B_{N,n}$ are the unit balls of Banach spaces in $W_{N,n}$, $W_N = \mathrm{ind}_{n \in \mathbb{N}} W_{N,n}$ and $U_{N,n}$ are the unit balls of the duals of $V_{N,n}$, $V_N = \mathrm{ind}_{n \in \mathbb{N}} V_{N,n}$.

We reduce the problem to tensorizing with the identity map.

Proposition 4.3 *Let $S : X \to Y$, $T : E \to E$ be continuous surjective operators. Then*

$$S \otimes T : X \varepsilon E \to Y \varepsilon E$$

is surjective if and only if

$$\mathrm{id}_Y \otimes T : Y \varepsilon E \to Y \varepsilon E, \qquad and \qquad S \otimes \mathrm{id}_E : X \varepsilon E \to Y \varepsilon E$$

are surjective.

Proof. Observe that

$$S \otimes T = (\mathrm{id}_Y \otimes T) \circ (S \otimes \mathrm{id}_E) = (S \otimes \mathrm{id}_E) \circ (\mathrm{id}_X \otimes T). \qquad \blacksquare$$

Lemma 4.4 *If X is a nuclear Fréchet space then $X = \mathrm{proj}_{N \in \mathbb{N}} X_N$ such that X_N are LS-spaces with c_0-steps. Therefore X'_N are projective limits of sequences of ℓ_1 Banach spaces.*

Proof. Clearly, by nuclearity, $X = \mathrm{proj}_{N \in \mathbb{N}} Y_N$, where $Y_N \simeq c_0$ and linking maps are compact.

If $T : c_0 \to c_0$ is a compact operator then there is a null sequence (x_n) such that

$$T(B_{c_0}) \subseteq \{(z_n) : |z_n| < x_n\}.$$

Therefore T factorizes through an LS-space (an inductive limit of c_0-Banach spaces):

$$V := \{(z_n) : \ \exists N \sup_{n \in \mathbb{N}} \frac{|z_n|}{\sqrt[N]{x_n}} < \infty\}.$$

Thus every linking map $i_N^{N+1} : Y_{N+1} \to Y_N$ factorizes through an LS-space of type V. $\blacksquare$

Proposition 4.5 *Assume that*

(4.1) $$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is a topologically exact sequence of PLS-spaces, E a PLS-space.

If either all X_N' or all E_N' are projective limits of Banach ℓ_1-spaces (in particular, if either X or E is a PLN-space or a nuclear Fréchet space) then

$$(4.2) \qquad 0 \longrightarrow X \varepsilon E \longrightarrow Y \varepsilon E \longrightarrow Z \varepsilon E \longrightarrow 0$$

is exact if $\mathrm{Proj}^1{}_{N \in \mathbb{N}}(X \varepsilon E) = 0$. If $\mathrm{Proj}^1{}_{N \in \mathbb{N}}(Y \varepsilon E) = 0$ then the above condition is also necessary.

Proof. It is known (see [22, p. 64]) that (4.1) is a projective limit of exact sequences

$$0 \longrightarrow X_N \xrightarrow{j_N} Y_N \xrightarrow{q_N} Z_N \longrightarrow 0,$$

where $Y = \mathrm{proj}_{N \in \mathbb{N}} Y_N$, $Z = \mathrm{proj}_{N \in \mathbb{N}} Z_N$ and Y_N, Z_N are LS-spaces. By [37, proof of 16.3.2] (comp. Theorem 4.2),

$$X \varepsilon E = \mathrm{proj}_{N \in \mathbb{N}} \ X_N \varepsilon E_N, \quad Y \varepsilon E = \mathrm{proj}_{N \in \mathbb{N}} \ Y_N \varepsilon E_N,$$

$$Z \varepsilon E = \mathrm{proj}_{N \in \mathbb{N}} \ Z_N \varepsilon E_N.$$

Since E_N are LS-spaces, i.e., Montel thus $X_N \varepsilon E_N = L_b(E_N', X_N)$ etc. Therefore the sequence

$$(4.3) \qquad 0 \longrightarrow X_N \varepsilon E_N \xrightarrow{j_N \otimes \mathrm{id}} Y_N \varepsilon E_N \xrightarrow{q_N \otimes \mathrm{id}} Z_N \varepsilon E_N \longrightarrow 0$$

is exact if every $T \in L(E_N', Z_N)$ lifts with respect to q_N to $S \in L(E_N', Y_N)$. By [6, Lemma 3.5] this is true (for the case of nuclear Fréchet spaces apply Lemma 4.4).

We have proved that (4.2) is a projective limit of exact sequences (4.3). The result follows from Palamodov's [63] or [83, Cor. 3.1.5] which implies existence of the following exact sequence:

$$0 \to X \varepsilon E \to Y \varepsilon E \to Z \varepsilon E \to$$
$$\to \mathrm{Proj}^1(X \varepsilon E) \to \mathrm{Proj}^1(Y \varepsilon E) \to \mathrm{Proj}^1(Z \varepsilon E) \to 0. \qquad \blacksquare$$

Corollary 4.6 *Let $S : X \to Y$, $T : E \to E$ be continuous open surjective operators between PLS-spaces. Assume that either E or $\ker T$ and $\ker S$ are PLN-spaces or nuclear Fréchet spaces. Then*

$$S \otimes T : X \varepsilon E \to Y \varepsilon E$$

is surjective whenever

$$\mathrm{Proj}^1(\ker S \varepsilon E) = \mathrm{Proj}^1(Y \varepsilon \ker T) = 0.$$

If, additionally, $\mathrm{Proj}^1(X \varepsilon E) = \mathrm{Proj}^1(Y \varepsilon E) = 0$ then this is also a necessary condition.

Proof. Apply Proposition 4.5 and Proposition 4.3. $\qquad \blacksquare$

Observe that:

Proposition 4.7 *If $T : X \to Y$ is a surjective continuous operator between PLS-spaces and $\mathrm{Proj}^1\, Y = 0$ then T is open.*

Proof. By Theorem 4.1, Y is ultrabornological. Since any PLS-space X is webbed, then by the De Wilde webbed open mapping theorem [57, 24.30], T is open. ∎

Finally, let us observe that vanishing of Proj^1 is somehow inherited by products.

Proposition 4.8 *Let X, $E^{(j)}$, $j \in \mathbb{N}$, be PLS-spaces. If $\mathrm{Proj}^1\left(X \varepsilon E^{(j)}\right) = 0$ for every $j \in \mathbb{N}$ then $\mathrm{Proj}^1\left(X \varepsilon \prod_{j \in \mathbb{N}} E^{(j)}\right) = 0$.*

Proof. This is an easy calculation since $X \varepsilon \prod_{j \in \mathbb{N}} E^{(j)} = \prod_{j \in \mathbb{N}} \left(X \varepsilon E^{(j)}\right)$ and if $X = \mathrm{proj}_{N \in \mathbb{N}}\, X_N$, $E^{(j)} = \mathrm{proj}\ _{N \in \mathbb{N}} E_N^{(j)}$. Then

$$\mathrm{Proj}^1 \left(X \varepsilon \prod_{j \in \mathbb{N}} E^{(j)} \right) = \prod_{N \in \mathbb{N}} \left(X_N \varepsilon \Big(\prod_{j \leq N} E_N^{(j)} \Big) \right) / \operatorname{im}\, \sigma,$$

$$\sigma : \prod_{N \in \mathbb{N}} \left(X_N \varepsilon \Big(\prod_{j \leq N} E_N^{(j)} \Big) \right) \to \prod_{N \in \mathbb{N}} \left(X_N \varepsilon \Big(\prod_{j \leq N} E_N^{(j)} \Big) \right).$$

∎

Conclusion: To decide if the ε-product of two surjective operators between PLN-spaces is surjective we need to know when Proj^1 of an ε-product of two PLN-spaces vanishes. We solve this problem in the next section.

5. Vanishing of Proj^1 for tensor products of PLS-spaces

Now, we will give criteria of vanishing of $\mathrm{Proj}^1\,(X \varepsilon E)$, where X, E are PLS-spaces. By Theorem 4.2, $X \varepsilon E$ is a PLS-space thus vanishing of Proj^1 is inherited by complete quotients (use Theorem 4.1). Since X and E are complemented subspaces of $X \varepsilon E$ a necessary condition for $\mathrm{Proj}^1\, X \varepsilon E = 0$ is that $\mathrm{Proj}^1\, X = \mathrm{Proj}^1\, E = 0$. Moreover, there is a very good condition (P_3^*) characterizing the vanishing of $\mathrm{Proj}^1\, V$ for PLS-spaces V in terms of inequalities between norms on functionals on V_N for a strongly reduced representation $V = \mathrm{proj}_{N \in \mathbb{N}}\, V_N$ (see [83, Cor. 3.3.10]). Unfortunately, for our purposes this is by far not enough since the space of functionals on $V_N = X_N \varepsilon E_N$ is too big to be handled and this condition gives no chance to "separate" the condition into suitable conditions on X and E.

A PLS space E is called *deeply reduced* if one (equivalently, each one) of its strongly reduced representing spectra of LS-spaces (E_N, i_N^{N+1}) satisfies the following condition

$$\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \exists\ n\ \forall\ m \geq n\ \exists\ k \geq m:$$

$$i_N^M E_{M,m} \subseteq \overline{i_N^K E_{K,k}}^{E_{N,k}} \cap \overline{E_{N,n}}^{E_{N,k}}.$$

If the inclusion holds only in the first (second) factor we call E *deeply reduced in columns (in rows)*.

Proposition 5.1 *If E is a deeply reduced in rows PLS-space then there is a strongly reduced representing spectrum $(\tilde{E}_N, i_N^{N+1})$ such that*

$$\forall\ N\ \exists\ n\ \forall\ m \geq n \qquad \tilde{E}_{N,m}\quad \text{is dense in}\quad \tilde{E}_{N,m+1}.$$

If, additionally, E is just deeply reduced then we may additionally assume that

$$\forall\ N\ \exists\ n\ \forall\ m \geq n \qquad i_N^{N+1}\tilde{E}_{N+1,m}\quad \text{is dense}\quad \tilde{E}_{N,m}.$$

Remark. The converse implications are trivial.

Examples. The space of real analytic functions $\mathscr{A}(\Omega)$ is deeply reduced since every compact subset of $\mathbb{R}^d$ has a complex neighbourhood basis consisting of pseudoconvex sets (see [14] or [31]). It follows also from [70, Satz 2.19] that $\mathscr{E}_{\{\omega\}}(\Omega)$ is deeply reduced for quasianalytic weights ω. Since for non-quasianalytic weights ω both $\mathscr{D}'_{(\omega)}(\Omega)$ and $\mathscr{E}_{\{\omega\}}(\Omega)$ are Köthe type PLS-spaces [76], [11], thus they are deeply reduced as all Köthe type PLS-spaces. It is also easy to show that every Fréchet Schwartz space is a deeply reduced PLS-space. On the other hand an LS-space is deeply reduced if and only if it is the dual of a countably normed Fréchet Schwartz space.

Proof of Proposition 5.1: Without loss of generality we may assume that $M = N + 1$. First, let us assume only deep reducedness in rows. Thus for $m \geq n$ we define

$$\tilde{E}_{N,m} := \overline{E_{N,n}}^{E_{N,m}} \subseteq E_{N,m}, \quad \tilde{E}_N := \text{ind}\ _{m \geq n}\tilde{E}_{N,m}.$$

Since

$$\forall\ N\ \exists\ n\ \forall\ m \geq n\ \exists\ k \geq m\quad i_N^{N+1}E_{N+1,m} \subseteq \tilde{E}_{N,k}$$

the spectra (E_N) and $(\tilde{E}_N)$ of LS-spaces are equivalent. Clearly, for $m \geq n$ the space $\tilde{E}_{N,m}$ is dense in $\tilde{E}_{N,m+1}$.

Now, assume that E is deeply reduced (also in columns!). By the above argument we may assume without loss of generality that for $m \geq n$ the space $E_{N,m}$ is dense in $E_{N,m+1}$ and that $k = m + 1$. Thus

$$\forall \ N \ \exists \ n(N) \ \forall \ m \geq n(N)$$

$$i_N^{N+1} E_{N+1,m} \subseteq \overline{i_N^{N+2} E_{N+2,m+1}}^{E_{N,m+1}} =: F_{N,m+1} \subseteq E_{N,m+1}.$$

We define $F_N := \text{ind}_{m \geq n} F_{N,m}$ and it is an LS-space such that the spectrum (F_N) is equivalent to (E_N). Since for $m \geq n(N+1)$ holds

$$i_N^{N+1} F_{N+1,m+1} \supseteq i_N^{N+1} i_{N+1}^{N+2} E_{N+2,m}$$

we have

$$(5.1) \qquad F_{N,m+1} \supseteq \overline{i_N^{N+1} F_{N+1,m+1}}^{F_{N,m+1}} \supseteq \overline{i_N^{N+2} E_{N+2,m}}^{E_{N,m}} = F_{N,m}.$$

Since for m big enough $E_{N+2,m}$ is dense in $E_{N+2,m+1}$ thus $i_N^{N+2} E_{N+2,m}$ is dense in $i_N^{N+2} E_{N+2,m+1}$ with respect to the topology of $E_{N,m+1}$. This means that $F_{N,m}$ is dense in $F_{N,m+1}$. By (5.1), also $i_N^{N+1} F_{N+1,m+1}$ is dense in $F_{N,m+1}$. ∎

We are ready to formulate the main result of this section.

Theorem 5.2 *Let X, E be PLS-spaces which are strongly reduced projective limits of LS-spaces:*

$$X = \text{proj}_{N \in \mathbb{N}} \ X_N = \text{proj}_{N \in \mathbb{N}} \ \text{ind}_{n \in \mathbb{N}} \ X_{N,n},$$
$$E = \text{proj}_{N \in \mathbb{N}} \ E_N = \text{proj}_{N \in \mathbb{N}} \ \text{ind}_{n \in \mathbb{N}} \ E_{N,n}.$$

If one of the following conditions hold:

(a) one of the spaces X or E is a deeply reduced PLN-space,

(b) both X and E are PLN-spaces,

(c) one of the spaces X or E is isomorphic to a Köthe type PLS-space of infinite order $\Lambda^\infty(A)$,

then the following conditions are equivalent:

(1) $\text{Proj}^1 (X \varepsilon E) = 0$;

(2) the pair (X, E) satisfies the condition (T), i.e.,

$$\forall \ N \ \exists \ M \geq N \ \forall \ K \geq M \ \exists \ n \ \forall \ m \geq n \ \exists \ k \geq m, S$$

$$\forall \ x \in X'_N, y \in E'_N :$$

$$\|x \circ i_N^M\|_{M,m}^* \|y \circ i_N^M\|_{M,m}^* \leq S \left(\|x\|_{N,n}^* \|y\|_{N,n}^* + \|x \circ i_N^K\|_{K,k}^* \|y \circ i_N^K\|_{K,k}^* \right).$$

(3) the pair (X, E) satisfies the condition (T_ε), i.e.,

$$\forall\, N\ \exists\, M \geq N\ \forall\, K \geq M\ \exists\, n\ \forall\, m \geq n, \varepsilon > 0\ \exists\, k \geq m, S$$

$$\forall\, x \in X'_N, y \in E'_N:$$

$$\|x \circ i_N^M\|_{M,m}^* \|y \circ i_N^M\|_{M,m}^* \leq \varepsilon \|x\|_{N,n}^* \|y\|_{N,n}^* + S\|x \circ i_N^K\|_{K,k}^* \|y \circ i_N^K\|_{K,k}^*.$$

Remarks. (1) Each one of the conditions imply that

$$\mathrm{Proj}^1\, X = \mathrm{Proj}^1\, E = 0$$

(comp. [83, 3.2.18]).

(2) Assumptions (a) and (b) are needed only to prove (3)$\Rightarrow$(1) or (2)$\Rightarrow$(1).

(3) Our result is much more general than the known results when one of the tensorized spaces is metrizable and the other dual metrizable (Vogt [78], Frerick [28], Frerick-Wengenroth [30, 74]) or recent results of Bonet and the author [6, 8].

(4) By [8, Th. 3.1, 3.4], the result is also true if one of the spaces X or E is a nuclear Fréchet space and, by [6, Th. 4.1], if one of the spaces is an LN-space. In [6] and [8] one of the spaces is represented by a "linearly ordered" family of Banach spaces – lack of such a "linear order" is the main difficulty in the proof below. We need the result as above in Theorem 5.2 having in mind applications to highly non-metrizable spaces $\mathscr{A}(U)$.

(5) Another approach to vanishing of Proj^1 on complicated spaces is given in [29].

(6) We use a convention $0 \cdot \infty = 0$.

Proof of Theorem 5.2. (1)$\Rightarrow$(2): Apply [83, Th. 3.2.18 or Th. 3.3.6] for functionals of the form $x \otimes y \in (X_N \varepsilon E_N)'$.

(2)$\Rightarrow$(3): The proof is inspired by [8, Th. 3.1 (2)$\Rightarrow$(3)]. Condition (T) implies by Braun and Vogt's lemma in the form given in [6, Lemma 4.4], the condition (P_X) and (P_Y) respectively:

$$\forall\, N_1\ \exists\, N \geq N_1\ \forall\, K \geq N\ \exists\, n_1\ \forall\, n \geq n_1, \gamma_X > 0\ \exists\, C_X, k \geq n$$

$$\forall\, x \in X'_{N_1}\ \|x \circ i_{N_1}^N\|_{N,n}^* \leq C_X \|x \circ i_{N_1}^K\|_{K,k}^* + \gamma_X \|x\|_{N_1,n_1}^*,$$

$$\forall\, \tilde{N}_1\ \exists\, \tilde{N} \geq \tilde{N}_1\ \forall\, \tilde{K} \geq \tilde{N}\ \exists\, \tilde{n}_1\ \forall\, \tilde{n} \geq \tilde{n}_1, \tilde{\gamma}_Y > 0\ \exists\, C_Y, \tilde{k} \geq \tilde{n}$$

$$\forall\, y \in E'_{\tilde{N}_1}\ \|y \circ i_{\tilde{N}_1}^{\tilde{N}}\|_{\tilde{N},\tilde{n}}^* \leq C_Y \|y \circ i_{\tilde{N}_1}^{\tilde{K}}\|_{\tilde{K},\tilde{k}}^* + \gamma_Y \|y\|_{\tilde{N}_1,\tilde{n}_1}^*.$$

Assume (T) in the form:

$$\forall\, N, \tilde{N}\ \exists\, M \geq N, \tilde{M} \geq \tilde{N}\ \forall\, K \geq M, \tilde{K} \geq \tilde{M}\ \exists\, n, \tilde{n}\ \forall\, m, \tilde{m}\ \exists\, S, k, \tilde{k}$$

$$\forall\, x \in X'_N, y \in E'_{\tilde{N}}:$$

$$\|x \circ i_N^M\|_{M,m}^* \|y \circ i_{\tilde{N}}^{\tilde{M}}\|_{\tilde{M},\tilde{m}}^* \leq S\left(\|x\|_{N,n}^* \|y\|_{\tilde{N},\tilde{n}}^* + \|x \circ i_N^K\|_{K,k}^* \|y \circ i_{\tilde{N}}^{\tilde{K}}\|_{\tilde{K},\tilde{k}}^*\right)$$

We take $x \in X'_{N_1}$, $y \in E'_{\tilde{N}_1}$. We apply (T) to $x \circ i^N_{N_1}$, $y \circ i^{\tilde{N}}_{\tilde{N}_1}$ and obtain the condition (T_1):

$$\|x \circ i^N_{N_1} \circ i^M_N\|^*_{M,m} \|y \circ i^{\tilde{N}}_{\tilde{N}_1} \circ i^{\tilde{M}}_{\tilde{N}}\|^*_{\tilde{M},\tilde{m}} \le$$
$$\le S\left(\|x \circ i^N_{N_1}\|^*_{N,n} \|y \circ i^{\tilde{N}}_{\tilde{N}_1}\|^*_{\tilde{N},\tilde{n}} + \|x \circ i^N_{N_1} \circ i^K_N\|^*_{K,k} \|y \circ i^{\tilde{N}}_{\tilde{N}_1} \circ i^{\tilde{K}}_{\tilde{N}}\|^*_{\tilde{K},\tilde{k}}\right).$$

If $\|y \circ i^{\tilde{N}}_{\tilde{N}_1}\|^*_{\tilde{N},\tilde{n}} \le \|y \circ i^{\tilde{K}}_{\tilde{N}_1}\|^*_{\tilde{K},\tilde{k}}$ then we apply (P_X) to (T_1) and choose $\gamma_X \le \varepsilon/(2S)$. Otherwise we apply first (P_Y) to (T_1) and then (P_X) to the term with constant SC_Y and take $\gamma_Y \le \varepsilon/(2S)$, $\gamma_X \le \varepsilon/(2SC_Y)$. This completes the proof of $(2) \Rightarrow (3)$ (details are left to the reader).

Before we prove the implication $(3) \Rightarrow (1)$ we need some auxiliary results.

Proposition 5.3 *If a pair of PLS-spaces (X, E) satisfies (T_ε) then if $(E_{N,n})$ are chosen to be reflexive for all N, n the same holds for all $y \in E'_{N,k}$ and the pair (X, E) satisfies also (TB_ε), i.e.,*

$$\forall\, N, \tilde{N}\ \exists\, M \ge N, \tilde{M} \ge \tilde{N}\ \forall\, K \ge M, \tilde{K} \ge \tilde{M}\ \exists\, n, \tilde{n}\ \forall\, m \ge n, \tilde{m} \ge \tilde{n},$$
$$\forall\, \varepsilon > 0\ \exists\, k \ge m, \tilde{k} \ge \tilde{m}, S\ \forall\, y \in E'_{\tilde{N},\tilde{k}}:$$
$$\|y \circ i^{\tilde{M}}_{\tilde{N}}\|^*_{\tilde{M},\tilde{m}} i^M_N B_{M,m} \subseteq \varepsilon\|y\|^*_{\tilde{N},\tilde{n}} B_{N,n} + S\|y \circ i^{\tilde{K}}_{\tilde{N}}\|^*_{\tilde{K},\tilde{k}} i^K_N B_{K,k}.$$

Moreover, if E is deeply reduced the above condition holds for $y \in \overline{E'_{\tilde{N},\tilde{k}}}$, where the closure is taken with respect to the standard topology of $E'_{\tilde{N},\tilde{n}} \cap E'_{\tilde{K},\tilde{k}}$:

Proof. By [8, Lemma 3.5 (b)], we obtain from (T_ε) the condition (TB_ε) but only for $y \in E'_{\tilde{N}}$.

Since $E_{\tilde{N},\tilde{k}} \hookrightarrow E_{\tilde{N}}$ is injective and $E_{\tilde{N},\tilde{k}}$, $E_{\tilde{N}}$ can be assumed reflexive, the dual map $\pi_{\tilde{k}} : E'_{\tilde{N}} \to E'_{\tilde{N},\tilde{k}}$ has a dense image. We define analogously $\pi^{\tilde{k}}_{\tilde{n}} : E'_{\tilde{N},\tilde{k}} \to E'_{\tilde{N},\tilde{n}}$. Let us take $z \in E'_{\tilde{N},\tilde{k}}$, $v \in B_{M,m}$. Then there is a sequence $(y_j) \subseteq E'_{\tilde{N}}$ such that $\pi_{\tilde{k}} y_j \to z$ in $E'_{\tilde{N},\tilde{k}}$. We get

$$\|\pi^{\tilde{k}}_{\tilde{m}} \pi_{\tilde{k}} y_j \circ i^{\tilde{M}}_{\tilde{N}}\|^*_{\tilde{M},\tilde{m}} i^M_N v = \varepsilon\|\pi^{\tilde{k}}_{\tilde{n}} \pi_{\tilde{k}} y_j\|^*_{\tilde{N},\tilde{n}} a_j + S\|\pi_{\tilde{k}} y_j \circ i^{\tilde{K}}_{\tilde{N}}\|^*_{\tilde{K},\tilde{k}} i^K_N b_j,$$

where $(a_j) \subseteq B_{N,n}$, $(b_j) \subseteq B_{K,k}$. Since sets $B_{N,n}$, $B_{K,k}$ are compact in X_N and X_K, and since every compact set of an LS-space is metrizable [15], we can find a suitable subsequence of j such that the corresponding subsequences of (a_j) and (b_j) are convergent to $a \in B_{N,n}$ and $b \in B_{K,k}$. We have proved that

$$\forall\, N, \tilde{N}\ \exists\, M \ge N, \tilde{M} \ge \tilde{N}\ \forall\, K \ge M, \tilde{K} \ge \tilde{M}$$
$$\exists\, n, \tilde{n}\ \forall\, m \ge n, \tilde{m} \ge \tilde{n}, \varepsilon > 0\ \exists\, k \ge m, \tilde{k} \ge \tilde{m},\ \exists\, S > 0\ \forall\, z \in E'_{\tilde{N},\tilde{k}}$$
$$\|\pi^{\tilde{k}}_{\tilde{m}} z \circ i^{\tilde{M}}_{\tilde{N}}\|^*_{\tilde{M},\tilde{m}} i^M_N B_{M,m} \subseteq \varepsilon\|\pi^{\tilde{k}}_{\tilde{n}} z\|^*_{\tilde{N},\tilde{n}} B_{N,n} + S\|z \circ i^{\tilde{K}}_{\tilde{N}}\|^*_{\tilde{K},\tilde{k}} i^K_N B_{K,k}.$$

If E is deeply reduced then we may (and we will) assume that $E'_{\tilde{N},\tilde{k}}$ injectively embeds continuously into $E'_{\tilde{N},\tilde{n}}$ and $E'_{\tilde{K},\tilde{k}}$. Clearly, then for $z \in \overline{E'_{\tilde{N},\tilde{k}}}$ we can find $y_j \in E'_{\tilde{N},\tilde{k}}$ which tends to z both in the norm of $E'_{\tilde{N},\tilde{n}}$ and $E'_{\tilde{K},\tilde{k}}$. The same argument as above shows the required inequality.

The first statement can be proved by the same trick. ∎

Proposition 5.4 *If a pair (X, E) of PLS-spaces satisfies (T_ε) for strongly reduced projective representations of X, E then either X is strict or E is deeply reduced in rows and either E is deeply reduced in columns or X is a Fréchet space.*

Proof. Let us assume that E is not deeply reduced in rows. Hence

$$\exists\ N\ \forall\ M\ \forall\ n\ \exists\ m\ \forall\ k\ \exists\ y \in E'_{N,k}\quad \pi_m^k(y \circ i_N^M) \neq 0\quad \text{but}\quad \pi_n^k y = 0,$$

where $\pi_m^k : E'_{M,k} \to E'_{M,m}, \pi_n^k : E'_{N,k} \to E'_{N,n}$ are restriction maps. By Proposition 5.3, the condition (TB_ε) holds, so we get

$$\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \forall\ m\ \exists\ k \geq m, S\quad i_N^M B_{M,m} \subseteq S i_N^K B_{K,k}.$$

Therefore

$$\forall\ N\ \exists\ M(N) \geq N\ \forall\ K \geq M\quad i_N^M X_M \subseteq i_N^K X_K.$$

We get inductively a sequence (N_j), $N_{j+1} = M(N_j)$, and the strict spectrum $(X_{N_j}/\ker i_{N_{j-1}}^{N_j})$ is equivalent to the given one.

If X is not a Fréchet space, then

$$\exists\ N_0\ \forall\ N \geq N_0\ \forall\ M \geq N\ \forall\ n\ \exists\ m_0\ \forall\ m \geq m_0\ \exists\ (x_j) \subseteq X'_N$$
$$\frac{\|x_j \circ i_N^M\|_{M,m}^*}{\|x_j\|_{N,n}^*} \to \infty\quad \text{for } j \to \infty.$$

Let us take (T_ε) and apply it for x_j and $y \in E'_{N,k}$ such that $y \circ i_N^K|_{E_{K,k}} = 0$ by Prop. 5.3. Thus for every $j > 0$ we get

$$\|y \circ i_N^M\|_{M,m}^* \cdot \|x_j \circ i_N^M\|_{M,m}^* \leq \varepsilon \|y\|_{N,n}^* \cdot \|x_j\|_{N,n}^*.$$

We have proved that for any $\varepsilon > 0$

$$\|y \circ i_N^M\|_{M,m}^* \leq \varepsilon \|y\|_{N,n}^*.$$

If $y \in E'_{N,k}$ belongs to the polar of $\varepsilon B_{N,n} + i_N^K E_{K,k}$ then it belongs to the polar of $i_N^M B_{M,m}$. Therefore, for every $\varepsilon > 0$ we get

$$i_N^M B_{M,m} \subseteq \overline{\varepsilon B_{N,n} + i_N^K E_{K,k}}^{E_{N,k}}.$$

Since $B_{N,n} \subseteq B_{N,k}$ we get

$$i_N^M B_{M,m} \subseteq \overline{i_N^K E_{K,k}}^{E_{N,k}} .$$

We have proved that E is deeply reduced in columns.   ■

Next result is a slight modification of [21, Lemma 2.2] (originally the result is also in [61]). We omit the proof based on the spectral theory of unbounded operators since it is a modification of the proof in [21, Lemma 2.2].

Theorem 5.5 *Let H, H_0, H_1 be Hilbert spaces and let U be a topological vector space. Let us assume that the following diagram of injective continuous linear maps commutes*

$$
\begin{array}{ccc}
H & \xrightarrow{\;j_1\;} & H_1 \\
{\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle u_1} \\
H_0 & \xrightarrow[\;u_0\;]{} & U.
\end{array}
$$

Let $j_1(H)$ and $j_0(H)$ be dense in H_1 and H_0 respectively. Moreover, let H be complete in the norm:

$$\|x\| := \left(\|j_1 x\|_{H_1}^2 + \|j_0 x\|_{H_0}^2 \right)^{1/2} .$$

Then for any $\varepsilon > 0$ there are: a set I, positive weights $v \colon I \to \mathbb{R}_+$, $w \colon I \to \mathbb{R}_+$ and isomorphisms

$$T : H \to \ell_2(I), \quad V : H_0 \to \ell_2(I, v), \quad W : H_1 \to \ell_2(I, w)$$

such that the following diagram commutes:

$$
\begin{array}{ccccc}
\ell_2(I) & \xrightarrow{\;\mathrm{id}\;} & \ell_2(I) & \xrightarrow{\;\beta\;} & \ell_2(I, w) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle W^{-1}} \\
\ell_2(I) & \xrightarrow{\;T^{-1}\;} & H & \xrightarrow{\;j_1\;} & H_1 \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle u_1} \\
\ell_2(I, v) & \xrightarrow[\;V^{-1}\;]{} & H_0 & \xrightarrow[\;u_0\;]{} & U,
\end{array}
$$

where α, β are natural embeddings and

$$\|T\| \cdot \|T^{-1}\|, \|V\| \cdot \|V^{-1}\|, \|W\| \cdot \|W^{-1}\| < 1 + \varepsilon.$$

Proof of Theorem 5.2. $(3) \Rightarrow (1)$: (a): Assume that E is a deeply reduced PLN-space. Without loss of generality we may assume that for every N, $n \in \mathbb{N}$ big enough:

- $E'_{N,n}$ is a Hilbert space;
- $E'_{N,n} \to E'_{N+1,n}$ and $E'_{N,n+1} \to E'_{N,n}$ are injective Hilbert-Schmidt maps of the Hilbert-Schmidt norm ν_2 and operator norm ≤ 1;
- $E'_{N,n+1}$ is dense in $E'_{N,n}$ (since $E_{N,n} \hookrightarrow E_{N,n+1}$ is injective).

Applying (T_ε) we get, by Proposition 5.3, for every $y \in \overline{E'_{N,k}}^{E'_{N,n} \cap E'_{K,k}}$ the following inclusion:

$$(5.2) \qquad \|y \circ i_N^M\|^*_{M,m} i_N^M B_{M,m} \subseteq \varepsilon \|y\|^*_{N,n} B_{N,n} + S\|y \circ i_N^K\|^*_{K,k} i_N^K B_{K,k}.$$

We define

$$H := \overline{E'_{N,k+1}}^{E'_{N,n+1} \cap E'_{K,k+1}} \subseteq \overline{E'_{N,k}}^{E'_{N,n} \cap E'_{K,k}}$$

here we equip H with the Hilbert norm from the intersection space $E'_{N,n+1} \cap E'_{K,k+1}$. Since

$$E'_{N,k+1} \subseteq H_1 := \left(\ker\left(i_N^K|_{E_{K,k+1}}\right)\right)^\perp \subseteq E'_{K,k+1},$$

we have

$$H = \overline{E'_{N,k+1}}^{E'_{N,n+1} \cap H_1}$$

and the image of H is dense in H_1. Obviously H is dense in $H_0 := E'_{N,n+1}$. If we define $U = E'_{K,n}$ then we are exactly in the situation described in Theorem 5.5.

The natural maps

$$\pi_0 : H_0 \to E'_{N,n}, \qquad \pi_1 : H_1 \to E'_{K,k}$$

are Hilbert-Schmidt injections of Hilbert-Schmidt norm ≤ 1. Moreover, by Prop. 5.3, the natural map $E'_{N,k+1} \to E'_{M,m}$ is continuous when the domain space is equipped with the Hilbert norm of $E'_{N,n+1} \cap E'_{K,k+1}$. It extends to a continuous linear map $r : H \to E'_{M,m}$.

Let $R \in W(U_{M,m}, B_{M,m})$. Using the map T obtained via Theorem 5.5, we define

$$t_i := R(rT^{-1}(e_i)) \in \|rT^{-1}e_i\|^*_{M,m} B_{M,m}.$$

Using (5.2) we find

$$\tilde{w}_i \in S\|\pi_1 j_1 T^{-1}(e_i)\|^*_{K,k} B_{K,k}, \quad \tilde{v}_i \in \varepsilon\|\pi_0 j_0 T^{-1}(e_i)\|^*_{N,n} B_{N,n}$$

such that

$$i_N^M t_i = i_N^K \tilde{w}_i + \tilde{v}_i, \qquad \text{for all } i.$$

We define

$$\tilde{W} : H_1 \to X_{K,k}, \quad \tilde{W}(x) := \sum_i \frac{1}{w_i^2} \langle Wx, e_i \rangle_{\ell_2(I,w)} \tilde{w}_i$$

and

$$\tilde{V} : H_0 \to X_{N,n}, \quad \tilde{V}(x) := \sum_i \frac{1}{v_i^2} \langle Vx, e_i \rangle_{\ell_2(I,v)} \tilde{v}_i.$$

We will show that they are well defined and continuous. Let $x \in H_1$,

$$\|\tilde{W}(x)\|_{K,k} \leq \sum_i \left| \frac{1}{w_i^2} \langle Wx, e_i \rangle_{\ell_2(I,w)} \right| \cdot \|\tilde{w}_i\|_{K,k}$$

$$\leq S \sum_i \left| \left\langle Wx, \frac{e_i}{w_i} \right\rangle_{\ell_2(I,w)} \right| \cdot \left\| \pi_1 j_1 T^{-1} \left(\frac{e_i}{w_i} \right) \right\|_{K,k}$$

$$\leq S \left(\sum_i \left| \left\langle Wx, \frac{e_i}{w_i} \right\rangle_{\ell_2(I,w)} \right|^2 \right)^{1/2} \cdot \left(\sum_i \left\| \pi_1 j_1 T^{-1} \left(\frac{e_i}{w_i} \right) \right\|_{K,k}^2 \right)^{1/2}$$

$$= S \|Wx\|_{\ell_2(I,w)} \nu_2(\pi_1 \circ W^{-1}) \leq S(1+\varepsilon) \|x\|_{H_1} (1+\varepsilon) \nu_2(\pi_1).$$

Analogously, for $x \in H_0$,

$$\|\tilde{V}(x)\|_{N,n} \leq \sum_i \left| \frac{1}{v_i^2} \langle Vx, e_i \rangle_{\ell_2(I,v)} \right| \cdot \left\| \pi_0 j_0 T^{-1} \left(\frac{e_i}{v_i} \right) \right\|_{N,n}^*$$

$$\leq \varepsilon(1+\varepsilon) \|x\|_{H_0} (1+\varepsilon) \nu_2(\pi_0).$$

We have proved that

$$\|\tilde{W}\| \leq S(1+\varepsilon)^2, \qquad \|\tilde{V}\| \leq \varepsilon(1+\varepsilon)^2.$$

Since H_1 is a closed Hilbert subspace of $E'_{K,k+1}$ we can extend $\tilde{W}$ with the same norm onto $E'_{K,k+1}$.

Let us take $x \in E'_N \subseteq E'_{N,k+1} \subseteq H$. We calculate

$$i_N^M \circ R \circ (i_N^M)' x = i_N^M \circ R \circ rx = i_N^M \circ R \circ r \circ T^{-1} \circ Tx$$

$$= \sum_i i_N^M \circ R \circ r \circ T^{-1}(e_i) \langle Tx, e_i \rangle_{\ell_2(I)} = \sum_i i_N^M t_i \langle Tx, e_i \rangle_{\ell_2(I)}$$

$$= \sum_i i_N^K \tilde{w}_i \langle Tx, e_i \rangle_{\ell_2(I)} + \sum_i \tilde{v}_i \langle Tx, e_i \rangle_{\ell_2(I)}$$

$$= \sum_i i_N^K \tilde{w}_i \langle Wj_1 x, e_i \rangle_{\ell_2(I)} + \sum_i \tilde{v}_i \langle Vj_0 x, e_i \rangle_{\ell_2(I)}$$

$$= i_N^K \tilde{W}(j_1 x) + \tilde{V}(j_0 x).$$

We have proved that

$$i_N^M \circ W(U_{M,m}, B_{M,m}) \circ (i_N^M)' \subseteq S(1+\varepsilon)^2 i_N^K \circ W(U_{K,k+1}, B_{K,k}) \circ (i_N^K)'$$
$$+\varepsilon(1+\varepsilon)^2 W(U_{N,n+1}, B_{N,n}).$$

By Theorem 4.2 and [83, Th. 3.2.14], this implies $\mathrm{Proj}^1 (X\varepsilon E) = 0$.

(b): By the Remark (4) after Theorem 5.2, we may assume that none of the spaces is Fréchet. If both are strict then as in the proof of Proposition 4.5 we show surjectivity of

$$i_N^{N+1} \otimes i_N^{N+1} : X_{N+1}\varepsilon E_{N+1} \to X_N \varepsilon E_N.$$

Thus the spectrum is strict and automatically $\mathrm{Proj}^1 X\varepsilon E = 0$. We may assume that X is neither strict nor Fréchet, then Prop. 5.4 implies that E is deeply reduced. Apply part (a).

(c): By Theorem 4.2 and [83, Th. 3.2.14], we need to prove

$$\forall\, N, \tilde{N}, \; \exists\, M \geq N, \tilde{M} \geq \tilde{N}\; \forall\, K \geq M, \tilde{K} \geq \tilde{M}$$

(5.3)
$$\exists\, n, \tilde{n}\; \forall\, m \geq n, \tilde{m} \geq \tilde{n}, \varepsilon > 0\; \exists\, k \geq m, \tilde{k} \geq \tilde{m}, \; \exists\, S > 0$$
$$i_N^M \circ W(U_{\tilde{M},\tilde{m}}, B_{M,m}) \circ (i_{\tilde{N}}^{\tilde{M}})' \subseteq S i_N^K \circ W(U_{\tilde{K},\tilde{k}}, B_{K,k}) \circ (i_{\tilde{N}}^{\tilde{K}})' +$$
$$+ \varepsilon W(U_{\tilde{N},\tilde{n}}, B_{N,n}).$$

We assume first that $E = \Lambda^\infty(B)$, $b_{1,n}(i) > 0$ for each n. Let e_i be the unit vector in E', then $\|e_i\|_{N,n}^* = 1/b_{N,n}(i)$. Thus, by (T_ε) for $y = e_i$ and $x \in X'_N$ we get

$$\frac{\|x \circ i_N^M\|_{M,m}^*}{b_{\tilde{M},\tilde{m}}(i)} \leq \varepsilon\frac{\|x\|_{N,n}^*}{b_{\tilde{N},\tilde{n}}(i)} + S\frac{\|x \circ i_N^K\|_{K,k}^*}{b_{\tilde{K},\tilde{k}}(i)}.$$

By [8, Lemma 3.5], we get

(5.4)
$$\frac{i_N^M B_{M,m}}{b_{\tilde{M},\tilde{m}}(i)} \subseteq \frac{2\varepsilon B_{N,n}}{b_{\tilde{N},\tilde{n}}(i)} + \frac{2S i_N^K B_{K,k}}{b_{\tilde{K},\tilde{k}}(i)}.$$

We identify

$$W(U_{\tilde{M},\tilde{m}}, B_{M,m}) \subseteq L(E'_{\tilde{M}}, X_M)$$

with a subset of the space of vector valued sequences:

$$L(E'_{\tilde{M}}, X_M) = \{u = (u(i))_{i\in\mathbb{N}} \subseteq X_M : \; \exists\, m, \tilde{m}\; \sup_i b_{\tilde{M},\tilde{m}}(i)\|u(i)\|_{M,m} < \infty\}.$$

In particular,

$$u = (u(i))_{i\in\mathbb{N}} \in W(U_{\tilde{M},\tilde{m}}, B_{M,m})$$

if and only if

$$u(i) \in \left(b_{\tilde{M},\tilde{m}}(i)\right)^{-1} B_{M,m} \quad \text{for every } i \in \mathbb{N}.$$

By (5.4), we get

$$i_N^M u(i) = 2\varepsilon v(i) + 2S i_N^K w(i) \quad \text{for each } i \in \mathbb{N},$$

where

$$v(i) \in \left(b_{\tilde{N}, \tilde{n}}(i)\right)^{-1} B_{N,n}, \qquad w(i) \in \left(b_{\tilde{K}, \tilde{k}}(i)\right)^{-1} B_{K,k}.$$

We define

$$v \in W(U_{\tilde{N}, \tilde{n}}, B_{N,n}) \subseteq L(E'_{\tilde{N}}, X_N), \quad w \in W(U_{\tilde{K}, \tilde{k}}, B_{K,k}) \subseteq L(E'_{\tilde{K}}, X_K)$$

by

$$v(x) := (v(i)x(i))_{i \in \mathbb{N}}, \quad w(z) := (w(i)z(i))_{i \in \mathbb{N}},$$

for

$$x = (x(i)) \in E'_{\tilde{N}}, \quad z = (z(i)) \in E'_{\tilde{K}}.$$

Obviously,

$$i_N^M \circ u \circ \left(i_{\tilde{N}}^{\tilde{M}}\right)' = 2\varepsilon v + 2S i_N^K \circ w \circ \left(i_{\tilde{N}}^{\tilde{K}}\right)',$$

which implies (5.3) with changed ε and S.

In the general case $E = \Lambda^\infty(B)$ is a countable product of spaces $E^{(j)}$ for which we have proved $\operatorname{Proj}^1\left(X\varepsilon E^{(j)}\right) = 0$. Apply Proposition 4.8. ∎

Next corollary is a generalization of the famous (DN)-(Ω) splitting theorem [57, Th. 30.1] or [82, Satz 1.9] which means, in particular, that if E is an LS-space, $E' \in (DN)$, and X is a Fréchet Schwartz space, $X \in (\Omega)$, then $\operatorname{Proj}^1\left(X\varepsilon E'\right) = 0$ whenever one of the spaces is nuclear. Observe that in that case both E and X have the dual interpolation estimate for big θ. It also generalizes results of Grothendieck [32, II.§4, no. 3, Prop. 15] and Vogt [77, Th. 4.9].

Corollary 5.6 *Let X, E be PLS-spaces. Assume that both spaces are PLN-spaces or one of them is a deeply reduced PLN-space. If E and X have the dual interpolation estimate for small (big) θ then $L_b(E'_b, X) = X\varepsilon E$ is ultrabornological and $\operatorname{Proj}^1(X\varepsilon E) = 0$.*

Remark. Results on inheritance of dual interpolation estimates for Schwartz (tensor) products of PLS-spaces were very recently obtained by Piszczek using the results of this paper, see [67].

Proof. Assume that

$$\forall\, N \ \exists\, M \geq N \ \forall\, K \geq M \ \exists\, n \ \forall\, m \geq n \ \exists\, \theta_0 \ \forall\, \theta \in]0, \theta_0[\ \exists\, k, C$$
$$\forall\, x \in X'_N, y \in E'_N$$
$$\|x \circ i_N^M\|_{M,m}^* \leq C \left(\|x\|_{N,n}^*\right)^\theta \cdot \left(\|x \circ i_N^K\|_{K,k}^*\right)^{1-\theta}$$
$$\|y \circ i_N^M\|_{M,m}^* \leq C \left(\|y\|_{N,n}^*\right)^\theta \cdot \left(\|y \circ i_N^K\|_{K,k}^*\right)^{1-\theta}.$$

Multiplying this inequalities we get easily (T). Apply Theorems 5.2 and 4.1. ∎

6. Conditions with one fixed space

The previous section gave a characterization of vanishing of $\mathrm{Proj}^1\, X \varepsilon E$ for a large class of PLS-spaces E, X. We fix one space E and we characterize those spaces X such that $\mathrm{Proj}^1\, X \varepsilon E = 0$

Corollary 6.1 *Let X be a PLS-space, U a real analytic manifold, then $\mathrm{Proj}^1\, X \varepsilon \mathscr{A}(U) = 0$ if and only if X satisfies the dual interpolation estimate for small θ (for U connected non-compact) or if and only if X satisfies $(P\overline{\overline{\Omega}})$ (for U compact).*

Proof.   Sufficiency for the non-compact case follows from Corollary 5.6 and 2.3 since $\mathscr{A}(U)$ is deeply reduced. For the compact case use Proposition 2.1 and [6, Cor. 3.8, 7.2].

Necessity. The function theoretic argument from the proof of [6, Th. 7.8] gives numbers $1 > a > b$ and a sequence of functionals $(g_j) \subset \mathscr{A}(\mathbb{R})'$ such that for $j \in \mathbb{N}$

$$1 = \|g_j\|_{N,n}^* > a^j = \|g_j\|_{M,m}^* > b^j = \|g_j\|_{K,k}^*.$$

Since, by Proposition 2.1, $\mathscr{A}(\mathbb{R})$ is a quotient of $\mathscr{A}(U)$ for non-compact connected U then we may assume that $(g_j) \subset \mathscr{A}(U)'$. Plugging the constructed functions into the condition (T) from Theorem 5.2 we get analogously as in the proof of [6, Th. 7.8] that for some $\eta > 0$, big r and all $y \in X_N'$

$$\|y \circ i_N^M\|_{M,m}^* \leq C \left(\|y \circ i_N^K\|_{K,k}^* r^{-\eta} + \|y\|_{N,n}^* r \right)$$

which is equivalent to $(P\underline{A})$.

On the other hand, $\mathscr{A}(U) \overset{comp}{\supseteq} \mathscr{A}(\mathbb{T}) \simeq \Lambda_0'(j)$ (see the proof of Proposition 2.1 and [19, Th. 5.4]) thus $\mathrm{Proj}^1\, X \varepsilon \Lambda_0'(j) = 0$. From [6, Cor. 3.8, 7.2] it follows that X has $(P\overline{\overline{\Omega}})$.   ∎

Corollary 6.2 *Let X be a PLS-space, $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open set, ω non-quasianalytic. $\mathrm{Proj}^1\, X \varepsilon \mathscr{E}_{\{\omega\}}(\Omega) = 0$ if and only if X satisfies $(P\overline{\overline{\Omega}})$. Analogously, $\mathrm{Proj}^1\, X \varepsilon \mathscr{D}'_{(\omega)}(\Omega) = 0$ if and only if X satisfies $(P\Omega)$.*

Proof.   Apply Proposition 4.8 and [6, Cor. 7.2, 3.8]. By [11, Prop. 4.9, 5.6],

$$\mathscr{E}_{\{\omega\}}(\Omega) \simeq \prod \Lambda_0(\alpha)', \qquad \mathscr{D}'_{(\omega)}(\Omega) \simeq \prod \Lambda_\infty(\beta)'.$$

   ∎

Remark.   The same holds for the space of currents $\mathscr{D}'(\Omega)$ over arbitrary C^∞-manifold Ω instead of $\mathscr{D}'_{(\omega)}(\Omega)$. In that case sufficiency follows from the

remark after Corollary 2.3. Necessity follows from the fact that $\mathscr{D}'(\mathbb{T})$ is isomorphic to a complemented subspace of $\mathscr{D}'(\Omega)$. Clearly $\mathscr{D}'(\mathbb{T})$ are just periodic currents on $\mathbb{R}$ so it is isomorphic to $\Lambda_0'(\log j)$.

Now, let us apply the results on the vanishing of Proj^1 to the surjectivity problem:

Corollary 6.3 *Let* $T : \mathscr{D}'(\Omega_1) \to \mathscr{D}'(\Omega_1)$, $S : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_2)$ *be surjective operators,* Ω_1, $\Omega_2 \subseteq \mathbb{R}^d$ *arbitrary open or* C^∞*-manifolds. Then*

$$T \otimes S : \mathscr{D}'(\Omega_1 \times \Omega_2) \to \mathscr{D}'(\Omega_1 \times \Omega_2)$$

is surjective if and only if $\ker T$ *and* $\ker S$ *have* $(P\Omega)$.

An analogous result holds for

$$T : \mathscr{D}'_{(\omega_1)}(\Omega_1) \to \mathscr{D}'_{(\omega_1)}(\Omega_1) \quad and \quad S : \mathscr{D}'_{(\omega_2)}(\Omega_2) \to \mathscr{D}'_{(\omega_2)}(\Omega_2).$$

Proof. Apply Cor. 4.6, Cor. 6.2 and the remark below it. Observe that

$$\mathrm{Proj}^1 \left(\mathscr{D}'_{(\omega_1)}(\Omega_1) \varepsilon \mathscr{D}'_{(\omega_2)}(\Omega_2) \right) = 0 \qquad \text{by Cor. 2.3 and Cor. 5.6.} \qquad \blacksquare$$

Next corollary (Theorem C) is crucial for the whole paper giving a criterion of the real analytic parameter dependence of linear operators on spaces of distributions or currents.

Corollary 6.4 *Let* $T : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ *be a surjective operator then*

$$T \otimes \mathrm{id}_{\mathscr{A}(U)} : \mathscr{D}'_{(\omega)}(\Omega) \varepsilon \mathscr{A}(U) \to \mathscr{D}'_{(\omega)}(\Omega) \varepsilon \mathscr{A}(U)$$

is surjective if and only if $\ker T$ *has the dual interpolation estimate for small* θ *(for non-compact connected* U*) or if and only if* $\ker T$ *has* $(P\overline{\overline{\Omega}})$ *(for compact* U*). Therefore surjectivity does not depend on the choice of an open set* $U \subseteq \mathbb{R}^n$.

Let us emphasize that the result is true, for instance, for variable coefficients differential operators. The dual interpolation estimate (2.1) is a kind of a very strong abstract "Hadamard three circle theorem" as explained in Section 1.

Proof. Apply Cor. 4.6 and Cor. 6.1. Observe that

$$\mathrm{Proj}^1 \ \mathscr{D}'_{(\omega)}(\Omega) \varepsilon \mathscr{A}(U) = 0 \qquad \text{by Cor. 5.6 and Cor. 2.3.} \qquad \blacksquare$$

Corollary 6.5 *Let* $T : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ *be a surjective operator and let* ω_1 *be a non-quasianalytic weight. Then*

$$T \otimes \mathrm{id}_{\mathscr{E}_{\{\omega_1\}}(U)} : \mathscr{D}'_{(\omega)}(\Omega) \varepsilon \mathscr{E}_{\{\omega_1\}}(U) \to \mathscr{D}'_{(\omega)}(\Omega) \varepsilon \mathscr{E}_{\{\omega_1\}}(U)$$

is surjective if and only if $\ker T$ *has* $(P\overline{\overline{\Omega}})$.

Proof. Apply Cor. 4.6, Cor. 6.2 and Cor. 2.3. $\qquad \blacksquare$

Clearly there are plenty of other results of that type – we leave their formulation to the reader. Just for fun we give the following example of surjectivity results for products of differential operators.

Corollary 6.6 *Let*

$$P_1(x, D) := \sum_{|\alpha| \leq m} a_\alpha(x)\partial_x^\alpha : \mathscr{D}'(\Omega_1) \to \mathscr{D}'(\Omega_1),$$

$$P_2(y, D) := \sum_{|\beta| \leq k} b_\beta(y)\partial_y^\beta : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_2)$$

be two linear surjective partial differential operators, $x \in \Omega_1$, $y \in \Omega_2$. The operator

$$P_1(x, D)P_2(y, D) = \sum_{|\alpha| \leq m, |\beta| \leq k} a_\alpha(x)b_\beta(y)\partial_x^\alpha\partial_y^\beta : \mathscr{D}'(\Omega_1 \times \Omega_2) \to \mathscr{D}'(\Omega_1 \times \Omega_2)$$

$(x, y) \in \Omega_1 \times \Omega_2$, is surjective if and only if $\ker P_1(x, D)$ and $\ker P_2(x, D)$ have $(P\Omega)$.

The above results are worth studying mostly if we have a method of evaluating the dual interpolation estimates or P-conditions for the kernel of the map. Using known descriptions of kernels of convolution operators we can prove the following characterizations:

Corollary 6.7 *Let ω be a non-quasianalytic weight and let $\mu \in \mathscr{D}'_{(\omega)}(\mathbb{R})$ be an ultradistribution with compact support. The convolution operator*

(6.1) $$T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\mathbb{R}, \mathscr{A}(U))$$

is surjective if and only if

(6.2) $$T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}) \to \mathscr{D}'_{(\omega)}(\mathbb{R})$$

has a linear continuous right inverse.

Remark. A characterization of existence of right inverse for T_μ in terms of $\hat{\mu}$ is given in [56, Th. 3.9, 2.1, 4.4, Prop. 4.6] and [42].

Proof. Sufficiency is obvious.

Necessity for distributions. A characterization of surjectivity of (6.2) is given in [27, Th. 3.4]. If (6.1) is surjective then (6.2) must be surjective. Then, by [27, Th. 2.10], $\ker T_\mu \subseteq \mathscr{D}'_{(\omega)}(\mathbb{R})$, $\dim \ker T_\mu = \infty$, is isomorphic to $\Lambda_{\infty,\infty}(\alpha, \beta)$, where $\alpha_j = |\operatorname{Im} a_j|$, $\beta_j = \omega(a_j)$ and (a_j) is a sequence with multiplicities of zeros of the Fourier transform $\hat{\mu}$ of μ. By Corollary 6.4 and Corollary 2.3, $\ker T_\mu$ is an LS-space. This completes the proof by [56, Th. 3.9 and Th. 2.1]. ∎

Unfortunately, it seems to be very difficult to check interpolation estimates for general operators or even differential operators with variable coefficients. In case of constant coefficients differential operators we may apply the Fundamental Principle of Ehrenpreis and Palamodov. This is the content of the next section.

7. Evaluation of the dual interpolation estimates via Phragmén-Lindelöf type conditions

First we need some notation and we introduce a family of Phragmén-Lindelöf type conditions. Assume from now on that V is an algebraic variety, $\Omega \subseteq \mathbb{R}^d$ is a convex open set and ω is a weight (in the sense of definitions of spaces $\mathscr{D}'_{(\omega)}(\Omega)$). We fix a convex compact exhaustion (K_N) of Ω. We define $PSH(V)$ to be the set of plurisubharmonic functions $u : V \to [-\infty, \infty[$, i.e., locally bounded from above and plurisubharmonic at regular points of V. We assume that the values in singular points of V are defined by $u(z) := \limsup_{\xi \in V_{reg}, \xi \to z} u(\xi)$ for $z \in V_{sing}$. For a convex compact set $\emptyset \neq K \subseteq \mathbb{R}^d$ we define its support function

$$h_K : \mathbb{R}^d \to \mathbb{R}, \quad h_K(\xi) := \sup\{\langle \xi, x \rangle : x \in K\},$$

moreover, for any $N \in \mathbb{N}$ we define $h_K := h_{K_N}$. Then

$$\mathrm{PSH}(V, N) := \{u \in \mathrm{PSH}(V) : \forall\, n \in \mathbb{N}\ \exists\, C_0\ \forall\, z \in V$$
$$u(z) \leq h_N(\mathrm{Im}\, z) - n\omega(z) + C_0\}.$$

We say that the variety V satisfies $\mathrm{IPL}(\Omega, \omega)$ if and only if

$$\forall\, N\ \exists\, M \geq N\ \forall\, K \geq M\ \exists\, n \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \forall\, \theta \in\,]0, 1[\ \exists\, k \in \mathbb{N}, C$$
$$\forall\, t \in \mathbb{R}\ \forall\, u \in \mathrm{PSH}(V, N) \qquad (a) + (b) \Rightarrow (c),$$

where

(a) $\forall\, z \in V \quad u(z) \leq h_N(\mathrm{Im}\, z) - n\omega(z) + t$;

(b) $\forall\, z \in V \quad u(z) \leq h_K(\mathrm{Im}\, z) - k\omega(z)$;

(c) $\forall\, z \in V \quad u(z) \leq h_M(\mathrm{Im}\, z) - m\omega(z) + \theta t + C$.

If all the conditions are considered only for $t \geq 0$ (or $t \leq 0$) we write $\mathrm{IPL}_+(\Omega, \omega)$ (or $\mathrm{IPL}_-(\Omega, \omega)$) and say that V satisfies the IPL condition for *positive* (*negative*) t. Similarly, if we replace the quantifier $\forall\, \theta \in\,]0, 1[$ by $\exists\, \theta_0 \in\,]0, 1[\ \forall\, \theta \in\,]0, \theta_0[$ we say that IPL is satisfied for *small* θ and denote it by $\mathrm{IPL}^0(\Omega, \omega)$. Analogously, if we put $\exists\, \theta_0 \in\,]0, 1[\ \forall\, \theta \in\,]\theta_0, 1[$ then we call the condition for *big* θ and denote it by $\mathrm{IPL}^1(\Omega, \omega)$. Of course, all the possible combinations of subscripts and superscripts are possible so we have defined nine versions of IPL.

It is easy to observe that $\mathrm{IPL}_+(\Omega, \omega)$ is equivalent to $\mathrm{IPL}^0_+(\Omega, \omega)$. Similarly, the condition $\mathrm{IPL}_-(\Omega, \omega)$ is equivalent to $\mathrm{IPL}^1_-(\Omega, \omega)$, hence $\mathrm{IPL}(\Omega, \omega)$ is equivalent to the combination of $\mathrm{IPL}^0_+(\Omega, \omega)$ and $\mathrm{IPL}^1_-(\Omega, \omega)$.

The idea of introducing Phragmén-Lindelöf properties to our theory is clearly inspired by Hörmander's characterization of surjectivity of linear partial differential operators with constant coefficients on the space of real analytic functions [35] and the impressive research of Meise, Taylor and Vogt on the existence of linear continuous right inverses for linear partial differential operators with constant coefficients on the space of smooth functions [48] and, especially, [55]. We apply many ideas from the latter paper. In view of Problem 9.1, the guiding principle is that everything true for the right inverse should be true for the real analytic parameter dependence. Surprisingly often that is right.

Remarks. (1) Clearly we may assume $\omega(0) \geq 1$.

(2) We may remove the constant C in the definitions of IPL conditions manipulating with k for $t \geq 0$ or n for $t \leq 0$.

Now, we formulate the main result of this section. Let us recall that for $\omega(z) = \log(2 + |z|)$ the space $\mathscr{D}'_{(\omega)}$ is just the standard space of distributions $\mathscr{D}'$.

Theorem 7.1 *Let $\Omega \subseteq \mathbb{R}^d$ be open convex and let $V = \{z : P(-z) = 0\}$ be the zero variety of the polynomial P. Let $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ be a linear partial differential operator with constant coefficients.*

- *(i) $\ker P(D)$ has the dual interpolation estimate if and only if V has $\mathrm{IPL}(\Omega, \omega)$.*
- *(ii) $\ker P(D)$ has a dual interpolation estimate for small θ (big θ) if and only if V has $\mathrm{IPL}^0(\Omega, \omega)$ ($\mathrm{IPL}^1(\Omega, \omega)$).*
- *(iii) $\ker P(D)$ has $(P\overline{\overline{\Omega}})$ if and only if V has $\mathrm{IPL}_+(\Omega, \omega)$ or, equivalently, $\mathrm{IPL}^0_+(\Omega, \omega)$.*
- *(iv) $\ker P(D)$ has $(P\Omega)$ if and only if V has $\mathrm{IPL}^1_+(\Omega, \omega)$.*
- *(v) $\ker P(D)$ has (PA) if and only if V has $\mathrm{IPL}_-(\Omega, \omega)$ or, equivalently, $\mathrm{IPL}^1_-(\Omega, \omega)$.*
- *(vi) if $\ker P(D)$ has $(P\underline{A})$ then V has $\mathrm{IPL}^0_-(\Omega, \omega)$.*

First, we reduce the result to irreducible varieties.

Lemma 7.2 *An algebraic variety has one of the* IPL *properties if and only if each of its irreducible component has the same property.*

Proof. Since each plurisubharmonic function on the variety is plurisubharmonic on each irreducible component (use [38, Th. 2.9.22]) sufficiency follows.

Necessity follows from the fact that every plurisubharmonic function on an irreducible component V_1 of an algebraic variety V extends to a plurisubharmonic function $\tilde{u}$ on the whole V by taking $\tilde{u}(z) = -\infty$ for $z \in V \setminus V_1$. ∎

We call a property (P) of PLS-spaces to be a *three space property of PLS-spaces* if for every short topologically exact sequence of PLS-spaces

$$0 \to X \to Y \to Z \to 0$$

the space Y has (P) whenever X and Z have (P).

Lemma 7.3 *The following properties are the three space properties of PLS-spaces: the dual interpolation estimate, the dual interpolation estimate for big θ and for small θ, $(P\overline{\overline{\Omega}})$, $(P\Omega)$ and (PA) .*

Proof. Observe that for a PLS-space X we have for stable α:

- $X \in (P\overline{\overline{\Omega}})$ if and only if $\mathrm{Proj}^1\, X\varepsilon\Lambda'_0(\alpha) = 0$ ($\Lambda_0(\alpha)$ equipped with ℓ_1-norms) [6, Cor. 3.8, 7.2];

- $X \in (P\Omega)$ if and only if $\mathrm{Proj}^1\, X\varepsilon\Lambda'_\infty(\alpha) = 0$ ($\Lambda_\infty(\alpha)$ equipped with ℓ_1-norms) [6, Cor. 3.8, 7.2];

- $X \in (PA)$ if and only if $\mathrm{Proj}^1\, X\varepsilon\Lambda_r(\alpha) = 0$ ($\Lambda_r(\alpha)$ equipped with ℓ_∞-norms) [8, Th. 4.4, 3.4];

- X has the dual interpolation estimate for small θ if and only if

$$\mathrm{Proj}^1\, X\varepsilon\mathscr{A}(\mathbb{R}) = 0$$

  (Cor. 6.1).

Other considered properties are combinations of the mentioned above. It suffices to show that for a fixed PLN-space E the condition $\mathrm{Proj}^1\, X\varepsilon E = 0$ is a three space property of PLS-spaces X.

Assume that $0 \to X \to Y \to Z \to 0$ is a short topologically exact sequence of PLS-spaces and $\mathrm{Proj}^1\, X\varepsilon E = \mathrm{Proj}^1\, Z\varepsilon E = 0$. As in the proof of Proposition 4.5 we obtain the following exact sequence:

$$0 \to X\varepsilon E \to Y\varepsilon E \to Z\varepsilon E \to$$
$$\to \mathrm{Proj}^1\, (X\varepsilon E) \to \mathrm{Proj}^1\, (Y\varepsilon E) \to \mathrm{Proj}^1\, (Z\varepsilon E) \to 0$$

and the result follows easily. ∎

Lemma 7.4 *Let P be a polynomial, $P = Q_1 \cdots Q_n$ its decomposition into irreducible factors. The space $\ker P(D) \subseteq \mathscr{D}'_{(\omega)}(\Omega)$ has one of the conditions $(P\overline{\overline{\Omega}})$, $(P\Omega)$, (PA) , the dual interpolation estimate for all (big, small) θ if and only if every space $\ker Q_j(D) \subseteq \mathscr{D}'_{(\omega)}(\Omega)$ has the same property for $j = 1, \ldots, n$.*

Please observe by the proof below that necessity holds also for $(P\underline{A})$.

Proof. As in [80, Lemma 5], we have a short exact sequence

(7.1) $0 \to \ker (P/Q_j)(D) \to \ker P(D) \to \ker Q_j(D) \to 0.$

If $\ker P(D)$ has one of the properties then $\mathrm{Proj}^1 \ker P(D) = 0$. By [83, Sec. 3.4.5, 3.4.6], $P(D)$ is surjective, thus all operators $Q_j(D)$ are surjective and $\mathrm{Proj}^1 \ker Q_j(D) = 0$. By Proposition 4.7, the sequence (7.1) is topologically exact. Hence $\ker Q_j(D)$ is a topological quotient of $\ker P(D)$ which implies necessity.

For sufficiency we apply Lemma 7.3 and inductively sequence (7.1). Let us note that if $\ker Q_j(D)$ satisfies one of the considered conditions then $\mathrm{Proj}^1 \ker Q_j(D) = 0$ and (7.1) is topologically exact as above. ∎

The crucial role in the proof of our result is played by the following approximation result of Meise, Taylor and Vogt [49, Th. 5.1] and Franken [26, Th. 8]. In fact, their formulation is much more complicated and based on the stronger assumption $|u(z)| \le L|z|$. The following result gives in the simplified form the essence of the result needed later on.

Theorem 7.5 *For any algebraic variety $V \subseteq \mathbb{C}^d$ there is a constant C_V such that for any $u \in \mathrm{PSH}(V)$ such that*

(7.2) $|u(z)| \le L|z| + L \qquad for \ z \in V$

there exists for every point $z \in V$ an entire function $f_z : \mathbb{C}^d \to \mathbb{C}$ such that

   (a) $u(z) \le \log |f_z(\zeta)| + L^2 + C_V \log(2 + |z|)$ for some $\zeta \in V$, $|\zeta - z| \le 2$;

   (b) $\forall \, \zeta \in V \ \exists \, \zeta' \in V \ \ \log |f_z(\zeta)| \le u(\zeta') + C_V \log(2 + |\zeta|)$ and $|\zeta - \zeta'| \le 1$.

Proof. Changing constants in the proof of [26, Th. 8], we get the same result under the weaker assumption (7.2). In [26] the functions f_z are defined only for parameters outside exceptional sets S_0 and E. To get them for all z we need two steps. First, for $z \in S_0$ we find $z_1 \in V \setminus S_0$, $|z - z_1| \le 1$, such that $u(z) \le u(z_1)$, see [26, Prop. 7 (iv)]. Then, by [26, (8) in the proof of Th. 10] and the remarks below as well as [26, Prop. 7 (v)(c)], for $z \in V \setminus S_0$

$$u(z) \le L^2 + u(z(\tau))$$

for some $\tau \in B \setminus E$ and $|z - z(\tau)| \le 1$. We define $f_z = f_\tau$, where f_τ is defined in [26, Th. 8]. Then we get the result by [26, Th. 8]. ∎

Proof of Theorem 7.1: By Lemmas 7.2 and 7.4 we may restrict our attention to irreducible polynomials P.

We show only the case (i). The other cases are very similar. By the Fundamental Principle for $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ (see [50, Lemma 3.5, 3.8], comp. [33, Sec. 4.3, 4.4]) we have

$$(\ker P(D))' = \{f \in H(V) : \ \exists N \ \forall n \quad \|f\|^*_{N,n} < \infty\},$$

where

$$\|f\|^*_{N,n} := \sup_{z \in V} |f(z)| \exp\left(-h_N(\operatorname{Im} z) + n\omega(z)\right).$$

Moreover, $\ker P(D) = \operatorname{proj}_{N \in \mathbb{N}} X_N$, where X_N are LN-spaces and

$$X'_N = \{f \in H(V) : \ \forall n \in \mathbb{N} \quad \|f\|^*_{N,n} < \infty\}.$$

The dual interpolation estimate for $\ker P(D) \subseteq \mathscr{D}'_{(\omega)}(\Omega)$ means that

$$\forall N' \ \exists M' \ \forall K' \ \exists n' \ \forall m', \theta \in \,]0,1[\ \exists k', C' \ \forall r > 0 \ \forall f \in X'_{N'} :$$

(7.3)
$$\frac{1}{r^\theta} \|f\|^*_{M',m'} \leq C \left(\|f\|^*_{K',k'} + \frac{1}{r} \|f\|^*_{N',n'} \right).$$

Taking logarithms on both sides and $t = \log r$ we get $\operatorname{IPL}(\Omega, \omega)$ with the above preambule of quantifiers but only for u of the form $u(z) = \log|f(z)| \in \operatorname{PSH}(V, N)$, $f \in H(V)$.

So it suffices to show that if $\operatorname{IPL}(\Omega, \omega)$ holds for such holomorphic functions then we have the same condition for $u \in \operatorname{PSH}(V)$.

Let us fix $\varepsilon > 0$. By Lemma 2.2, we construct a non-quasianalytic weight σ, $\omega = o(\sigma)$, and a plurisubharmonic function $v : \mathbb{C}^d \to [-\infty, \infty[$ such that for $z \in \mathbb{C}^d$

$$-D\sigma(z) \leq v(z) \leq \varepsilon |\operatorname{Im} z| - \sigma(z).$$

Obviously for some $\tilde{D}$:

(7.4)
$$-\tilde{D}|z| - \tilde{D} \leq v(z) \leq \varepsilon |z|.$$

Since $\omega = o(\sigma)$, we get an increasing sequence $(D_1(p))_{p \in \mathbb{N}}$ such that

(7.5)
$$\forall z \in V \qquad v(z) \leq \varepsilon |\operatorname{Im} z| - p\omega(z) + D_1(p).$$

Let us recall that there is a constant A such that

(7.6)
$$\omega(x+1) \leq A\omega(x) + A \qquad \text{for all } x$$

and we may assume that $\log(2 + |z|) \leq \omega(z)$ for any z.

We take arbitrary N, $B(0, \varepsilon) \subseteq K_N$. We take $N' := N$ and choose from (7.3) M', $M := M'$, then for any $K \geq M$, $K' := K$ we find n' from (7.3), $n := An' + AC_V$ (C_V comes from Theorem 7.5). Then for any m, $m' := A^2 m + A^2 C_V$ we choose for suitable θ a constant k' from (7.3), $k := Ak' + AC_V$.

We take arbitrary $t \geq 0$, $u \in \mathrm{PSH}(V, N)$ such that for $z \in V$ we have:

$$(7.7) \qquad \begin{aligned} u(z) &\leq h_N(\mathrm{Im}\, z) - n\omega(z) + t, \\ u(z) &\leq h_K(\mathrm{Im}\, z) - k\omega(z). \end{aligned}$$

We define

$$\tilde{u}(z) := \max\left(u(z), v(z) - D_1(k) \right).$$

Using (7.4) and the second estimate in (7.7), let us observe that for some $L > 0$ not depending on t we get:

$$|\tilde{u}(z)| \leq L|z| + L.$$

By (7.5), $\tilde{u} \in \mathrm{PSH}(V, N)$ since $u \in \mathrm{PSH}(V, N)$. For $z \in V$ and $t \geq 0$ we get by (7.5):

$$(7.8) \quad \begin{aligned} \tilde{u}(z) &\leq \max\left(h_N(\mathrm{Im}\, z) - n\omega(z) + t, h_N(\mathrm{Im}\, z) - n\omega(z) + D_1(n) - D_1(k) \right) \\ &\leq h_N(\mathrm{Im}\, z) - n\omega(z) + t \end{aligned}$$

and

$$(7.9) \qquad \tilde{u}(z) \leq h_K(\mathrm{Im}\, z) - k\omega(z).$$

We apply to the function $\tilde{u}$ Theorem 7.5, then we find entire functions f_z. Let us take $\zeta \in V$, then, by (7.8) and (7.6) for $C_N := \max_{|x| \leq 1} h_N(x)$:

$$\begin{aligned} \log|f_z(\zeta)| &\leq \max\{\tilde{u}(\zeta') : |\zeta' - \zeta| \leq 1\} + C_V \log(2 + |\zeta|) \leq \\ &\leq h_N(\mathrm{Im}\, \zeta) + \max_{|x| \leq 1} h_N(x) - n\omega(|\zeta| - 1) + t + C_V \log(2 + |\zeta|) \leq \\ &\leq h_N(\mathrm{Im}\, \zeta) + C_N - n\left(\frac{\omega(\zeta)}{A} - 1 \right) + t + C_V \omega(\zeta) \\ &\leq h_N(\mathrm{Im}\, \zeta) - n'\omega(\zeta) + t + C_N + n. \end{aligned}$$

On the other hand, (7.9) implies:

$$\log|f_z(\zeta)| \leq h_K(\mathrm{Im}\, \zeta) - k'\omega(\zeta) + C_K + k.$$

Analogously, since $\tilde{u} \in \mathrm{PSH}(V, N)$ we get also $\log|f_z| \in \mathrm{PSH}(V, N)$. We apply IPL for logarithms of holomorphic functions to get

$$\log|f_z(\zeta)| \leq h_M(\mathrm{Im}\, \zeta) - m'\omega(\zeta) + \theta t + \max(C_N + n, C_K + k) + C'.$$

Let us take $z \in V$, then by (a) in Theorem 7.5, for $C'' := \max(C_N + n, C_K + k) + C' + L^2$ and some $\zeta \in V$, $|\zeta - z| \leq 2$ we get

$$u(z) \leq \log |f_z(\zeta)| + L^2 + C_V \log(2 + |z|)$$

$$\leq h_M(\operatorname{Im} z) + \max_{|x| \leq 2} h_M(x) - m'\omega(|z| - 2) + \theta t + C'' + C_V \log(2 + |z|)$$

$$\leq h_M(\operatorname{Im} z) + 2C_M - m'\Big(\frac{\omega(z)}{A^2} - \frac{1}{A} - 1\Big) + \theta t + C_V\omega(z) + C''$$

$$\leq h_M(\operatorname{Im} z) - m\omega(z) + \theta t + 2C_M + m'/A + m' + C''.$$

We have finished the proof for $t \geq 0$ with constant $C = 2C_M + m'/A + m' + C''$.

For the case $t \leq 0$ we use

$$u(z) \leq h_N(\operatorname{Im} z) - n\omega(z),$$
$$u(z) \leq h_K(\operatorname{Im} z) - k\omega(z) - t.$$

The proof is similar getting

$$u(z) \leq h_M(\operatorname{Im} z) - m\omega(z) + (\theta - 1)t + \mathrm{const}. \qquad \blacksquare$$

We will need the following auxiliary lemma (comp. [55, Lemma 2.10]).

Lemma 7.6 *Let V be an algebraic variety, $A > 1$. Assume that for some fixed convex compact sets $N \Subset M \Subset \mathbb{R}^d$ the following condition holds:*

$$\exists\, n \;\, \forall\, m \;\, \exists\, \theta_0(m) \;\, \forall\, \theta \in\,]\theta_0(m), 1[\;\, \exists\, k(m, \theta), C \;\, \forall\, t \geq 0 \;\, \forall\, u \in \mathrm{PSH}(V, N)$$
$$(a) + (b) \Rightarrow (c),$$

where

(a) $\forall\, z \in V$ $u(z) \leq h_N(\operatorname{Im} z) - n\omega(z) + t$;
(b) $\forall\, z \in V$ $u(z) \leq A h_M(\operatorname{Im} z) - k(m, \theta)\omega(z)$;
(c) $\forall\, z \in V$ $u(z) \leq h_M(\operatorname{Im} z) - m\omega(z) + \theta t + C$.

Then

$$\forall\, p \geq 2 \;\, \exists\, n(p) > n \;\, \forall\, m \;\, \exists\, \theta_0(m, p) \;\, \forall\, \theta \in\,]\theta_0(m, p), 1[\;\, \exists\, k(m, \theta, p), C$$
$$\forall\, t \geq 0 \;\, \forall\, u \in \mathrm{PSH}(V, N) \qquad (a_p) + (b_p) \Rightarrow (c_p),$$

where

(a_p) $\forall\, z \in V$ $u(z) \leq h_N(\operatorname{Im} z) - n(p)\omega(z) + t$;
(b_p) $\forall\, z \in V$ $u(z) \leq A^p h_M(\operatorname{Im} z) - k(m, \theta, p)\omega(z)$;
(c_p) $\forall\, z \in V$ $u(z) \leq h_M(\operatorname{Im} z) - m\omega(z) + \theta t + C$.

An analogous result holds for other IPL-*type conditions, i.e., for $t \leq 0$ or for small θ.*

Proof. We go by induction with respect to p for u applying the inductive hypothesis for $v := u/A$. $\qquad \blacksquare$

We show that we can manipulate with the set Ω:

Proposition 7.7 *Let $\Omega \subseteq \mathbb{R}^d$ be an open convex set, V an arbitrary algebraic variety satisfying one of the $\mathrm{IPL}(\Omega, \omega)$ conditions. Then for every $s > 0$ and $a \in \mathbb{R}^d$ the variety V satisfies the same IPL condition with the sets*

$$s\Omega = \{sx : x \in \Omega\}, \quad \Omega + a = \{x + a : x \in \Omega\}, \quad -\Omega = \{-x : x \in \Omega\}.$$

Proof. Let us observe that for any compact set K

$$h_{sK}(x) = sh_K(x), \quad h_{K+a}(x) = h_K(x) + \langle x, a \rangle, \quad h_{-K}(x) = h_K(-x).$$

Observe also that if $u(z)$ is plurisubharmonic then $u(z)/s$, $u(z) + \langle \operatorname{Im} z, a \rangle$, $u(-z)$ are also plurisubharmonic functions with respect to z. The conclusion follows easily. ∎

Proposition 7.8 *Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \mathbb{R}^d$ be an increasing sequence of open convex sets such that an algebraic variety V satisfies $\mathrm{IPL}(\Omega_j, \omega)$ for any $j \in \mathbb{N}$. Then V satisfies $\mathrm{IPL}(\Omega, \omega)$ for $\Omega = \bigcup_j \Omega_j$. The same result holds for other IPL-conditions.*

Proof. By Proposition 7.7, without loss of generality we may assume that $0 \in \Omega_1$. Let $K_N \Subset \Omega$ arbitrary, then $K_N \Subset \Omega_j$ for some j. By $\mathrm{IPL}(\Omega_j, \omega)$ there is $K_M \Subset \Omega_j$ and $K_K, K_M \Subset K_K \Subset \Omega_j$ such that $0 \in \operatorname{Int} K_M$ and

$$\exists\, n \ \forall\, m \ \exists\, \theta_0(m) \ \forall\, \theta \in\,]\theta_0(m), 1[\ \exists\, k(m, \theta), C \ \forall\, t \ \forall\, u \in \mathrm{PSH}(V, N)$$
$$(a) + (b) \Rightarrow (c),$$

where

(a) $\quad \forall\, z \in V \qquad u(z) \le h_N(\operatorname{Im} z) - n\omega(z) + t$,

(b) $\quad \forall\, z \in V \qquad u(z) \le h_K(\operatorname{Im} z) - k\omega(z)$,

(c) $\quad \forall\, z \in V \qquad u(z) \le h_M(\operatorname{Im} z) - m\omega(z) + \theta t + C$.

There is $A > 1$, $AK_M \Subset K_K$, so that $Ah_M(\operatorname{Im} z) \le h_K(\operatorname{Im} z)$ and thus assumptions of Lemma 7.6 are satisfied. Let us take any $L \Subset \Omega$, clearly for some $p \in \mathbb{N}$ we have $A^p K_M \supseteq L$. The conclusion follows from Lemma 7.6 for $\mathrm{IPL}(\Omega, \omega)$. ∎

Now, we give a reformulation of IPL conditions.

Corollary 7.9 *(i) The condition $\mathrm{IPL}(\Omega, \omega)$ is equivalent to*

$$\forall\, N \ \exists\, M \ \forall\, \rho > 0 \ \exists\, n \ \forall\, m \ \forall\, \theta \in\,]0, 1[\ \exists\, k, C \ \forall\, t \in \mathbb{R} \ \forall\, u \in \mathrm{PSH}(V, N)$$
$$(a) + (b) \Rightarrow (c),$$

where

(a) $\forall\, z \in V$ $u(z) \le h_N(\operatorname{Im} z) - n\omega(z) + t$;

(b) $\forall\, z \in V$ $u(z) \le \rho|\operatorname{Im} z| - k\omega(z)$;

(c) $\forall\, z \in V$ $u(z) \le h_M(\operatorname{Im} z) - m\omega(z) + \theta t + C$.

(ii) The condition $\operatorname{IPL}(\mathbb{R}^d, \omega)$ *is equivalent to*

$$\exists\, A > 1 \;\; \forall\, \rho > 0 \;\; \exists\, n \;\; \forall\, m \;\; \forall\, \theta \in\,]0, 1[\;\; \exists\, k, C \;\; \forall\, t \in \mathbb{R} \;\; \forall\, u \in \operatorname{PSH}(V, N)$$
$$(a) + (b) \Rightarrow (c),$$

where $K_N = \overline{B}(0, 1)$

(a) $\forall\, z \in V$ $u(z) \le |\operatorname{Im} z| - n\omega(z) + t$;

(b) $\forall\, z \in V$ $u(z) \le \rho|\operatorname{Im} z| - k\omega(z)$;

(c) $\forall\, z \in V$ $u(z) \le A|\operatorname{Im} z| - m\omega(z) + \theta t + C$.

On the other hand $\operatorname{IPL}(B(0, 1), \omega)$ *is equivalent to*

$$\forall\, A > 1 \;\; \forall\, \rho > 0 \;\; \exists\, n \;\; \forall\, m \;\; \forall\, \theta \in\,]0, 1[\;\; \exists\, k, C \;\; \forall\, t \in \mathbb{R} \;\; \forall\, u \in \operatorname{PSH}(V, N)$$
$$(a) + (b) \Rightarrow (c),$$

where (a), (b) and (c) are the same as above.

(iii) The same results hold for other IPL*-conditions.*

Proof. (i): The condition implies the original $\operatorname{IPL}(\Omega, \omega)$ since

$$h_K(\operatorname{Im} z) \le \rho|\operatorname{Im} z|$$

for some ρ. For the other direction, by Proposition 7.7 and its proof we may assume without loss of generality that $0 \in K_N \Subset K_M \Subset \Omega$. The result follows from Lemma 7.6. The proof of other cases is similar. ∎

8. Application to parameter dependence of solutions of linear partial differential operators on spaces of distributions

We consider now the case of linear partial differential operators with constant coefficients $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ convex, ω a non-quasianalytic weight such that $\log(2 + |z|) = o(\omega(z))$ or $\omega(z) = \log(2 + |z|)$. First we observe that the only essential condition is $(P\overline{\overline{\Omega}})$ or, equivalently, the dual interpolation estimate for small θ.

Proposition 8.1 *Every algebraic variety has* $\operatorname{IPL}^1(\Omega, \omega)$ *and* $\operatorname{IPL}_-(\Omega, \omega)$.

The result follows also by [6, Cor. 8.4] and [8, Prop. 5.4] from Theorem 7.1 at least for $\omega(z) = \log(2 + |z|)$ but we give below a much simpler direct proof.

Proof. By Prop. 7.7, we may assume that $0 \in K_1 \Subset \Omega$. Since $\mathrm{IPL}_-(\Omega, \omega)$ is equivalent to $\mathrm{IPL}^1_-(\Omega, \omega)$ it suffices to prove $\mathrm{IPL}^1(\Omega, \omega)$.

We observe that for a given N, $M > N$ there is $\varepsilon > 0$ such that $K_M \supseteq K_N + B(0, \varepsilon)$. We choose K arbitrary and find θ so close to 1 that $(1 - \theta)h_K(\mathrm{Im}\, z) < \varepsilon |\mathrm{Im}\, z|$ for any $z \in \mathbb{C}^d$. Moreover, we find k so big that for fixed m, n, we have $m \le \theta n + (1 - \theta)k$. Then

$$\theta(h_N(\mathrm{Im}\, z) - n\omega(z) + t) + (1 - \theta)(h_K(\mathrm{Im}\, z) - k\omega(z)) \le$$
$$\le h_M(\mathrm{Im}\, z) - m\omega(z) + \theta t. \qquad \blacksquare$$

Corollary 8.2 *Let $P \in \mathbb{C}[z_1, \ldots, z_d]$ be an arbitrary polynomial, $V = \{z : P(-z) = 0\}$ its zero variety, $\Omega \subseteq \mathbb{R}^d$ a convex open set. If*

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$$

is a linear partial differential operator, then $\ker P(D)$ *has the dual interpolation estimate for big θ, i.e., (PA) and $(P\Omega)$. Moreover, the following assertions are equivalent:*

- *(a) V has $\mathrm{IPL}(\Omega, \omega)$ or, it has one of the equivalent conditions: $\mathrm{IPL}^0(\Omega, \omega)$, $\mathrm{IPL}_+(\Omega, \omega)$, $\mathrm{IPL}^0_+(\Omega, \omega)$;*
- *(b) $\ker P(D) \subseteq \mathscr{D}'_{(\omega)}(\Omega)$ has the dual interpolation estimate for all θ (or, equiv., small θ);*
- *(c) $\ker P(D)$ has $(P\overline{\overline{\Omega}})$;*
- *(d) $P(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U))$ is surjective;*
- *(e) $P(D) : \mathscr{D}'_{(\omega)}(\Omega, \Lambda'_0(\alpha)) \to \mathscr{D}'_{(\omega)}(\Omega, \Lambda'_0(\alpha))$ is surjective for some (all) stable (regular) α;*
- *(f) $P(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{E}_{\{\omega_1\}}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{E}_{\{\omega_1\}}(U))$ is surjective for some (all) non-quasianalytic weights ω_1.*

Remark. The first part of the Corollary 8.2 is essentially known (see [6, Cor. 8.4], [8, Prop. 5.4]) but the proof here is much simpler.

Proof. The first part and the equivalences of (a)–(c) follows from Theorem 7.1 and Proposition 8.1. (d)⇔(b): Corollary 6.4. (c)⇔(e): [6, Cor. 3.9, Th. 4.1, Th. 7.1]. (c)⇔(f): Corollary 6.5. $\qquad \blacksquare$

By [80, Th. 14, Th. 3], we know that the kernel of $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ for convex $\Omega \subseteq \mathbb{R}^d$, $d > 1$, has never $(\overline{\overline{\Omega}})$ and the same holds for arbitrary $\Omega \subseteq \mathbb{R}^d$, $d > 1$, for elliptic P. Thus we prove part (a) of Theorem A (use Corollary 2.3, Corollary 6.4 and Corollary 8.2):

Corollary 8.3 *Let $d > 1$. If P is hypoelliptic and $\Omega \subseteq \mathbb{R}^d$ is convex or P elliptic and $\Omega \subseteq \mathbb{R}^d$ arbitrary then*

$$P(D) : \mathscr{D}'(\Omega, \mathscr{A}(U)) \to \mathscr{D}'(\Omega, \mathscr{A}(U))$$

is never surjective.

Corollary 8.4 *For any pseudoconvex domain $\Omega \subseteq \mathbb{C}^d$, d arbitrary, the Cauchy-Riemann operator*

$$\bar{\partial} : \mathscr{D}'^{(0,0)}(\Omega, \mathscr{A}(U)) \to \ker \bar{\partial} \subset \mathscr{D}'^{(0,1)}(\Omega, \mathscr{A}(U))$$

is never surjective.

Let us mention that Langenbruch and the author have proved recently [20, Cor. 4.9] that no elliptic $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ has real analytic parameter dependence for any open $\Omega \subseteq \mathbb{R}^d$, $d > 1$.

Proof. Corollary 6.4 holds also for $\bar{\partial} : \mathscr{D}'^{(0,p)}(\Omega) \to \ker \bar{\partial} \subset \mathscr{D}'^{(0,p+1)}(\Omega)$ for $p \geq 0$ since $\bar{\partial}$ is open onto its image (for a simple functional analytic proof of that fact see [23, proof of Cor. 2]). It is easily seen that $\ker \bar{\partial}$ has a quotient isomorphic to the space $H(U)$ of holomorphic functions over one dimensional complex open set (just take restrictions to a one dimensional complex line going through Ω). If $\ker \bar{\partial}$ has $(P\overline{\overline{\Omega}})$, equivalently, $(\overline{\overline{\Omega}})$ then $H(U)$ would have $(\overline{\overline{\Omega}})$, a contradiction with Corollary 8.3. ∎

Remark. Since, by [65] or [68], the scalar Cauchy-Riemann complex splits from the second term on, the Cauchy-Riemann operator:

$$\bar{\partial} : \mathscr{D}'^{(0,p)}(\Omega, \mathscr{A}(U)) \to \ker \bar{\partial} \subset \mathscr{D}'^{(0,p+1)}(\Omega, \mathscr{A}(U))$$

has even a continuous linear right inverse (which implies surjectivity) for $p \geq 1$.

The next result (a part of Theorem B) is a surprisingly easy consequence of the whole previous theory and it shows that the real analytic parameter dependence is less restrictive if considered for distributions on the whole plane.

Theorem 8.5 *Let $P \in \mathbb{C}[z_1, \ldots, z_d]$ be an arbitrary polynomial and $\Omega \subseteq \mathbb{R}^d$ an open convex set, ω a suitable weight. If*

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U))$$

is surjective then

$$P(D) : \mathscr{D}'_{(\omega)}(\mathbb{R}^d, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\mathbb{R}^d, \mathscr{A}(U))$$

is surjective as well.

Moreover, if v is an arbitrary vector such that the hyperplane

$$H_c(v) := \{x : \langle x, v \rangle = c\}$$

is tangent to the boundary of Ω at x_0 (the point of smoothness of the boundary) then

$$P(D) : \mathscr{D}'_{(\omega)}(H_+(v), \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(H_+(v), \mathscr{A}(U))$$

is surjective as well, for $H_+(v) := \{x : \langle x, v \rangle > 0\}$.

Proof.   By Corollary 8.2, it suffices to prove that $\mathrm{IPL}^0_+(\Omega, \omega)$ implies $\mathrm{IPL}^0_+(\mathbb{R}^d, \omega)$ and $\mathrm{IPL}^0_+(H_+(v), \omega)$. This follows by Prop. 7.7 and Prop 7.8 since if $0 \in \Omega$ then $\mathbb{R}^d = \bigcup_{n>0} n\Omega$ and if $x_0 = 0$ then $H_+(v) = \bigcup_{n>0} n\Omega$. ■

The second part of Theorem B, one of the main and most surprising results of the paper requires a longer proof.

Theorem 8.6 *Let $P \in \mathbb{C}[z_1, \ldots, z_d]$ be an arbitrary polynomial and P_p its principal part, $\Omega \subseteq \mathbb{R}^d$ open convex set, ω arbitrary weight. If*

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U))$$

is a surjective operator then

$$P_p(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$$

has a linear continuous right inverse, in particular,

$$P_p(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U))$$

is surjective and has a linear continuous right inverse.

It implies that real analytic parameter dependence is a very restrictive property.

Corollary 8.7 *Let $P \in \mathbb{C}[z_1, \ldots, z_d]$ be an arbitrary polynomial and P_p its principal part, $\Omega \subseteq \mathbb{R}^d$ bounded open convex set with C^1-boundary, ω arbitrary weight. If*

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U))$$

is a surjective operator then P_p is proportional to a product of real linear forms.

Proof. Apply Theorem 8.6 and [48, Th. 3.8], [55, Th. 3.3]. ■

Proof of Theorem 8.6. By Corollary 8.2, the zero variety $V := \{z : P(-z) = 0\}$ of P has $\mathrm{IPL}^0_+(\Omega, \omega)$. Let V_p denote the zero variety of P_p. By [52, Th. 5.5] and [55, Th. 3.3], if V_p has the so-called condition $\mathrm{HPL}(\Omega, loc)$ at zero then

$$P_p : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$$

has a continuous linear right inverse (and, of course, the same holds for the vector valued version of $P_p(D)$). So it suffices to show that $\mathrm{IPL}^0_+(\Omega, \omega)$ for V implies $\mathrm{HPL}(\Omega, loc)$ at zero for V_p, i.e., that there are open sets $U_1 \subseteq U_2 \subseteq U_3 \Subset \mathbb{C}^d$, $0 \in U_1$ such that

$$\forall \; N \;\; \exists \; M, \delta > 0 \;\; \forall \; u \in \mathrm{PSH}(U_3 \cap V) \quad (\alpha) + (\beta) \Rightarrow (\gamma),$$

where

(α) $u(z) \le h_N(\mathrm{Im}\, z) + \delta$ for $z \in U_3 \cap V$;

(β) $u(z) \le 0$ for $z \in U_2 \cap \mathbb{R}^d \cap V$;

(γ) $u(z) \le h_M(\mathrm{Im}\, z)$ for $z \in U_1 \cap V$.

We will use some ideas of [55, Th. 4.1]. Let us define a maximal function

$U_E(z; h, V, D) :=$
$$\sup\{u(z) : u \in \mathrm{PSH}(V \cap D), u \le h \text{ on } D \cap V, u \le 0 \text{ on } E \cap V\},$$

where D is a domain in $\mathbb{C}^d$, h a function on D, $E \subseteq D$. For $0 < \varepsilon < 1$ we define

$$K_\varepsilon := \{z \in \mathbb{C}^d : |\mathrm{Im}\, z| \le \varepsilon |z| \text{ or } |z| \le \varepsilon\} \cap \bar{B}(0, 2),$$
$$E = \mathbb{R}^d \cap \bar{B}(0, 2) = \bigcap_{\varepsilon > 0} K_\varepsilon,$$
$$V_j := \{z/j : z \in V\} \quad \text{for } j \in \mathbb{N}.$$

Our aim is to estimate for some $M_1, \delta > 0$ and all $z \in B(0, 1) \cap V$:

$$U_E(z, h_N \circ \mathrm{Im} + \delta, V, B(0, 3)) \le h_{M_1}(\mathrm{Im}\, z)$$

which implies $\mathrm{HPL}(\Omega, loc)$ for V at zero.

Let us take any N, without loss of generality (see Prop. 7.7) we may assume that $K_N \supset B(0, 1)$. We find $N_1 > N$ such that

$$h_{N_1}(\mathrm{Im}\, z) \ge h_N(\mathrm{Im}\, z) + \eta |\mathrm{Im}\, z| > |\mathrm{Im}\, z|$$

for suitable η, $0 < \eta < 1$, small enough.

Choose $\delta > 0$, $\delta < \eta/2$. By $\mathrm{IPL}_+^0(\Omega, \omega)$ and Corollary 7.9, there is M such that for each $\rho > 0$ if $\varphi \in \mathrm{PSH}(V, N_1)$ and for all $z \in V$

$$\varphi(z) \leq h_{N_1}(\mathrm{Im}\, z) - n\omega(z) + t,$$
$$\varphi(z) \leq \rho|\mathrm{Im}\, z| - k\omega(z)$$

then for all $z \in V$

(8.1) $$\varphi(z) \leq h_M(\mathrm{Im}\, z) - m\omega(z) + \theta t + C_{\theta, m, \rho}.$$

Choose an arbitrary ε, $0 < \varepsilon < 1$, ρ_1 so large that

$$h_{N_1}(\mathrm{Im}\, z) \leq \rho_1|\mathrm{Im}\, z|, \quad \rho := \rho(\varepsilon) := 1 + \rho_1 + \delta\varepsilon^{-2} + \eta\varepsilon^{-2}.$$

Let us fix $j \in \mathbb{N}$. Take $u \in \mathrm{PSH}(V_j \cap B(0, 3))$ satisfying

$$u(z) \leq h_N(\mathrm{Im}\, z) + \delta \qquad \text{for } V_j \cap B(0, 3),$$
$$u(z) \leq 0 \qquad \text{for } z \in K_\varepsilon \cap V_j.$$

Take an arbitrary fixed $w = (w_1, \ldots, w_d) \in V_j \cap B(0, 1)$. Define for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$

$$\psi : V \cap \{z \in \mathbb{C}^d : |z - \mathrm{Re}\,(jw)| \leq j\} \to [-1, \infty[$$

by

$$\psi(z) := \max\left\{ ju(z/j) - \frac{\eta j}{2} \mathrm{Re}\left[\sum_{p=1}^d \left(\frac{z_p}{j} - \mathrm{Re}\, w_p\right)^2\right], h_{N_1}(\mathrm{Im}\, z)\right\}.$$

We extend ψ to a plurisubharmonic function on the whole V. Indeed, observe that for

$$z \in V, \quad |z - \mathrm{Re}\,(jw)| = j, \quad \text{i.e., } |\mathrm{Im}\, z| \leq j,$$

we get

$$ju(z/j) - \frac{\eta j}{2} \mathrm{Re}\left[\sum_{p=1}^d \left(\frac{z_p}{j} - \mathrm{Re}\, w_p\right)^2\right] \leq$$
$$\leq jh_N\left(\frac{\mathrm{Im}\, z}{j}\right) + j\delta + \eta j\left(\frac{|\mathrm{Im}\, z|}{j} - \frac{1}{2}\right) \leq h_{N_1}(\mathrm{Im}\, z).$$

Hence, we can extend ψ by taking

$$\psi(z) := h_{N_1}(\mathrm{Im}\, z) \qquad \text{for } |z - \mathrm{Re}\,(jw)| \geq j.$$

For $|z - \mathrm{Re}\,(jw)| \leq j$ we have $\left|\frac{\mathrm{Im}\, z}{j}\right| \leq 1$ and

(8.2) $$\psi(z) \leq \max\left(h_N(\mathrm{Im}\, z) + j\delta + \eta j\left|\frac{\mathrm{Im}\, z}{j}\right|^2, h_{N_1}(\mathrm{Im}\, z)\right)$$
$$\leq h_{N_1}(\mathrm{Im}\, z) + j(\delta + \eta).$$

Combining the two inequalities above we get

(8.3) $\psi(z) \leq h_{N_1}(\operatorname{Im} z) + j(\delta + \eta)$ for $z \in V$.

We will prove now that

(8.4) $\psi(z) \leq \rho|\operatorname{Im} z|$ for $z \in V$.

We consider separately three cases, $z \in V$:

1° $|z - \operatorname{Re}(jw)| \geq j$;

2° $|z - \operatorname{Re}(jw)| < j$, $z/j \in K_\varepsilon$;

3° $|z - \operatorname{Re}(jw)| < j$, $z/j \notin K_\varepsilon$.

1°. Clearly $\psi(z) \leq h_{N_1}(\operatorname{Im} z) \leq \rho_1|\operatorname{Im} z| \leq \rho|\operatorname{Im} z|$.

2°. Since $|\operatorname{Im} z/j| \leq 1$ and $z/j \in K_\varepsilon$, $u(z/j) \leq 0$ and

$$\psi(z) \leq \max\left\{ -\frac{\eta j}{2} \operatorname{Re}\left[\sum_{p=1}^{d} \left(\frac{z_p}{j} - \operatorname{Re} w_p \right)^2 \right], h_{N_1}(\operatorname{Im} z) \right\}$$

$$\leq \max\{\eta|\operatorname{Im} z|, h_{N_1}(\operatorname{Im} z)\} \leq \rho|\operatorname{Im} z|.$$

3°. As in (8.2) we observe that $\psi(z) \leq h_{N_1}(\operatorname{Im} z) + j(\delta + \eta)$. Since $j \leq |z|/\varepsilon$ and $|z| \leq |\operatorname{Im} z|/\varepsilon$ we get

$$\psi(z) \leq h_{N_1}(\operatorname{Im} z) + \frac{(\delta + \eta)|\operatorname{Im} z|}{\varepsilon^2} \leq \rho|\operatorname{Im} z|.$$

We have proved (8.4) for all $z \in V$.

Assume that $0 < \gamma < 1$, $(1 + \gamma)K_M \Subset K_{M_1} \Subset \Omega$. By Lemma 2.2, there is a non-quasianalytic weight σ such that $\omega = o(\sigma)$, $\omega \leq \sigma$ and there is a plurisubharmonic function φ_k such that

$$-C_k \sigma(z) \leq \varphi_k(z) \leq \gamma|\operatorname{Im} z| - k\sigma(z)$$

for $k \in \mathbb{N}$ and some $C_k > 0$. This is the first place where we have to assume that ω is a non-quasianalytic weight.

We define a plurisubharmonic function $\tilde{\psi}_k$ on V (k depends on θ as in $\mathrm{IPL}_+^0(\Omega, \omega)$):

$$\tilde{\psi}_k(z) := \frac{\psi(z) + \varphi_{2k}(z)}{1 + \gamma}.$$

From (8.3) it follows for $z \in V$

$$\tilde{\psi}_k(z) \leq \frac{1}{1 + \gamma} \cdot (h_{N_1}(\operatorname{Im} z) + j(\delta + \eta) + \gamma|\operatorname{Im} z| - 2k\sigma(z))$$

$$\leq h_{N_1}(\operatorname{Im} z) + j(\delta + \eta) - k\omega(z)$$

and analogously $\tilde{\psi}_k \in \mathrm{PSH}(V, N_1)$. Similarly, by (8.4), it follows for $z \in V$

$$\tilde{\psi}_k(z) \leq \frac{1}{1+\gamma} \cdot (\rho|\operatorname{Im} z| + \gamma|\operatorname{Im} z| - 2k\sigma(z)) \leq \rho|\operatorname{Im} z| - k\omega(z).$$

By (8.1), i.e., by $\mathrm{IPL}_+^0(\Omega, \omega)$,

$$\tilde{\psi}_k(z) \leq h_M(\operatorname{Im} z) - m\omega(z) + \theta j(\delta + \eta) + C_{\theta,m,\rho} \qquad \text{for } z \in V.$$

By the definition of $\tilde{\psi}_k$ we get for $z \in V$

$$\psi(z) \leq (1+\gamma)(h_M(\operatorname{Im} z) - m\omega(z) + \theta j(\delta + \eta) + C_{\theta,m,\rho}) + C_{2k}\sigma(z).$$

Let us take $z = jw \in V$, then $|z - \operatorname{Re}(jw)| = |\operatorname{Im}(jw)| \leq j|w| \leq j$ and $ju(z/j) \leq \psi(z)$. Hence

$$ju(w) = ju(z/j)$$
$$\leq h_{M_1}(\operatorname{Im}(jw)) - (1+\gamma)(m\omega(jw) - \theta j(\delta + \eta) - C_{\theta,m,\rho}) + C_{2k}\sigma(jw).$$

We get for all $w \in V_j \cap B(0,1)$

$$U_{K_\varepsilon}(w, h_N \circ \operatorname{Im} + \delta, V_j, B(0,3)) \leq$$
$$\leq h_{M_1}(\operatorname{Im} w) - (1+\gamma)(m\omega(jw)/j - \theta(\delta + \eta) - C_{\theta,m,\rho}/j) + C_{2k}\sigma(jw)/j.$$

Since V_j converge to V_p in the sense of [51, Def. 4.3], by [51, Th. 4.4], we get as $j \to \infty$ for $z \in V \cap B(0,1)$ the following inequality for regular points $z \in V \cap B(0,1)$

$$U_{K_\varepsilon}(z, h_N \circ \operatorname{Im} + \delta, V, B(0,3)) \leq h_{M_1}(\operatorname{Im} z) + \theta(1+\gamma)(\delta + \eta)$$

since $\omega(t)/t, \sigma(t)/t \to 0$ as $t \to \infty$ (here non-quasianaliticity of ω and σ is used). On singular points we define plurisubharmonic functions via upper limits, so the same inequality holds for singular points as well. Since this is true for any $\theta \in]0, \theta_0[$ we get for $z \in V \cap B(0,1)$

$$U_{K_\varepsilon}(z, h_N \circ \operatorname{Im} + \delta, V, B(0,3)) \leq h_{M_1}(\operatorname{Im} z).$$

By [51, Prop. 4.2],

$$U_E(z, h_N \circ \operatorname{Im} + \delta, V, B(0,3)) \leq h_{M_1}(\operatorname{Im} z),$$

since $E = \bigcap_{\varepsilon>0} K_\varepsilon$. ∎

Now, we show that the real analytic parameter dependence of solutions implies (is equivalent) to the existence of a linear continuous right inverse in many cases. We start with homogeneous polynomials and part (d) of Theorem A.

Theorem 8.8 *Let $P \in \mathbb{C}[z_1, \ldots, z_d]$ be an arbitrary homogenenous polynomial, $\Omega \subseteq \mathbb{R}^d$ an open convex set, ω an arbitrary non-quasianalytic weight. The operator*

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U))$$

is surjective if and only if

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega) \quad (or, \; equivalently, \; P(D) : C^{\infty}(\Omega) \to C^{\infty}(\Omega))$$

has a continuous linear right inverse. In particular, for bounded convex Ω with C^1-boundary it holds if and only if P is proportional to a product of real linear forms.

Remarks. (1) This result shows a dramatic difference between real analytic and holomorphic or smooth dependence — the latter two holds always, see [8].

(2) By [48], [52, Th. 5.6], for homogeneous P the existence of a right inverse on $\mathscr{D}'_{(\omega)}(\Omega)$ or $\mathscr{E}_{\{\omega\}}(\Omega)$, ω non-quasianalytic, does not depend on ω and it is equivalent to the existence of a right inverse on $C^{\infty}(\Omega)$.

In fact, the above result can be reformulated as a new elegant characterization of the existence of a continuous linear right inverse.

Proof of Theorem 8.8: The result follows immediately from Theorem 8.6, [48, Th. 3.8] and [55, Th. 3.3]. One can also give a simpler direct proof – we will sketch it. By [52, Th. 5.5], it suffices to show for homogeneous varieties V that $\mathrm{IPL}^0_+(\Omega, \omega)$ implies $\mathrm{PL}(\Omega, \omega)$ (see [55, Def. 2.5, Prop. 2.8]), i.e.,

$$\forall \, N \; \exists \, M \geq N \; \forall \, K \geq M \; \exists \, B > 0 \; \forall \, u \in \mathrm{PSH}(V) \qquad (\alpha) + (\beta) \Rightarrow (\gamma),$$

where

(α) $u(z) \leq h_N(\mathrm{Im}\, z) + C$ for some C and any $z \in V$,

(β) $u(z) \leq h_K(\mathrm{Im}\, z)$ for any $z \in V$,

(γ) $u(z) \leq h_M(\mathrm{Im}\, z) + B\omega(z)$ for any $z \in V$.

Assume that $K_N \supseteq B(0,1)$ (use Proposition 7.7) and $u \in \mathrm{PSH}(V)$ satisfies (α) and (β). Let ε, $0 < \varepsilon < 1$, be so small that $(1 + \varepsilon)K_M \Subset K_{M_1} \Subset \Omega$. By Lemma 2.2, we find a non-quasianalytic weight $\sigma \geq \omega$, $\omega = o(\sigma)$, and $\varphi_k \in \mathrm{PSH}(\mathbb{C}^d)$ such that

$$-C_k \sigma(z) \leq \varphi_k(z) \leq \varepsilon |\mathrm{Im}\, z| - 2k\sigma(z).$$

We apply $\mathrm{IPL}^0_+(\Omega, \omega)$ to

$$v_{r,k}(z) := \frac{(1/r)u(rz) + \varphi_k(z)}{1 + \varepsilon} \qquad \text{for } r > 0$$

and then taking $r \to 0$ and $\theta \to 0$ we get

$$u(\zeta) \le h_{M_1}(\operatorname{Im}\zeta)$$

since $\sigma(t)/t, \omega(t)/t \to 0$ as $t \to \infty$. ∎

Before we consider further cases we need the following lemma, a strengthening of [55, Lemma 4.7].

Lemma 8.9 *Assume that V is an algebraic variety in $\mathbb{C}^d$ that satisfies $\mathrm{IPL}^0_+(\mathbb{R}^d, \omega)$ in the form described in Corollary 7.9. Let $M > 0$, then there is a constant C such that*

$$|\operatorname{Im} w| \le C\omega(w).$$

for any $w = (w_1, \ldots, w_d) \in V$ satisfying

$$|\operatorname{Im} w| \le M|\operatorname{Im} z|$$

for all z in the connected component V_w of w in the set

$$V \cap \{z \in \mathbb{C}^d : |z - w| < s|\operatorname{Im} w|\}$$

for some $s \ge 4A + 4$.

Proof. Assume that there is a sequence $(w_j)_{j \in \mathbb{N}}$ satisfying assumptions of Lemma for fixed M such that $\frac{|\operatorname{Im} w_j|}{\omega(w_j)} \to \infty$. Since $\omega \ge 1$, $|w_j| \to \infty$ and then we can construct a function $h : [0, \infty[\to [0, \infty[$ such that $h(|w_j|) = |\operatorname{Im} w_j|$ for infinitely many j and $\omega = o(h)$. By Lemma 2.2 there is a non-quasianalytic weight $\sigma \ge \omega$, $\omega = o(\sigma)$, $\sigma = o(h)$. Clearly, then $\frac{|\operatorname{Im} w_j|}{\sigma(w_j)} \to \infty$. Summarizing, it suffices to prove

$$|\operatorname{Im} w| \le C\sigma(w).$$

We define $u : V_w \to \mathbb{R}$ for $z = (z_1, \ldots, z_d)$ by

$$u(z) := \max\left\{(2A+1)|\operatorname{Im} w| - \frac{1}{2s|\operatorname{Im} w|}\operatorname{Re}\left[\sum_{p=1}^d (z_p - w_p)^2\right], |\operatorname{Im} z|\right\}.$$

For $z \in V$ with $|z - w| = s|\operatorname{Im} w|$ we get

$$(2A+1)|\operatorname{Im} w| - \frac{s|\operatorname{Im} w|}{2}\operatorname{Re}\left[\sum_{p=1}^d \left(\frac{z_p - w_p}{s|\operatorname{Im} w|}\right)^2\right] \le$$

$$\le (2A+1)|\operatorname{Im} w| + s|\operatorname{Im} w|\left(\frac{|\operatorname{Im}(z-w)|}{s|\operatorname{Im} w|} - \frac{1}{2}\right) \le |\operatorname{Im} z|.$$

Hence we can extend u to a plurisubharmonic function on V defining $u(z) = |\operatorname{Im} z|$ outside V_w. Take $\rho = (A+1)M + 1$, m arbitrary, $\theta \leq \frac{1}{2(2+2A)}$ and find k from $\operatorname{IPL}^0_+(\mathbb{R}^d, \omega)$. Then we define, by Lemma 2.2, a plurisubharmonic function φ on $\mathbb{C}^d$ such that

$$-C\sigma(z) \leq \varphi(z) \leq |\operatorname{Im} z| - 2k\sigma(z).$$

We take a function
$$v(z) := (u(z) + \varphi(z))/2.$$

Clearly, outside V_w we have

$$v(z) \leq |\operatorname{Im} z| - k\sigma(z),$$

thus $v \in \operatorname{PSH}(V, \overline{B}(0,1))$. Moreover since on V_w we have $|\operatorname{Im} w| \leq M|\operatorname{Im} z|$ thus

$$(8.5) \quad (2A+1)|\operatorname{Im} w| - \frac{s|\operatorname{Im} w|}{2}\operatorname{Re}\left[\sum_{p=1}^{d}\left(\frac{z_p - w_p}{s|\operatorname{Im} w|}\right)^2\right] \leq$$

$$\leq (2A+1)|\operatorname{Im} w| + |\operatorname{Im} z| + |\operatorname{Im} w| \leq ((2A+2)M+1)|\operatorname{Im} z|.$$

Clearly

$$v(z) \leq ((A+1)M+1)|\operatorname{Im} z| - k\sigma(z) \leq \rho|\operatorname{Im} z| - k\omega(z).$$

On the other hand, by (8.5),

$$v(z) \leq |\operatorname{Im} z| + (A+1)|\operatorname{Im} w| - k\sigma(z) \leq |\operatorname{Im} z| + (A+1)|\operatorname{Im} w| - n\omega(z).$$

Taking $t = (A+1)|\operatorname{Im} w|$ we get by $\operatorname{IPL}^0_+(\mathbb{R}^d, \omega)$

$$v(z) \leq A|\operatorname{Im} z| - m\omega(z) + (1/4)|\operatorname{Im} w|,$$

therefore

$$u(z) \leq 2A|\operatorname{Im} z| - 2m\omega(z) + (1/2)|\operatorname{Im} w| + C\sigma(z).$$

Applying the above inequality for $z = w$ we get $|\operatorname{Im} w| \leq 2C\sigma(w)$.   ∎

Proposition 8.10 *Let $d \in \mathbb{N}$, $P \in \mathbb{C}[z_1, \ldots, z_d, z_{d+1}, z_{d+2}]$ be a polynomial of the form*

$$P(z_1, \ldots, z_{d+2}) = \sum_{j=1}^{d} \varepsilon_j z_j^2 + \lambda z_{d+1} + \mu z_{d+2},$$

where $\varepsilon_j = \pm 1$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\mu \in \mathbb{R}$. Then the zero variety $V(P)$ has no $\operatorname{IPL}^0_+(\mathbb{R}^{d+2}, \omega)$ for any non-quasianalytic weight ω, $\omega(t) = o(t^{(1/2)})$.

Proof. We take any non-quasianalytic weight σ, $\omega = o(\sigma)$, $\sigma \geq \omega$, $\sigma(t) = o(t^{1/2})$, comp. Lemma 2.2. Exactly as in the proof of [53, Lemma 4] we prove the inequality (2) there using Lemma 8.9 instead of [55, Lemma 4.7] and we get

$$R \sin \frac{\delta}{2} \leq C\omega(R^2) \qquad \text{for } R \geq R_0,$$

a contradiction. ∎

We are ready to solve the case of polynomials of order two and we prove Theorem A (c).

Theorem 8.11 *If $P \in \mathbb{C}[z_1, \ldots, z_d]$ is an arbitrary polynomial of order two such that there exists an open convex set $\Omega \subseteq \mathbb{R}^d$ for which*

$$P(D) : \mathscr{D}'(\Omega, \mathscr{A}(U)) \to \mathscr{D}'(\Omega, \mathscr{A}(U))$$

is surjective then $P(D)$ is as in Theorem A (c).

It is proved in [53, Th. 2] for P of order two that $P(D) : \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ has a continuous linear right inverse if and only if the necessary condition above holds:

Corollary 8.12 *If $P \in \mathbb{C}[z_1, \ldots, z_d]$ is an arbitrary polynomial of second order then*

$$P(D) : \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U)) \to \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U))$$

is surjective if and only if

$$P(D) : \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$$

has a continuous linear right inverse.

Proof of Theorem 8.11. By Theorem 8.5 and Theorem 8.6, if

$$P(D) : \mathscr{D}'(\Omega, \mathscr{A}(U)) \to \mathscr{D}'(\Omega, \mathscr{A}(U))$$

is surjective then

$$P(D) : \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U)) \to \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U))$$

is surjective and

$$P_p(D) : \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$$

has a continuous linear right inverse.

By [53, Lemma 2], there is $\lambda \in \mathbb{C}$ such that λP_p is real. As in the proof of [53, Lemma 3 (i)], the real analytic parameter dependence on $\mathbb{R}^d$ does not change if we make either a real linear invertible change of variables or a complex translation of variables in the symbol P. By [53, Lemma 2], doing such changes of variables we may assume that P is one of the forms:

(i) $P(z) = \sum_{j=1}^r z_j^2 - \sum_{j=r+1}^s z_j^2 + c$, $c \in \mathbb{C}$;

(ii) $P(z) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{s} z_j^2 + \lambda z_{s+1}$, $\lambda \in \mathbb{C} \setminus \{0\}$;

(iii) $P(z) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{s} z_j^2 + i z_{s+1} + z_{s+2}$.

By Proposition 8.10, P cannot be of the form (iii) and in the form (ii) λ must be real. Since

$$P_p(z) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{s} z_j^2$$

and $P_p(D)$ has a continuous linear right inverse, then by [53, Prop. 2], either $r = s = 1$ or $s > r$. If in case (ii) $r = s = 1$ then

$$P(z) = z_1^2 + \lambda z_2 \qquad \lambda \in \mathbb{R}.$$

Of course, if $P(D)$ has the real analytic parameter dependence on $\mathbb{R}^d$, $d > 2$, then it has such a dependence for $\mathbb{R}^2$ and the zero variety $V(P) \subseteq \mathbb{C}^2$ has $\mathrm{IPL}_+^0(\mathbb{R}^2, \log(2 + |\cdot|))$, see Cor. 8.2.

We will show by Lemma 8.9 that this is impossible. Take a point

$$z_r := (-ir, -r^2/\lambda) \in V(P) \qquad \text{for } r \in \mathbb{R}.$$

Let us fix $s > 0$, then for $(-z, z^2/\lambda) \in V(P)$ if we have

(8.6) $\qquad |(-z, z^2/\lambda) - (-ir, -r^2/\lambda)| \leq s |\operatorname{Im} z_r| = s|r|,$

then $|z^2 + r^2| \leq \lambda s |r|$. Hence for r big enough and suitable small δ:

$|z| \geq r/2, \qquad \operatorname{Arg} z \in [\pi/2 - \delta/2, \pi/2 + \delta/2] \cap [-\pi/2 - \delta/2, -\pi/2 + \delta/2].$

Therefore if (8.6) holds then

$$|\operatorname{Im} z| > (1/2)|z| \geq r/4 = |\operatorname{Im} z_r|/4.$$

We have proved assumptions of Lemma 8.9, thus

$$r = |\operatorname{Im} z_r| \leq O(\log(2 + |z_r|));$$

a contradiction. ∎

Finally, we consider polynomials of two variables.

We call a polynomial P (ω)-*hyperbolic with respect to a vector v* [54] for a non-quasianalytic weight ω if $P_p(v) \neq 0$ (i.e., v is non-characteristic) and $P(D)$ admits a fundamental solution $E \in \mathscr{D}'_{(\omega)}(\mathbb{R}^d)$ with $\operatorname{supp} E \subseteq \overline{H_+(v)}$, where

$$H_+(v) := \{x \in \mathbb{R}^d : \langle x, v \rangle > 0\}.$$

A polynomial is (ω)-*hyperbolic* if it is (ω)-hyperbolic with respect to at least one non-characteristic vector. It is known that for $\omega(t) = \log(2 + |t|)$, i.e., for classical distributions $\mathscr{D}'$, (ω)-hyperbolicity is just the classical hyperbolicity [54, Remark 2.2 (c)], [36, Def. 12.3.3, Th. 12.3.1, 12.5.1]. We are ready to prove Theorem A (b).

Theorem 8.13 *Let $P \in \mathbb{C}[z_1, z_2]$ be an arbitrary polynomial. The following assertions are equivalent:*

(a) $P(D) : \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U)) \to \mathscr{D}'_{(\omega)}(\Omega, \mathscr{A}(U))$ is surjective for some open convex set $\Omega \subseteq \mathbb{R}^2$;

(b) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ has a continuous linear right inverse for all open convex sets $\Omega \subseteq \mathbb{R}^2$;

(c) $P(D)$ is (ω)-hyperbolic or, equivalently, (ω)-hyperbolic with respect to all non-characteristic directions.

Remarks. 1. The paper [48] contains also the proof of (b)$\Leftrightarrow$(c) for the classical space of distributions $\mathscr{D}'(\Omega)$. Unfortunately the proof of necessity contained there is based on [48, Lemma 4.10] for which one can find a counterexample $P(z_1, z_2, z_3) = z_1 + z_2 z_3$. The general case (b)$\Leftrightarrow$(c) is contained in [52, Th. 5.14]

2. By [16, Th. 1], hyperbolicity with respect to all non-characteristic directions is equivalent to the fact that P and the principal part P_p are equally strong and P_p is proportional to a product of linear functions with real coefficients.

Proof. (b)$\Rightarrow$(a) Obvious. The equivalence of two conditions in part (c) is proved in [52, Th. 5.14]. (c)$\Rightarrow$(b) is known [48, Th. 4.11], [52, Th. 5.14]. We will prove (a)$\Rightarrow$(c):

Let us assume that P is irreducible. Let $\vec{v}$ be a non-characteristic direction of P. Without loss of generality we may assume that $\vec{v} = (1, 0)$.

Therefore, we may assume that

$$P(z_1, z_2) = z_1^m + \sum_{j=0}^{m-1} e_j(z_2) z_1^j \qquad \text{for all } (z_1, z_2) \in \mathbb{C}^2.$$

Since P_p is a homogenenous polynomial of two variables, we find k, $0 \le k \le m$, $0 \ne \alpha_j \in \mathbb{C}$ for $1 \le j \le k$ (if $k > 0$) such that

$$P_p(z_1, z_2) = z_1^{m-k} \prod_{j=1}^{k} (z_1 - \alpha_j z_2) \qquad \text{for all } (z_1, z_2) \in \mathbb{C}^2.$$

We assume that $1 \le k < m$ (other cases are even simpler). By Theorem 8.6 and 8.5,

$$P_p(D) : \mathscr{D}'_{(\omega)}(\mathbb{R}^2) \to \mathscr{D}'_{(\omega)}(\mathbb{R}^2)$$

has a continuous linear right inverse. By [48, 4.7], for every irreducible factor Q of P_p

$$\dim_{\mathbb{R}} V(Q) \cap \mathbb{R}^2 = 1,$$

thus $\alpha_j \in \mathbb{R}$ for $1 \le j \le k$.

Note that $V(P)$ can be represented via Puiseux expansion. Therefore there exists a constant $B > 0$ such that

$$V(P) \cap \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| \leq B\}$$

is compact and

$$V(P) \cap \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| \geq B\}$$

is a union of disjoint branches (W_j) each of them has the following expansion: there is $q_j \in \mathbb{N}$ such that for $|z_2| \geq B$ elements of $(z_1, z_2) \in W_j$ are of the form $(s_j(z_2), z_2)$,

$$(8.7) \qquad s_j(z_2) = \sum_{l=-\infty}^{q_j} a_{jl} z_2^{l/q_j}.$$

Let us observe that

$$\frac{1}{t}(s_j(tz_2), tz_2) \to (a_{jq_j} z_2, z_2) \in V(P_p), \qquad \text{as } t \to \infty,$$

for any $z_2 \in \mathbb{C}$. Thus a_{jq_j} must be equal to some α_j or to 0 and so it is real. It is easily seen that there exists a constant B_1 such that for any $z = (z_1, z_2) \in V(P)$

$$(8.8) \qquad\qquad |z| \leq B_1(|z_2| + 1).$$

Let us observe that for $B \leq |z_2| \leq B + 1$

$$|\operatorname{Im}(s_j(z_2) - a_{jq_j} z_2)| \leq C$$

for some C not dependent on j.

That is why the function u defined for $z = (z_1, z_2) \in W_j$, $|z_2| \geq B$ by the formula:

$$u(z) := \max \left(|\operatorname{Im}(s_j(z_2) - a_{jq_j} z_2)| - C, 0 \right)$$

and $u(z) := 0$ for $z = (z_1, z_2) \in V(P) \setminus W_j$ or $z = (z_1, z_2) \in V(P), |z_2| \leq B$ is a plurisubharmonic function on $V(P)$.

We choose ι, λ such that $\alpha := \lambda/q_\iota$ is the biggest among the numbers l/q_j such that $a_{jl} \neq 0$, $l < q_j$, j arbitrary.

We will show that $t^\alpha = O(\omega(t))$ and this will be the most part of the proof below. Clearly it suffices to consider $0 < \alpha < 1$. If $t^\alpha \neq O(\omega(t))$ then there is a function $h : [0, \infty[\to [0, \infty[$ such that $\omega = o(h)$ and for some sequence $t_j \to \infty$, $h(t_j) = t_j^\alpha$, $\frac{h(t_j)}{\omega(t_j)} \to \infty$.

We start with some useful estimates for u. First,

$$(8.9) \qquad\qquad u(z) \leq D|z_2|^\alpha$$

for all $z = (z_1, z_2) \in V(P)$. Since a_{jq_j} are real

$$(8.10) \qquad u(z) \leq |\operatorname{Im} z_1| + D_1|\operatorname{Im} z_2| \qquad \text{whenever } D_1 := \max_j |a_{jq_j}|.$$

We need also estimates from below. If $a_{\iota\lambda} \notin \mathbb{R}$ then for $z_2 \in \mathbb{R}$:

$$| \operatorname{Im} (s_\iota(z_2) - a_{\iota q_\iota} z_2) | = | \operatorname{Im} \sum_{l=-\infty}^{\lambda} a_{\iota l} z_2^{l/q_\iota} | \geq D_1 |z_2|^\alpha$$

for sufficiently big $|z_2|$. If $a_{\iota\lambda} \in \mathbb{R}$, $z_2 \in \mathbb{R}$, then

$$z_2^\alpha = |z_2|^\alpha \exp(i(\operatorname{Arg} z_2 + 2\pi p)\alpha), \qquad p = 0, 1, \ldots, q_\iota - 1,$$

where $\operatorname{Arg} z_2 = 0$ for $z_2 > 0$ and $= \pi$ for $z_2 < 0$. Choosing appropriately p and $\operatorname{Arg} z_2$ we get for all suitable big $|z_2|$ that

$$| \operatorname{Im} a_{\iota\lambda} z_2^\alpha | \geq D_3 |z_2|^\alpha.$$

Summarizing, there is a constant $E > 0$ such that for big real z_2 we have

$$(8.11) \qquad | \operatorname{Im} (s_\iota(z_2) - a_{\iota q_\iota} z_2) | \geq E |z_2|^\alpha \qquad \text{for suitable choice of } s_\iota(z_2).$$

By Corollary 8.2, $V(P)$ has $\operatorname{IPL}_+^0(\mathbb{R}^2, \omega)$ in the form described in Corollary 7.9 (ii). Let us fix $w \in \mathbb{R}$, $B \leq |w| = R$, $\mu = 4A \geq 2$. Define for $z = (z_1, z_2) \in V(P)$:

$$\tilde{u}(z) = \begin{cases} \max\left(\mu u(z) - \frac{R}{2} \operatorname{Re}\left[\left(\frac{z_2-w}{R}\right)^2\right], |\operatorname{Im} z|\right) & \text{for } |z_2 - w| \leq R, \\ |\operatorname{Im} z| & \text{otherwise on } V(P). \end{cases}$$

For $|z_2 - w| = R$ we have by (8.9)

$$\mu u(z) - \frac{R}{2} \operatorname{Re}\left[\left(\frac{z_2-w}{R}\right)^2\right] =$$

$$= \mu u(z) + \frac{R}{2}\left[\left|\operatorname{Im}\left(\frac{z_2-w}{R}\right)\right|^2 - \left|\operatorname{Re}\left(\frac{z_2-w}{R}\right)\right|^2\right] \leq$$

$$\leq \mu D |z_2|^\alpha + |\operatorname{Im} z_2| - R/2 \leq |\operatorname{Im} z|$$

for sufficiently big R. Hence $\tilde{u}$ is a plurisubharmonic function on $V(P)$. Analogously, we obtain for $z \in V(P)$

$$(8.12) \qquad\qquad \tilde{u}(z) \leq |\operatorname{Im} z| + \mu D 2^\alpha R^\alpha.$$

We take $\rho := (\mu + \mu D_1 + 3)/2$, m arbitrary, $\theta = E/(2^{\alpha+2}D)$. We choose by Lemma 2.2 a weight σ, $\omega(z) \leq \sigma(z)$, $\omega = o(\sigma)$, $\sigma = o(h)$ and we construct a plurisubharmonic function φ such that

$$-C_k \sigma(z) \leq \varphi(z) \leq |\operatorname{Im} z| - 2k(\theta)\sigma(z).$$

We define
$$v(z) := (\tilde{u}(z) + \varphi(z))/2.$$
Then, by (8.12)
$$v(z) \le |\operatorname{Im} z| - k(\theta)\sigma(z) + \mu D 2^{\alpha-1} R^\alpha.$$
Clearly $v \in \mathrm{PSH}(V, B(0,1))$ and
$$v(z) \le |\operatorname{Im} z| - n\omega(z) + \mu D R^\alpha 2^{\alpha-1}.$$
On the other hand, by (8.10) since for $|z_2 - w| \le R$ we have
$$-\frac{R}{2} \operatorname{Re}\left[\left(\frac{z_2 - w}{R}\right)^2\right] \le |\operatorname{Im} z_2|$$
we get
$$\tilde{u}(z) \le \mu|\operatorname{Im} z_1| + \mu D_1|\operatorname{Im} z_2| + |\operatorname{Im} z_2| + |\operatorname{Im} z|,$$
therefore,
$$v(z) \le \rho|\operatorname{Im} z| - k(\theta)\omega(z).$$
By $\mathrm{IPL}^0_+(\mathbb{R}^2, \omega)$,
$$v(z) \le A|\operatorname{Im} z| - m\omega(z) + \theta\mu D R^\alpha 2^{\alpha-1}$$
and
$$\tilde{u}(z) \le 2A|\operatorname{Im} z| - 2m\omega(z) + 2^\alpha \theta\mu D R^\alpha + C_k\sigma(z).$$
Taking $z = (z_1, w) \in W_j$ we get $\mu u(z) \le \tilde{u}(z)$ and
$$u(z) \le \frac{1}{2}|\operatorname{Im} z| + 2^\alpha \theta D R^\alpha + C_k\sigma(z)/\mu.$$
Moreover, $u(z) = \max(|\operatorname{Im} z| - C, 0)$ since $z_2 = w \in \mathbb{R}$, $a_{jq_j} \in \mathbb{R}$, hence either $|\operatorname{Im} z| \le C$ or
$$|\operatorname{Im} z| = u(z) + C \le \frac{1}{2}|\operatorname{Im} z| + 2^\alpha \theta D R^\alpha + C_k\sigma(z)/\mu + C.$$
Therefore
$$|\operatorname{Im} z| \le 2^{\alpha+1}\theta D R^\alpha + 2C_k\sigma(z)/\mu + 2C.$$
Using (8.11) and $|w| = R$ we get by choosing suitable z_1
$$E R^\alpha \le 2^{\alpha+1}\theta D R^\alpha + 2C_k\sigma(z)/\mu + 2C.$$
Since $\theta = E/(2^{\alpha+2}D)$,
$$R^\alpha \le \frac{4C_k}{\mu E}\sigma(z) + \frac{4C}{E}.$$
By (8.8) $\sigma(z) = O(\sigma(z_2)) = o(h(R))$; a contradiction.

We have proved that $t^\alpha = O(\omega(t))$.

Again applying (8.7) and the fact that all a_{jq_j} are real we get for $z = (z_1, z_2) \in V(P)$, $z_2 \in \mathbb{R}$,

$$| \operatorname{Im} z_1 | \leq F(1 + \omega(z_2))$$

for some constant F which implies (ω)-hyperbolicity with respect to $\vec{v}$ by the definition (see [54, Prop. 2.7]).

For reducible P it follows that each of its irreducible factors is (ω)-hyperbolic with respect to each non-characteristic direction. The conclusion follows by [54, Lemma 2.5]. ■

Clearly, a careful reader will find several further results on the parameter dependence following our theory, for instance, for ultradifferential dependence instead of real analytic dependence — we omit details.

9. Open problems

We collect some open problems suggested by the theory described in this paper. The first main problem is the following:

Problem 9.1 *Is it true that for arbitrary polynomial P the positive solution of the real analytic parameter dependence problem for $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is equivalent to the existence of continuous linear right inverse for the same operator?*

This problem is exactly equivalent to the following question (comp. [6, Problem 9.5]):

Problem 9.2 *Can we characterize existence of a continuous linear right inverse for $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ by the dual interpolation estimate for the kernel of $P(D)$?*

There are two tempting particular cases of Problem 9.1: $\Omega = H_+$ a half space and Ω an open bounded set with C^1-boundary. By Theorem 8.5 and [48, Cor. 2.10], a positive solution of the first case implies a positive solution for the second one. In view of [48, Prop. 3.2] the problem reduces to the question of hyperbolicity with respect to the normal direction to the boundary of a H_+.

By [23, Th. 1, Th. 3] (comp. [22]), $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ has a continuous linear right inverse if and only if the kernel $\ker P(D)$ is a strict PLS-space. The same holds for any operator $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ with a translation invariant kernel or the so-called s'-friendly kernel. Therefore Problem 9.1 is very close to the conjecture that *every translation invariant subspace of $\mathscr{D}'_{(\omega)}(\Omega)$ with $(P\overline{\overline{\Omega}})$ (or, with the dual interpolation estimate) is strict.* We do not know any counterexample although there is an example of

a PLS-Köthe sequence space which is s'-friendly non-strict subspace of $\mathscr{D}'$ with $(P\overline{\overline{\Omega}})$ and which is neither LB-space nor a Fréchet space.

Of course, one can consider Problem 9.1 and Problem 9.2 for ultradistributions $\mathscr{D}'_{(\omega)}(\Omega)$ and ultradifferentiable functions $\mathscr{E}_{\{\omega\}}(\Omega)$ as well.

There are interesting cases not covered by Theorem 5.2.

Problem 9.3 *Characterize* $\mathrm{Proj}^1\, X \varepsilon E = 0$ *for a PLN-space* X *and a PLS-space* E *in terms of conditions* (T) *or* (T_ε).

The problem is open even for Fréchet Schwartz E.

A truly challenging problem is to characterize in terms of the symbol kernels of linear partial differential operators with *variable* coefficients on $\mathscr{D}'(\Omega)$ or $\mathscr{A}(\Omega)$ which have the dual interpolation estimate for small θ since this would solve the real analytic parameter dependence problem for partial differential operators with variable coefficients via Corollary 6.4. Creating a suitable theory would require a nice description of the kernel — so it would be the analogue of the fundamental principle for partial differential operators with variable coefficients. It seems that we are very far from that.

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