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h^1, bmo, blo and Littlewood-Paley g-functions with non-doubling measures

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Abstract

Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which satisfies the growth condition that there exist constants $C_0 > 0$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, $\mu(B(x, r)) \le C_0 r^n$, where $B(x, r)$ is the open ball centered at x and having radius r. In this paper, we introduce a local atomic Hardy space $h^{1,\infty}_{\mathrm{atb}}(\mu)$, a local BMO-type space $\mathrm{rbmo}\,(\mu)$ and a local BLO-type space $\mathrm{rblo}\,(\mu)$ in the spirit of Goldberg and establish some useful characterizations for these spaces. Especially, we prove that the space $\mathrm{rbmo}\,(\mu)$ satisfies a John-Nirenberg inequality and its predual is $h^{1,\infty}_{\mathrm{atb}}(\mu)$. We also establish some useful properties of $\mathrm{RBLO}\,(\mu)$ and improve the known characterization theorems of $\mathrm{RBLO}\,(\mu)$ in terms of the natural maximal function by removing the assumption on the regularity condition. Moreover, the relations of these local spaces with known corresponding function spaces are also presented. As applications, we prove that the inhomogeneous Littlewood-Paley g-function $g(f)$ of Tolsa is bounded from $h^{1,\infty}_{\mathrm{atb}}(\mu)$ to $L^1(\mu)$, and that $[g(f)]^2$ is bounded from $\mathrm{rbmo}\,(\mu)$ to $\mathrm{rblo}\,(\mu)$.

1. Introduction

Recall that a *non-doubling measure* μ on $\mathbb{R}^d$ means that μ is a nonnegative Radon measure which only satisfies the following growth condition, namely, there exist constants $C_0 > 0$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu\,(B(x, r)) \le C_0 r^n, \tag{1.1}$$

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where $B(x,r)$ is the open ball centered at x and having radius r. Such a measure μ is not necessary to be doubling, which is a key assumption in the classical theory of harmonic analysis. In recent years, it was shown that many results on the Calderón-Zygmund theory remain valid for non-doubling measures; see, for example, [11, 12, 13, 14, 16, 17, 18, 10, 7, 26]. One of the main motivations for extending the classical theory to the non-doubling context was the solution of several questions related to analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [19, 20, 22] or survey papers [21, 23, 24, 25] for more details.

In particular, Tolsa [17] developed a Littlewood-Paley theory with non-doubling measures for functions in $L^p(\mu)$ when $p \in (1,\infty)$ and used this Littlewood-Paley decomposition to establish some $T(1)$ theorems. One of the main purposes of this paper is to investigate behaviors of the inhomogeneous Littlewood-Paley g-functions of Tolsa in [17] at the extremal cases, namely, in the cases when $p = 1$ or $p = \infty$. To this end, in this paper, we first introduce a local atomic Hardy space $h^{1,\infty}_{\rm atb}(\mu)$, a local BMO-type space $\mathrm{rbmo}\,(\mu)$ and a local BLO-type space $\mathrm{rblo}\,(\mu)$ in the spirit of Goldberg [4] and establish some useful characterizations for these spaces. Especially, we prove that the space $\mathrm{rbmo}\,(\mu)$ satisfies a John-Nirenberg inequality and its predual is $h^{1,\infty}_{\rm atb}(\mu)$. We also improve Theorem 2 and Theorem 3 of [7] on the characterization of $\mathrm{RBLO}\,(\mu)$ in terms of the natural maximal function by removing the assumption on the regularity condition there. Moreover, relations of these local spaces with the Hardy space $H^1(\mu)$ and the BMO-type space $\mathrm{RBMO}\,(\mu)$ of Tolsa in [16] and the BLO-type space $\mathrm{RBLO}\,(\mu)$ of Jiang in [7] are also presented. As applications, we prove that the inhomogeneous Littlewood-Paley g-function $g(f)$ of Tolsa is bounded from $h^{1,\infty}_{\rm atb}(\mu)$ to $L^1(\mu)$, and that $[g(f)]^2$ is bounded from $\mathrm{rbmo}\,(\mu)$ to $\mathrm{rblo}\,(\mu)$. We mention that when $\mathbb{R}^d$ is not an initial cube (see [17, Definition 3.4] or Definition 2.2 below) which implies $\mu(\mathbb{R}^d) = \infty$, we proved in [27] that the homogeneous Littlewood-Paley g-function $\dot{g}(f)$ of Tolsa is bounded from the Hardy space $H^1(\mu)$ to $L^1(\mu)$, and that if $f \in \mathrm{RBMO}\,(\mu)$, then $[\dot{g}(f)]^2$ is either infinite everywhere or finite almost everywhere, and in the latter case, $[\dot{g}(f)]^2$ is bounded from $\mathrm{RBMO}\,(\mu)$ to $\mathrm{RBLO}\,(\mu)$. This result generalizes the corresponding result of Leckband [9] in replacing $L^\infty(\mathbb{R}^d)$ by $\mathrm{BMO}\,(\mathbb{R}^d)$, even when μ is the d-dimensional Lebesgue measure and $\dot{g}(f)$ is the classical Littlewood-Paley g-function. Also, to the best of our knowledge, even when μ is the d-dimensional Lebesgue measure, both the space $\mathrm{rblo}\,(\mu)$ and the boundedness of the inhomogeneous Littlewood-Paley g-function $g(f)$ from $h^{1,\infty}_{\rm atb}(\mu)$ to $L^1(\mu)$ and from $\mathrm{rbmo}\,(\mu)$ to $\mathrm{rblo}\,(\mu)$ are new. An interesting open problem is if $\dot{g}(f)$ and $g(f)$ can characterize the Hardy space $H^1(\mu)$ and $h^{1,\infty}_{\rm atb}(\mu)$, respectively.

We remark that some other variants of local atomic Hardy space and local BMO-type space in the sense of Goldberg were also introduced in [26]. However, it seems that they are not natural for the boundedness of the inhomogeneous Littlewood-Paley g-function. A *new idea* used in this paper is to *classify cubes of* $\mathbb{R}^d$ *by using the coefficients* $\delta(\cdot,\cdot)$ *of Tolsa* [17] (see [17, Definition 3.2] or Definition 2.1 below); while in [26], cubes are classified by their side lengths as in the case of Euclidean spaces in [4]. To be precise, in this paper, using the coefficients $\delta(\cdot,\cdot)$, we introduce a class $\mathcal{D}$ of cubes, which have "large" side lengths in the sense that if μ is the d-dimensional Lebesgue measure, then $Q \in \mathcal{D}$ if and only if the side length of Q is no less than C, where C is a positive constant independent of Q. We then use $\mathcal{D}$ to define our local Hardy space, local BMO-type space and local BLO-type space.

It is well-known that the coefficients $\delta(\cdot,\cdot)$ of Tolsa describe well the geometric properties of cubes of $\mathbb{R}^d$; see Lemma 3.1 in [17] (or Lemma 2.1 below). These properties play key roles in the whole theory of analysis associated with non-doubling measures. Using these coefficients, Tolsa in [17, 18] further found suitable variants of dyadic cubes, which are now called *cubes of generations*. These cubes of generations are the basis of the construction on approximations of the identity of Tolsa in [17]. Another *novelty* of this paper is that we introduce a quantity, which further clarifies the geometric relations between general cubes and "dyadic" cubes of Tolsa in [17, 18]; see Lemma 2.2 below. These properties together with the known properties of "dyadic" cubes (see, for example, Lemma 3.4 and Lemma 4.2 in [17]) are key tools used in this paper.

The organization of this paper is as follows. In Section 2, we recall some necessary definitions and notation, including the definitions of the spaces $H^{1,p}_{\mathrm{atb}}(\mu)$, RBMO$(\mu)$, RBLO$(\mu)$, approximations of the identity and the inhomogeneous Littlewood-Paley g-function. We also remark that the space RBLO(μ) used in this paper is a slight variant of the corresponding one in [7]. Section 3 is divided into two parts. In Section 3.1, we introduce the spaces rbmo(μ) and $h^{1,p}_{\mathrm{atb}}(\mu)$ with $p \in (1,\infty]$, and obtain some basic properties of these spaces, including the John-Nirenberg inequality, the duality between rbmo(μ) and $h^{1,p}_{\mathrm{atb}}(\mu)$, and the relations between $H^{1,p}_{\mathrm{atb}}(\mu)$ and $h^{1,p}_{\mathrm{atb}}(\mu)$ and between RBMO(μ) and rbmo(μ). In Section 3.2, we first establish some useful properties for RBLO(μ) and improve Theorem 2 and Theorem 3 of [7] on the characterization of RBLO(μ) in terms of the natural maximal function by removing the assumption on the regularity condition there, and we then introduce the space rblo(μ) and establish the relation between RBLO(μ) and rblo(μ). Moreover, similar to the result in [7], we also obtain a characterization of rblo(μ) by a local maximal operator. In

Section 4, applying the results in Section 3, we establish the boundedness of the Littlewood-Paley g-function $g(f)$ from $h^{1,p}_{\rm atb}(\mu)$ to $L^1(\mu)$, and prove that if $f \in {\rm rbmo}\,(\mu)$, then $[g(f)]^2$ belongs to ${\rm rblo}\,(\mu)$ with norm no more than $C\|f\|^2_{{\rm rbmo}\,(\mu)}$, where C is a positive constant independent of f. As a corollary, we also obtain the boundedness of the Littlewood-Paley g-function $g(f)$ from ${\rm rbmo}\,(\mu)$ to ${\rm rblo}\,(\mu)$.

We finally make some convention. Throughout the paper, we always denote by C a *positive constant* which is independent of main parameters, but it may vary from line to line. *Constant with subscript such as* C_1, does not change in different occurrences. The *notation* $Y \lesssim Z$ means that there exists a positive constant C such that $Y \le CZ$, while $Y \gtrsim Z$ means that there exists a positive constant C such that $Y \ge CZ$. The *symbol* $A \sim B$ means that $A \lesssim B \lesssim A$. For $p \in (1,\infty)$, denote by p' the *conjugate index* of p, namely, $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, for any $D \subset \mathbb{R}^d$, we denote by χ_D the *characteristic function* of D. We also set $\mathbb{N} := \{1,\, 2,\, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

2. Preliminaries

Throughout this paper, by a *cube* $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the axes and centered at some point of ${\rm supp}\,(\mu)$, and we denote its side length by $l(Q)$ and its center by x_Q. If $\mu(\mathbb{R}^d) < \infty$, we also regard $\mathbb{R}^d$ as a cube. Let α, β be two positive constants, $\alpha \in (1,\infty)$ and $\beta \in (\alpha^n, \infty)$. We say that a cube Q is an $(\alpha,\, \beta)$-*doubling cube* if it satisfies $\mu(\alpha Q) \le \beta\mu(Q)$, where and in what follows, given $\lambda > 0$ and any cube Q, λQ denotes the *cube concentric with* Q *and having side length* $\lambda l(Q)$. It was pointed out by Tolsa (see [16, pp. 95-96] or [17, Remark 3.1]) that if $\beta > \alpha^n$, then for any $x \in {\rm supp}\,(\mu)$ and any $R > 0$, there exists some $(\alpha,\, \beta)$-doubling cube Q centered at x with $l(Q) \ge R$, and that if $\beta > \alpha^d$, then for μ-almost everywhere $x \in \mathbb{R}^d$, there exists a sequence of $(\alpha,\, \beta)$-doubling cubes $\{Q_k\}_{k\in\mathbb{N}}$ centered at x with $l(Q_k) \to 0$ as $k \to \infty$. Let $\rho \in (1,\infty)$. Throughout this paper, *we always take* $\beta_\rho := \rho^{d+1}$. For any cube Q, let $\widetilde{Q}^\rho$ be the *smallest* (ρ, β_ρ)-*doubling cube* which has the form $\rho^k Q$ with $k \in \mathbb{N} \cup \{0\}$. If $\rho = 2$, we denote the *cube* $\widetilde{Q}^\rho$ simply by $\widetilde{Q}$. Moreover, by a *doubling cube* Q, we always mean a $(2, 2^{d+1})$-doubling cube.

Given two cubes Q, $R \subset \mathbb{R}^d$, let Q_R be the smallest cube concentric with Q containing Q and R. The following coefficients were first introduced by Tolsa in [16]; see also [17, 18].

Definition 2.1. *Given two cubes* Q, $R \subset \mathbb{R}^d$, *we define*

$$\delta(Q,R) := \max\left\{ \int_{Q_R\setminus Q} \frac{1}{|x-x_Q|^n}\, d\mu(x),\ \int_{R_Q\setminus R} \frac{1}{|x-x_R|^n}\, d\mu(x) \right\}.$$

We may treat points $x \in \mathbb{R}^d$ as if they were cubes (with side length $l(x) = 0$). So, for any x, $y \in \mathbb{R}^d$ and cube $Q \subset \mathbb{R}^d$, the symbols $\delta(x, Q)$ and $\delta(x, y)$ make sense.

The following useful properties of $\delta(\cdot, \cdot)$, which were proved by Tolsa in [18, pp. 320-321] (see also [17, Lemma 3.1]), play important roles in the whole paper.

Lemma 2.1. *There exists a positive constant C, which only depends on C_0, n, d and ρ, such that the following properties hold:*

(a) *If $l(Q) \sim l(R)$ and $\operatorname{dist}(Q, R) \le Cl(Q)$, then $\delta(Q, R) \le C$. Moreover, $\delta(Q, \eta Q) \le C_0 2^n \eta^n$ for any $\eta \in (1, \infty)$.*

(b) *Let $\rho \in (1, \infty)$ and $Q \subset R$ be concentric cubes such that there exist no (ρ, β_ρ)-doubling cubes of the form $\rho^k Q$, $k \ge 0$, with $Q \subset \rho^k Q \subset R$. Then $\delta(Q, R) \le C$.*

(c) *If $Q \subset R$, then $\delta(Q, R) \le C[1 + \log \frac{l(R)}{l(Q)}]$.*

(d) *There exists an $\epsilon_0 > 0$ such that if $P \subset Q \subset R$, then*

$$\delta(P, R) = \delta(P, Q) + \delta(Q, R) \pm \epsilon_0.$$

In particular, $\delta(P, Q) \le \delta(P, R) + \epsilon_0$ and $\delta(Q, R) \le \delta(P, R) + \epsilon_0$. Moreover, if P and Q are concentric, then $\epsilon_0 = 0$.

(e) *For any P, Q, $R \subset \mathbb{R}^d$, $\delta(P, R) \le C + \delta(P, Q) + \delta(Q, R)$.*

We now recall the notion of cubes of generations in [17, 18].

Definition 2.2. *We say that $x \in \mathbb{R}^d$ is a stopping point (or stopping cube) if $\delta(x, Q) < \infty$ for some cube $Q \ni x$ with $0 < l(Q) < \infty$. We say that $\mathbb{R}^d$ is an initial cube if $\delta(Q, \mathbb{R}^d) < \infty$ for some cube Q with $0 < l(Q) < \infty$. The cubes Q such that $0 < l(Q) < \infty$ are called transit cubes.*

Remark 2.1. *In* [17, p. 67]*, it was pointed out that if $\delta(x, Q) < \infty$ for some transit cube Q containing x, then $\delta(x, Q') < \infty$ for any other transit cube Q' containing x. Also, if $\delta(Q, \mathbb{R}^d) < \infty$ for some transit cube Q, then $\delta(Q', \mathbb{R}^d) < \infty$ for any transit cube Q'.*

Let A be some *big positive constant.* In particular, as in [17, 18], we assume that A is much bigger than the constants ϵ_0, ϵ_1 and γ_0, which appear, respectively, in Lemma 3.1, Lemma 3.2 and Lemma 3.3 of [17]. Moreover, the constants A, ϵ_0, ϵ_1 and γ_0 depend only on C_0, n, d and ρ. In what follows, for $\epsilon > 0$ and $a, b \in \mathbb{R}$, the *notation* $a = b \pm \epsilon$ does not mean any precise equality but the estimate $|a - b| \le \epsilon$.

Definition 2.3. *Assume that $\mathbb{R}^d$ is not an initial cube. Let $\rho \in (1,\infty)$ and $\beta_\rho := \rho^{d+1}$. We fix some (ρ,β_ρ)-doubling cube $R_0 \subset \mathbb{R}^d$. This will be our 'reference' cube. For each $j \in \mathbb{N}$, let R_{-j} be some (ρ,β_ρ)-doubling cube concentric with R_0, containing R_0, and such that $\delta(R_0, R_{-j}) = jA \pm \epsilon_1$ (which exists because of* [17, Lemma 3.3]*). If Q is a transit cube, we say that Q is a cube of generation $k \in \mathbb{Z}$ if it is a (ρ,β_ρ)-doubling cube, and for some cube R_{-j} containing Q we have $\delta(Q, R_{-j}) = (j+k)A \pm \epsilon_1$. If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation $k \in \mathbb{Z}$ if for some cube R_{-j} containing x we have $\delta(Q, R_{-j}) \le (j+k)A + \epsilon_1$.*

We remark that the definition of cubes of generations is proved in [17, p. 68] to be independent of the chosen reference cubes $\{R_{-j}\}_{j\in\mathbb{Z}_+}$ in the sense modulo some small errors.

Definition 2.4. *Assume that $\mathbb{R}^d$ is an initial cube. Let $\rho \in (1,\infty)$ and $\beta_\rho := \rho^{d+1}$. Then we choose $\mathbb{R}^d$ as our 'reference' cube: If Q is a transit cube, we say that Q is a cube of generation $k \ge 1$, if Q is (ρ,β_ρ)-doubling and $\delta(Q, \mathbb{R}^d) = kA \pm \epsilon_1$. If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation $k \ge 1$ if $\delta(x, \mathbb{R}^d) \le kA + \epsilon_1$. Moreover, for all $k \le 0$, we say that $\mathbb{R}^d$ is a cube of generation k.*

In what follows, we also regard that $\mathbb{R}^d$ is a cube centered at all the points $x \in \operatorname{supp}(\mu)$. Using [17, Lemma 3.2], it is easy to verify that for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, there exists a (ρ,β_ρ)-doubling cube of generation k centered at x; see [17, p. 68]. Throughout this paper, for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, we denote by $Q_{x,k}$ a *fixed (ρ,β_ρ)-doubling cube centered at x of generation k*. By [27, Proposition 2.1] and Definition 2.4, it follows that for any $x \in \operatorname{supp}(\mu)$, $l(Q_{x,k}) \to \infty$ as $k \to -\infty$.

Remark 2.2. *We should point out that when $\mathbb{R}^d$ is an initial cube, cubes of generations in* [17] *were not assumed to be doubling. However, by using* [17, Lemma 3.2]*, it is easy to check that doubling cubes of generations exist even in this case. Moreover, it is not so difficult to verify that $(2, 2^{d+1})$-doubling cubes in* [17] *can be replaced by (ρ,β_ρ)-doubling cubes, where $\rho \in (1,\infty)$ and $\beta_\rho = \rho^{d+1}$.*

In [17], Tolsa constructed an *approximation of the identity* $S := \{S_k\}_{k=-\infty}^{\infty}$ related to $(2, 2^{d+1})$-*doubling cubes* $\{Q_{x,k}\}_{x\in\mathbb{R}^d,\, k\in\mathbb{Z}}$, which are integral operators given by kernels $S_k(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying the following properties:

(A-1) $S_k(x,y) = S_k(y,x)$ for all x, $y \in \mathbb{R}^d$;

(A-2) For any $k \in \mathbb{Z}$ and any $x \in \operatorname{supp}(\mu)$, if $Q_{x,k}$ is a transit cube, then

$$\int_{\mathbb{R}^d} S_k(x,y)\, d\mu(y) = 1;$$

(A-3) If $Q_{x,k}$ is a transit cube, then $\operatorname{supp}(S_k(x,\cdot))\subset Q_{x,k-1}$;

(A-4) If $Q_{x,k}$ and $Q_{y,k}$ are transit cubes, then there exists a positive constant C such that

$$0\le S_k(x,y)\le \frac{C}{[l(Q_{x,k})+l(Q_{y,k})+|x-y|]^n}; \tag{2.1}$$

(A-5) If $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ are transit cubes, and $x,\,x'\in Q_{x_0,k}$ for some $x_0\in\operatorname{supp}(\mu)$, then there exists a positive constant C such that

$$|S_k(x,y)-S_k(x',y)|\le C\frac{|x-x'|}{l(Q_{x_0,k})}\frac{1}{[l(Q_{x,k})+l(Q_{y,k})+|x-y|]^n}. \tag{2.2}$$

Moreover, Tolsa also pointed out that (A-1) through (A-5) also hold if any of $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ is a stopping cube, and that (A-1), (A-3) through (A-5) also hold if any of $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ coincides with $\mathbb{R}^d$, except that (A-2) is replaced by (A-2)': if $Q_{x,k}=\mathbb{R}^d$ for some $x\in\operatorname{supp}(\mu)$, then $S_k:=0$. In what follows, without loss of generality, for any $x\in\operatorname{supp}(\mu)$, we *always assume that $Q_{x,k}$ is not a stopping cube*, since the proofs for stopping cubes are similar. Moreover, in what follows, when we mention the approximation of the identity S, we always mean that they are associated with $(2,2^{d+1})$-*doubling cubes.*

For $k\in\mathbb{Z}$, let $D_k:=S_k-S_{k-1}$. We also use D_k to denote the *corresponding integral operator with kernel D_k*. The *inhomogeneous Littlewood-Paley g-function $g(f)$* is defined by

$$g(f)(x):=\left[|S_1(f)(x)|^2+\sum_{k=2}^{\infty}|D_k(f)(x)|^2\right]^{1/2}.$$

We next recall the notions of the spaces $H^1(\mu)$ and $\mathrm{RBMO}(\mu)$ in [16] and the space $\mathrm{RBLO}(\mu)$ in [7].

Definition 2.5. *Given $f\in L^1_{\mathrm{loc}}(\mu)$, we set*

$$\mathcal{M}_\Phi(f)(x):=\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d}f\varphi\,d\mu\right|,$$

where the notation $\varphi\sim x$ means that $\varphi\in L^1(\mu)\cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)}\le 1$,

(ii) $0\le\varphi(y)\le\frac{1}{|y-x|^n}$ *for all $y\in\mathbb{R}^d$, and*

(iii) $|\nabla\varphi(y)| \le \frac{1}{|y-x|^{n+1}}$ *for all* $y \in \mathbb{R}^d$, *where* $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$.

Definition 2.6. *The Hardy space* $H^1(\mu)$ *is the set of all functions* $f \in L^1(\mu)$ *satisfying that* $\int_{\mathbb{R}^d} f\,d\mu = 0$ *and* $\mathcal{M}_\Phi(f) \in L^1(\mu)$. *Moreover, we define the norm of* $f \in H^1(\mu)$ *by*

$$\|f\|_{H^1(\mu)} := \|f\|_{L^1(\mu)} + \|\mathcal{M}_\Phi(f)\|_{L^1(\mu)}.$$

On the Hardy space, Tolsa established the following atomic characterization (see [16, 18]).

Definition 2.7. *Let* $\eta \in (1,\infty)$ *and* $p \in (1,\infty]$. *A function* $b \in L^1_{\rm loc}(\mu)$ *is called a p-atomic block if*

(i) *there exists some cube* R *such that* $\operatorname{supp}(b) \subset R$,

(ii) $\int_{\mathbb{R}^d} b(x)\,d\mu(x) = 0$,

(iii) *for* $j = 1, 2$, *there exist functions* a_j *supported on cubes* $Q_j \subset R$ *and numbers* $\lambda_j \in \mathbb{R}$ *such that* $b = \lambda_1 a_1 + \lambda_2 a_2$, *and*

$$\|a_j\|_{L^p(\mu)} \le [\mu(\eta Q_j)]^{1/p-1}[1 + \delta(Q_j, R)]^{-1}. \tag{2.3}$$

Then we define $|b|_{H^{1,p}_{\rm atb}(\mu)} := |\lambda_1| + |\lambda_2|$.

A function $f \in L^1(\mu)$ *is said to belong to the space* $H^{1,p}_{\rm atb}(\mu)$ *if there exist p-atomic blocks* $\{b_i\}_{i\in\mathbb{N}}$ *such that* $f = \sum_{i=1}^\infty b_i$ *with* $\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\rm atb}(\mu)} < \infty$. *The* $H^{1,p}_{\rm atb}(\mu)$ *norm of* f *is defined by*

$$\|f\|_{H^{1,p}_{\rm atb}(\mu)} := \inf\left\{\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\rm atb}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f *in p-atomic blocks as above.*

Remark 2.3. *It was proved in* [16, 18] *that the definition of* $H^{1,p}_{\rm atb}(\mu)$ *in* [16] *is independent of the chosen constant* $\eta \in (1,\infty)$, *and for any* $p \in (1,\infty)$, *all the atomic Hardy spaces* $H^{1,p}_{\rm atb}(\mu)$ *coincide with* $H^{1,\infty}_{\rm atb}(\mu)$ *with equivalent norms. Moreover, Tolsa proved that* $H^{1,\infty}_{\rm atb}(\mu)$ *coincides with* $H^1(\mu)$ *with equivalent norms (see* [18, Theorem 1.2]*). Thus, in the rest of this paper, we identify the atomic Hardy space* $H^{1,p}_{\rm atb}(\mu)$ *with* $H^1(\mu)$, *and when we use the atomic characterization of* $H^1(\mu)$, *we always assume* $\eta = 2$ *and* $p = \infty$ *in Definition* 2.7.

Definition 2.8. *Let η, $\rho \in (1, \infty)$ and $\beta_\rho := \rho^{d+1}$. A function $f \in L^1_{\rm loc}(\mu)$ is said to be in the space* RBMO(μ) *if there exists some constant $\widetilde{C} \geq 0$ such that for any cube Q centered at some point of* supp(μ),

$$\frac{1}{\mu(\eta Q)} \int_Q \left| f(y) - m_{\widetilde{Q}^\rho}(f) \right| d\mu(y) \leq \widetilde{C},$$

and for any two (ρ, β_ρ)-doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \leq \widetilde{C}[1 + \delta(Q, R)],$$

where $m_Q(f)$ denotes the mean of f over cube Q, namely,

$$m_Q(f) := \frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y).$$

Moreover, we define the RBMO(μ) *norm of f to be the minimal constant $\widetilde{C}$ as above and denote it by $\|f\|_{{\rm RBMO}(\mu)}$.*

Remark 2.4. *It was proved by Tolsa in* [16] *that the definition of* RBMO(μ) *is independent of the choices of η and ρ. As a result, unless explicitly pointed out, in what follows, when we mention* RBMO(μ) *we always take $\rho = \eta = 2$ in Definition* 2.8.

The following space RBLO(μ) is a slight variant of the corresponding space introduced by Jiang in [7]. In fact, we will show that they both coincide with equivalent norms (see Proposition 3.10 below). It is obvious that $L^\infty(\mu) \subset$ RBLO(μ).

Definition 2.9. *Let η, $\rho \in (1, \infty)$ and $\beta_\rho := \rho^{d+1}$. A function $f \in L^1_{\rm loc}(\mu)$ is said to belong to the space* RBLO(μ) *if there exists some constant $\widetilde{C} \geq 0$ such that for any cube Q centered at some point of* supp(μ),

$$\frac{1}{\mu(\eta Q)} \int_Q \left[f(x) - \operatorname*{essinf}_{\widetilde{Q}^\rho} f \right] d\mu(x) \leq \widetilde{C}, \tag{2.4}$$

and for any two (ρ, β_ρ)-doubling cubes $Q \subset R$,

$$\operatorname*{essinf}_Q f - \operatorname*{essinf}_R f \leq \widetilde{C}[1 + \delta(Q,\, R)]. \tag{2.5}$$

The minimal constant $\widetilde{C}$ as above is defined to be the norm of f in the space RBLO(μ) *and denoted by $\|f\|_{{\rm RBLO}(\mu)}$.*

Remark 2.5. *(i) In* [7], *the space* RBLO(μ) *was defined in the following way, namely, a function* $f \in L^1_{\mathrm{loc}}(\mu)$ *is said to belong to the space* RBLO(μ) *if there exists a nonnegative constant* $\widetilde{C}$ *such that for any* $(4\sqrt{d}, (4\sqrt{d})^{n+1})$*-doubling cube* Q*,*

$$\frac{1}{\mu(Q)} \int_Q \left[f(x) - \operatorname*{essinf}_Q f\right] d\mu(x) \le \widetilde{C},$$

and for any $(4\sqrt{d}, (4\sqrt{d})^{n+1})$*-doubling cubes* $Q \subset R$*,*

$$m_Q(f) - m_R(f) \le \widetilde{C}[1 + \delta(Q, R)].$$

The minimal constant $\widetilde{C}$ *as above is defined to be the norm of* f *in the space* RBLO(μ)*.*

(ii) Due to the observation of Tolsa on the existence of small (ρ, ρ^{d+1})*-doubling cubes, where* $\rho > 1$*, it seems that it is convenient in applications to replace* $(4\sqrt{d}, (4\sqrt{d})^{n+1})$*-doubling cubes by* (ρ, ρ^{d+1})*-doubling cubes in the definition of* RBLO(μ) *in (i). Moreover, Definition* 2.9 *is convenient in proving that the definition of the space* RBLO(μ) *is independent of the choice of the constants* η, $\rho \in (1, \infty)$*; see Proposition* 3.8 *and Proposition* 3.9 *below.*

To introduce our local spaces, a new idea is to introduce a *special set of cubes via the coefficients of Tolsa* in [16, 17], which is a *key point*. To be precise, in the case that $\mathbb{R}^d$ is not an initial cube, letting $\{R_{-j}\}_{j\in\mathbb{Z}_+}$ be the cubes as in Definition 2.3, we then define the *set*

$$\mathcal{D} := \big\{Q \subset \mathbb{R}^d : \text{ there exists a cube } P \subset Q \text{ and } j \in \mathbb{Z}_+ \text{ such that } P \subset R_{-j} \text{ with } \delta(P, R_{-j}) \le (j+1)A + \epsilon_1\big\}.$$

If $\mathbb{R}^d$ is an initial cube, we define the *set*

$$\mathcal{D} := \big\{Q \subset \mathbb{R}^d : \text{ there exists a cube } P \subset Q \text{ such that } \delta(P, \mathbb{R}^d) \le A + \epsilon_1\big\}.$$

It is easy to see that *if* $Q \in \mathcal{D}$*, then any* R *containing* Q *is also in* $\mathcal{D}$ and the definition of the set $\mathcal{D}$ is *independent of the chosen reference cubes* $\{R_{-j}\}_{j\in\mathbb{Z}_+}$ in the sense modulo some small error (the error is no more than $2\epsilon_1 + \epsilon_0$); see also [17, p. 68]. Moreover, the following observation implies that in the case that μ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$, then for any cube $Q \subset \mathbb{R}^d$, $Q \in \mathcal{D}$ if and only if $l(Q) \gtrsim 1$. Based on this observation, we can think that our local spaces are the local spaces in the spirit of Goldberg [4].

Proposition 2.1. *Let* μ *be the* d*-dimensional Lebesgue measure on* $\mathbb{R}^d$*. Then for any cube* $Q \subset \mathbb{R}^d$*,* $Q \in \mathcal{D}$ *if and only if* $l(Q) \ge a_0$*, where* a_0 *is a positive constant independent of* Q*.*

Proof. In this case, we choose $\{R_{-j}\}_{j\in\mathbb{Z}_+}$ as the cubes centered at the origin with side length 2^j. We first see the sufficiency. For any cube Q with $l(Q) \geq a_0$, it is easy to see that there exists a nonnegative constant $\widetilde{C}$, which depends only on d, and $j \in \mathbb{Z}_+$ such that $Q \subset R_{-j}$ and

$$\delta(Q, R_{-j}) \leq \int_{B(x_Q,\sqrt{d}l(R_{-j}))\setminus B(x_Q,\frac{a_0}{2})} \frac{1}{|x - x_Q|^d}\, dx \leq (j+1)A,$$

where $A \geq \widetilde{C}\log(2\sqrt{d}\max(1/a_0, 1))$. Thus $Q \in \mathcal{D}$.

Conversely, if $Q \in \mathcal{D}$, then there exists a cube $Q' \subset Q$ and $j \in \mathbb{Z}_+$ such that $Q' \subset R_{-j}$ and $\delta(Q', R_{-j}) \leq (j+1)A + \epsilon_1$. To finish the proof, it suffices to verify that $l(Q') \gtrsim 1$. Moreover, we only need to consider the case that $l(Q') < (\sqrt{d})^{-1}$. Let ω_{d-1} be the $(d-1)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^d$. By Definition 2.1, we have

$$\delta(Q', R_{-j}) = \int_{Q'_{R_{-j}}\setminus Q'} \frac{1}{|x - x_{Q'}|^d} dx.$$

From this and the fact that

$$\{x \in \mathbb{R}^d : \sqrt{d}l(Q')/2 \leq |x - x_{Q'}| \leq l(R_{-j})/2\} \subset (Q'_{R_{-j}} \setminus Q'),$$

it follows that

$$\delta(Q', R_{-j}) \geq \omega_{d-1} \int_{\frac{\sqrt{d}l(Q')}{2}}^{\frac{l(R_{-j})}{2}} \frac{1}{r}\, dr = \omega_{d-1} \log\left(\frac{l(R_{-j})}{\sqrt{d}l(Q')}\right),$$

which implies that

$$l(R_{-j}) \lesssim 2^{\delta(Q', R_{-j})/\omega_{d-1}} l(Q') \lesssim 2^{(jA)/\omega_{d-1}} l(Q'). \tag{2.6}$$

On the other hand, since $l(R_0) = 1$,

$$\begin{aligned}\delta(R_0, R_{-j}) &= \int_{R_{-j}\setminus R_0} \frac{1}{|x|^d}\, dx \\ &\leq \omega_{d-1} \int_{\frac{1}{2}}^{\sqrt{d}l(R_{-j})} \frac{1}{r}\, dr = \omega_{d-1} \log\left(2\sqrt{d}l(R_{-j})\right),\end{aligned}$$

which together with Definition 2.3 yields that

$$l(R_{-j}) \gtrsim 2^{\delta(R_0, R_{-j})/\omega_{d-1}} \gtrsim 2^{(jA)/\omega_{d-1}}. \tag{2.7}$$

Combining (2.6) and (2.7) implies that $l(Q') \gtrsim 1$, which completes the proof of Proposition 2.1. ■

In what follows, for any cube R and $x \in R \cap \operatorname{supp}(\mu)$, let H_R^x be the *largest integer* k such that $R \subset Q_{x,k}$. The following properties on H_R^x are useful in applications.

Lemma 2.2. *The following properties hold:*

(a) *For any cube R and $x \in R \cap \operatorname{supp}(\mu)$, $Q_{x, H_R^x+1} \subset 3R$ and $5R \subset Q_{x, H_R^x-1}$.*

(b) *For any cube R, $x \in R \cap \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$ with $k \geq H_R^x + 2$, $Q_{x,k} \subset \frac{7}{5}R$.*

(c) *For any cube $R \subset \mathbb{R}^d$ and $x, y \in R \cap \operatorname{supp}(\mu)$, $|H_R^x - H_R^y| \leq 1$.*

(d) *For any cube R and $x \in R \cap \operatorname{supp}(\mu)$, $H_R^x \geq 0$ when $R \notin \mathcal{D}$. Moreover, if $R \in \mathcal{D}$, then $H_R^x \leq 1$ when $\mathbb{R}^d$ is not an initial cube, and $0 \leq H_R^x \leq 1$ when $\mathbb{R}^d$ is an initial cube.*

(e) *When $k \geq 2$, for any $x \in \operatorname{supp}(\mu)$, $Q_{x,k} \notin \mathcal{D}$.*

(f) *For any cube $R \notin \mathcal{D}$ and $x \in R \cap \operatorname{supp}(\mu)$, if any cube $R' \subset Q_{x, H_R^x+2}$, then $R' \notin \mathcal{D}$.*

(g) *For any cube R and $x \in R \cap \operatorname{supp}(\mu)$, there exists a positive constant C such that $\delta(R, Q_{x,H_R^x}) \leq C$ and $\delta(Q_{x,H_R^x+1}, R) \leq C$.*

Proof. We first verify (a). For any $x \in R \cap \operatorname{supp}(\mu)$, by the definition of H_R^x together with the decreasing property of $Q_{x,k}$ in k, we know that $R \subset Q_{x, H_R^x}$ and $R \not\subset Q_{x, H_R^x+1}$, which imply that $l(R) \leq l(Q_{x, H_R^x})$ and $l(Q_{x, H_R^x+1}) \leq 2l(R)$. These facts together with the fact that $l(Q_{x, H_R^x}) \leq \frac{1}{10} l(Q_{x, H_R^x-1})$ (see [17, p. 69]) imply (a).

To see (b), for any $x \in R \cap \operatorname{supp}(\mu)$, by the fact that $l(Q_{x, H_R^x+2}) \leq \frac{1}{10} l(Q_{x, H_R^x+1})$ (see [17, p. 69]) together with the fact that $l(Q_{x, H_R^x+1}) \leq 2l(R)$, we have $l(Q_{x, H_R^x+2}) \leq \frac{1}{5} l(R)$. Thus, $Q_{x, H_R^x+2} \subset \frac{7}{5}R$, which together with the decreasing property of $Q_{x,k}$ in k again verifies (b).

For any $R \subset \mathbb{R}^d$ and $x, y \in R \cap \operatorname{supp}(\mu)$, it is clear that $y \in Q_{x, H_R^x} \cap Q_{y, H_R^y}$. Then Lemma 4.2 in [17] together with the definition of H_R^x implies that $R \subset Q_{x, H_R^x} \subset Q_{y, H_R^x-1}$. This shows that $H_R^y \geq H_R^x - 1$. Symmetrically, we have $H_R^y \leq H_R^x + 1$, which verifies (c).

We now verify (d). Assume that $R \notin \mathcal{D}$. By similarity, we only consider the case that $\mathbb{R}^d$ is not an initial cube. Recall that

$$\sigma := 100\epsilon_0 + 100\epsilon_1 + (12)^{n+1} C_0$$

(see [17, p. 69]).

Then by Lemma 2.1 (a), we see that for any cube Q, $\delta(Q, 3Q) \le 6^n C_0 < \sigma \ll A$ (see also [17, p. 69]). Now assume that $3R \subset R_{-j}$ for some $j \in \mathbb{Z}_+$. If $R \notin \mathcal{D}$, by the conclusion of (a) and Lemma 2.1 (d), we have that

$$\begin{aligned}(j+1)A + \epsilon_1 < \delta(R, R_{-j}) &= \delta(R, 3R) + \delta(3R, R_{-j})\\ &< \sigma + \delta(Q_{x, H_R^x+1}, R_{-j}) + \epsilon_0\\ &= \sigma + (H_R^x + 1 + j)A \pm 4\epsilon_1 + \epsilon_0.\end{aligned}$$

This estimate together with the fact that $\epsilon_0 \le \epsilon_1 \ll \sigma \ll A$ implies that $H_R^x \ge 0$.

If $\mathbb{R}^d$ is not an initial cube and $R \in \mathcal{D}$, by the definitions of $\mathcal{D}$ and cubes of generations, there exists a cube $Q' \subset R$ and $j_1, j_2 \in \mathbb{Z}_+$ such that $Q' \subset R_{-j_1}$ with $\delta(Q', R_{-j_1}) \le (j_1+1)A + \epsilon_1$ and $Q_{x, H_R^x} \subset R_{-j_2}$ with $\delta(Q_{x, H_R^x}, R_{-j_2}) = (j_2 + H_R^x)A \pm \epsilon_1$. Let $j := \max(j_1, j_2)$. By Lemma 2.1, we have $\delta(Q_{x, H_R^x}, R_{-j}) = (j + H_R^x)A \pm 4\epsilon_1$. Lemma 2.1 (d) together with the definition of $Q \in \mathcal{D}$ and the fact that $\epsilon_0 \le \epsilon_1$ (see [17, p. 67]) implies that $\delta(Q_{x, H_R^x}, R_{-j}) \le (j+1)A + 4\epsilon_1$. On the other hand, if $H_R^x \ge 2$, then by the fact that $\epsilon_1 \ll A$,

$$\delta(Q_{x, H_R^x}, R_{-j}) = (j + H_R^x)A \pm 4\epsilon_1 > (j+1)A + 4\epsilon_1.$$

This is a contradiction, which verifies that $H_R^x \le 1$ when $R \in \mathcal{D}$.

Similarly, if $\mathbb{R}^d$ is an initial cube, then for any cube $R \in \mathcal{D}$ and $x \in R \cap \operatorname{supp}(\mu)$, we also have that $H_R^x \le 1$. On the other hand, recall that if $\mathbb{R}^d$ is an initial cube, then for any cube R, $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$ with $k \le 0$, $Q_{x,k} = \mathbb{R}^d$. Therefore, obviously, $H_R^x \ge 0$, which verify (d).

To see (e), by similarity, we only consider the case that $\mathbb{R}^d$ is an initial cube. Assume that $Q_{x,k} \in \mathcal{D}$. By the definition, there exists a cube $Q \subset Q_{x,k}$ such that $\delta(Q, \mathbb{R}^d) \le A + \epsilon_1$. By Lemma 2.1 (d), we then have

$$A + \epsilon_1 \ge \delta(Q, \mathbb{R}^d) = \delta(Q, Q_{x,k}) + \delta(Q_{x,k}, \mathbb{R}^d) \pm \epsilon_0 \ge kA - \epsilon_1 - \epsilon_0,$$

which is impossible when $k \ge 2$, since $A \gg \epsilon_1 \ge \epsilon_0$. Thus, $Q_{x,k} \notin \mathcal{D}$, which completes the proof of (e).

To prove (f), we only consider the case that $\mathbb{R}^d$ is not an initial cube, since the argument for the case that $\mathbb{R}^d$ is an initial cube is similar. If any cube $R' \in \mathcal{D}$, by the definition of $\mathcal{D}$, there exists a cube $R'' \subset R'$ and $j_1 \in \mathbb{Z}_+$ such that $R'' \subset R_{-j_1}$ and $\delta(R'', R_{-j_1}) \le (j_1+1)A + \epsilon_1$. By Definition 2.3, there exists an $j_2 \in \mathbb{Z}_+$ such that $Q_{x, H_R^x+2} \subset R_{-j_2}$ and

$$\delta(Q_{x, H_R^x+2}, R_{-j_2}) \le (j_2+1)A + \epsilon_1.$$

Let $j := \max\{j_1, j_2\}$. By the assumption that $R' \subset Q_{x,\,H_R^x+2}$, we also have that $R'' \subset Q_{x,\,H_R^x+2}$, which combining with Lemma 2.1 (d) implies that

$$\begin{aligned}(j+1)A + 3\epsilon_1 + 2\epsilon_0 &\geq \delta(R'', R_{-j}) + \epsilon_0 \geq \delta(Q_{x,\,H_R^x+2}, R_{-j})\\ &= (H_R^x + 2 + j)A \pm 2\epsilon_1 \pm \epsilon_0,\end{aligned}$$

where we used the fact that

$$\delta(R_{-j}, R_{-j_i}) = (j - j_i)A \pm 2\epsilon_1.$$

This together with the choice of the constant A shows that $H_R^x < 0$, which contradicts to (d). Thus, $R' \notin \mathcal{D}$, which completes the proof of (f).

Finally, by the properties (a) and (b) above and Lemma 2.1, we have

$$\delta(2R, Q_{x,\,H_R^x-1}) \leq \epsilon_0 + \delta(Q_{x,\,H_R^x+2},\, Q_{x,\,H_R^x-1}) \lesssim 1,$$

and hence,

$$\delta(R,\, Q_{x,\,H_R^x}) \leq \epsilon_0 + \delta(R, Q_{x,\,H_R^x-1}) \lesssim 1 + \delta(R, 2R) + \delta(2R, Q_{x,\,H_R^x-1}) \lesssim 1.$$

Also, the above property (a) and Lemma 2.1 imply that

$$\begin{aligned}\delta(Q_{x,H_R^x+1},\, R) &\lesssim 1 + \delta(Q_{x,H_R^x+1}, 3R) + \delta(3R, R)\\ &\lesssim 1 + \delta(Q_{x,H_R^x+1}, Q_{x,H_R^x-1}) \lesssim 1,\end{aligned}$$

which verifies (g) and hence, completes the proof of Lemma 2.2. ∎

3. The spaces $\mathrm{rbmo}\,(\mu)$, $\mathrm{rblo}\,(\mu)$ and $h^{1,p}_{\mathrm{atb}}(\mu)$ with $p \in (1, \infty]$

In Section 3.1, we introduce a local atomic Hardy space $h^{1,p}_{\mathrm{atb}}(\mu)$ and a local BMO-type space $\mathrm{rbmo}\,(\mu)$. After presenting some basic properties of these spaces, we then prove that the space $\mathrm{rbmo}\,(\mu)$ satisfies a John-Nirenberg inequality and its predual is $h^{1,\infty}_{\mathrm{atb}}(\mu)$. Moreover, we also establish the relation between $H^{1,\infty}_{\mathrm{atb}}(\mu)$ and $h^{1,\infty}_{\mathrm{atb}}(\mu)$ and between $\mathrm{RBMO}\,(\mu)$ and $\mathrm{rbmo}\,(\mu)$. In Section 3.2, we introduce a local BLO-type space $\mathrm{rblo}\,(\mu)$ and establish some characterizations of both $\mathrm{RBLO}\,(\mu)$ and $\mathrm{rblo}\,(\mu)$. In particular, the relation between $\mathrm{RBLO}\,(\mu)$ and $\mathrm{rblo}\,(\mu)$ is presented.

3.1. The spaces $\mathrm{rbmo}\,(\mu)$ and $h^{1,p}_{\mathrm{atb}}(\mu)$ for $p \in (1, \infty]$

We begin with the definition of $\mathrm{rbmo}\,(\mu)$.

Definition 3.1. *Let $\eta \in (1, \infty)$, $\rho \in [\eta, \infty)$ and $\beta_\rho := \rho^{d+1}$. A function $f \in L^1_{\rm loc}(\mu)$ is said to be in the space* $\mathrm{rbmo}_{\eta,\rho}(\mu)$*, if there exists a nonnegative constant $\widetilde{C}$ such that for any cube $Q \notin \mathcal{D}$,*

$$\frac{1}{\mu(\eta Q)} \int_Q \left| f(y) - m_{\widetilde{Q}^\rho}(f) \right| d\mu(y) \le \widetilde{C},$$

that for any two (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|m_Q(f) - m_R(f)| \le \widetilde{C}[1 + \delta(Q, R)],$$

and that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q |f(y)|\, d\mu(y) \le \widetilde{C}. \tag{3.1}$$

Moreover, we define the $\mathrm{rbmo}_{\eta,\rho}(\mu)$ *norm of f by the minimal constant $\widetilde{C}$ as above and denote it by $\|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}$.*

It follows from Definition 2.8 that for any fixed $\eta \in (1, \infty)$ and any $\rho \in [\eta, \infty)$, $\mathrm{rbmo}_{\eta,\rho}(\mu) \subset \mathrm{RBMO}(\mu)$. Moreover, from the propositions below, we will see that the space $\mathrm{rbmo}_{\eta,\rho}(\mu)$ enjoys properties similar to the space $\mathrm{RBMO}(\mu)$, including that the definition of the space $\mathrm{rbmo}_{\eta,\rho}(\mu)$ is independent of $\eta \in (1, \infty)$ and $\rho \in [\eta, \infty)$. First of all, we have the basic properties, whose proofs are left to the reader.

Proposition 3.1. *Let $\eta \in (1, \infty)$ and $\rho \in [\eta, \infty)$. The following properties hold:*

(i) $\mathrm{rbmo}_{\eta,\rho}(\mu)$ *is a Banach space.*

(ii) $L^\infty(\mu) \subset \mathrm{rbmo}_{\eta,\rho}(\mu) \subset \mathrm{RBMO}(\mu)$. *Moreover, for all $f \in L^\infty(\mu)$, $\|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)} \le 2\|f\|_{L^\infty(\mu)}$, and there exists a positive constant C such that for all $f \in \mathrm{rbmo}_{\eta,\rho}(\mu)$,*

$$\|f\|_{\mathrm{RBMO}(\mu)} \le C\|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}.$$

(iii) *If $f \in \mathrm{rbmo}_{\eta,\rho}(\mu)$, then $|f| \in \mathrm{rbmo}_{\eta,\rho}(\mu)$ and there exists a positive constant C such that for all $f \in \mathrm{rbmo}_{\eta,\rho}(\mu)$,*

$$\||f|\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)} \le C\|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}.$$

(iv) *If $f, g \in \mathrm{rbmo}_{\eta,\rho}(\mu)$, then* $\min(f, g)$, $\max(f, g) \in \mathrm{rbmo}_{\eta,\rho}(\mu)$, *and there exists a positive constant C such that for all $f, g \in \mathrm{rbmo}_{\eta,\rho}(\mu)$,*

$$\|\min(f, g)\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)} \le C(\|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)} + \|g\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)})$$

and

$$\|\max(f, g)\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)} \le C(\|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)} + \|g\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}).$$

We now introduce another equivalent norm for the space $\mathrm{rbmo}_{\eta,\,\rho}\,(\mu)$. Let $\eta \in (1,\infty)$. Suppose that for a given $f \in L^1_{\mathrm{loc}}\,(\mu)$, there exist a nonnegative constant $\widetilde{C}$ and a collection of numbers $\{f_Q\}_Q$ such that

$$\sup_{Q\notin\mathcal{D}} \frac{1}{\mu(\eta Q)} \int_Q |f(y) - f_Q|\,d\mu(y) \le \widetilde{C}, \tag{3.2}$$

that for any two cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|f_Q - f_R| \le \widetilde{C}[1 + \delta(Q, R)], \tag{3.3}$$

and that for any cube $Q \in \mathcal{D}$,

$$|f_Q| \le \widetilde{C}. \tag{3.4}$$

We then define the *norm* $\|f\|_{*,\,\eta} := \inf\{\widetilde{C}\}$, where the infimum is taken over all the constants $\widetilde{C}$ as above and all the numbers $\{f_Q\}_Q$ satisfying (3.2) through (3.4).

With a minor modification of the proof for Lemma 2.6 in [16], we have the following conclusion.

Proposition 3.2. *The norms $\|\cdot\|_{*,\,\eta}$ for $\eta \in (1,\infty)$ are equivalent.*

Proof. Let $\eta_1 > \eta_2 > 1$ be fixed. Obviously, $\|f\|_{*,\,\eta_1} \le \|f\|_{*,\,\eta_2}$. To prove the converse, we need to show that for a fixed collection of numbers $\{f_Q\}_Q$ satisfying (3.2) through (3.4) with η and $\widetilde{C}$ respectively replaced by η_1 and $\|f\|_{*,\,\eta_1}$, we have that for any $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(\eta_2 Q)} \int_Q |f(y) - f_Q|\,d\mu(y) \lesssim \|f\|_{*,\,\eta_1}.$$

Fix $\rho \in [\eta_1,\infty)$ and $\beta_\rho = \rho^{d+1}$. For any cube $Q \notin \mathcal{D}$ and any $x \in \mathrm{supp}\,(\mu) \cap Q$, we choose $Q'_{x,\,2}$ as follows. If $l(Q_{x,\,H^x_Q+2}) \le \frac{\eta_2-1}{10\eta_1} l(Q)$, we then let $Q'_{x,\,2} = Q_{x,\,H^x_Q+2}$. Otherwise, let k_0 be the maximal negative integer such that $\rho^{k_0} l(Q_{x,\,H^x_Q+2}) \le \frac{\eta_2-1}{10\eta_1} l(Q)$ and we then let $Q'_{x,\,2}$ be the biggest $(\rho,\,\beta_\rho)$-doubling cube centered at x with side length $\rho^k l(Q_{x,\,H^x_Q+2})$ with $k \le k_0$. By Lemma 2.1, we have $\delta(Q'_{x,\,2},\, Q_{x,\,H^x_Q+2}) \lesssim 1$. From Lemma 2.2 (d), (e) and (f), it follows that $Q'_{x,\,2} \notin \mathcal{D}$. By the Besicovitch covering theorem, there exists a subsequence of cubes $\{Q'_{x_i,\,2}\}_i$ which still covers $Q \cap \mathrm{supp}\,(\mu)$ and has a bounded overlap. For any i, by Lemma 2.2 (g) and $Q \notin \mathcal{D}$, we have

$$\begin{aligned}|f_{Q'_{x_i,\,2}} - f_Q| &\le |f_{Q'_{x_i,\,2}} - f_{Q_{x_i,\,H^{x_i}_Q}}| + |f_Q - f_{Q_{x_i,\,H^{x_i}_Q}}| \\ &\lesssim [1 + \delta(Q'_{x_i,\,2}, Q_{x_i,\,H^{x_i}_Q}) + \delta(Q, Q_{x_i,\,H^{x_i}_Q})]\|f\|_{*,\,\eta_1} \lesssim \|f\|_{*,\,\eta_1}.\end{aligned}$$

From this estimate together with the facts that for each i, $Q'_{x_i,2} \notin \mathcal{D}$ and $Q_{x_i,2}$ is (ρ, β_ρ)-doubling and that $\rho \geq \eta_1$, we see that

$$\int_{Q'_{x_i,2}} |f(x) - f_Q| \, d\mu(x) \leq \int_{Q'_{x_i,2}} |f(x) - f_{Q'_{x_i,2}}| \, d\mu(x) + \mu(Q'_{x_i,2})|f_{Q'_{x_i,2}} - f_Q|$$
$$\lesssim \mu(Q'_{x_i,2})\|f\|_{*,\eta_1}.$$

Therefore, from the facts that $\{Q'_{x_i,2}\}_i$ are almost disjoint and that $Q'_{x_i,2} \subset \eta_2 Q$ for all i, it follows that

$$\int_Q |f(x) - f_Q| \, d\mu(x) \leq \sum_i \int_{Q'_{x_i,2}} |f(x) - f_Q| \, d\mu(x) \lesssim \mu(\eta_2 Q)\|f\|_{*,\eta_1},$$

which completes the proof of Proposition 3.2. ■

Based on Proposition 3.2, from now on, we write $\|\cdot\|_*$ *instead of* $\|\cdot\|_{*,\eta}$.

Proposition 3.3. *Let* $\eta \in (1, \infty)$, $\rho \in [\eta, \infty)$ *and* $\beta_\rho := \rho^{d+1}$. *Then the norms* $\|\cdot\|_*$ *and* $\|\cdot\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}$ *are equivalent.*

Proof. Suppose that $f \in L^1_{\mathrm{loc}}(\mu)$. We first show that

$$\|f\|_* \lesssim \|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}. \tag{3.5}$$

For any cube Q, let $f_Q := m_{\widetilde{Q}^\rho}(f)$ if $\widetilde{Q}^\rho \notin \mathcal{D}$, and otherwise, let $f_Q := 0$. For any $Q \notin \mathcal{D}$, if $\widetilde{Q}^\rho \notin \mathcal{D}$, by Definition 3.1, we have

$$\frac{1}{\mu(\eta Q)} \int_Q |f(y) - f_Q| \, d\mu(y) \leq \|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}.$$

If $\widetilde{Q}^\rho \in \mathcal{D}$, then $f_Q = 0$. The (ρ, β_ρ)-doubling property of $\widetilde{Q}^\rho$ together with Definition 3.1 and the assumption that $\rho \geq \eta$ further yields that

$$\frac{1}{\mu(\eta Q)} \int_Q |f(y) - f_Q| \, d\mu(y)$$
$$\leq \frac{1}{\mu(\eta Q)} \int_Q \left|f(y) - m_{\widetilde{Q}^\rho}(f)\right| \, d\mu(y) + \frac{\mu(Q)}{\mu(\eta Q)} \left|m_{\widetilde{Q}^\rho}(f)\right|$$
$$\lesssim \|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}.$$

Notice that $Q \subset \widetilde{Q}^\rho$. If $Q \in \mathcal{D}$, then $\widetilde{Q}^\rho \in \mathcal{D}$ and $f_Q = 0$. Obviously, $|f_Q| \lesssim \|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}$. Therefore (3.5) is reduced to showing that for any two cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|f_Q - f_R| \lesssim [1 + \delta(Q, R)]\|f\|_{\mathrm{rbmo}_{\eta,\rho}(\mu)}. \tag{3.6}$$

To show (3.6), we first claim that for any $f \in \mathrm{rbmo}_{\eta,\,\rho}(\mu)$ and any cubes $Q \subset R$,

$$\left| m_{\widetilde{Q}^\rho}(f) - m_{\widetilde{R}^\rho}(f) \right| \lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{rbmo}_{\eta,\,\rho}(\mu)}. \tag{3.7}$$

If $Q \in \mathcal{D}$, then $\widetilde{Q}^\rho \in \mathcal{D}$ and $\widetilde{R}^\rho \in \mathcal{D}$. In this case, (3.7) follows directly from Definition 3.1. If $Q \notin \mathcal{D}$, to verify (3.7), we consider two cases.

Case (i) $l(\widetilde{R}^\rho) \geq l(\widetilde{Q}^\rho)$. In this case, $\widetilde{Q}^\rho \subset 2\widetilde{R}^\rho$. Let $R_0 := \widetilde{2\widetilde{R}^\rho}$. It follows from Lemma 2.1 that $\delta(\widetilde{R}^\rho, R_0) \lesssim 1$ and $\delta(\widetilde{Q}^\rho, R_0) \lesssim 1 + \delta(Q, R)$. Therefore if neither $\widetilde{Q}^\rho$ nor $\widetilde{R}^\rho$ are in $\mathcal{D}$, then

$$\begin{aligned} \left| m_{\widetilde{Q}^\rho}(f) - m_{\widetilde{R}^\rho}(f) \right| &\leq \left| m_{\widetilde{Q}^\rho}(f) - m_{R_0}(f) \right| + \left| m_{R_0}(f) - m_{\widetilde{R}^\rho}(f) \right| \\ &\lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{rbmo}_{\eta,\,\rho}(\mu)}. \end{aligned}$$

If both $\widetilde{Q}^\rho$ and $\widetilde{R}^\rho$ are in $\mathcal{D}$, then by $\rho \geq \eta$ and the (ρ, β_ρ)-doubling property of $\widetilde{Q}^\rho$ and $\widetilde{R}^\rho$,

$$\left| m_{\widetilde{Q}^\rho}(f) - m_{\widetilde{R}^\rho}(f) \right| \leq \left| m_{\widetilde{Q}^\rho}(f) \right| + \left| m_{\widetilde{R}^\rho}(f) \right| \lesssim \|f\|_{\mathrm{rbmo}_{\eta,\,\rho}(\mu)}.$$

Thus we only need to consider the case that only one of $\widetilde{Q}^\rho$ and $\widetilde{R}^\rho$ is in $\mathcal{D}$. By similarity, we may assume that $\widetilde{Q}^\rho \in \mathcal{D}$ while $\widetilde{R}^\rho \notin \mathcal{D}$. Since $\widetilde{Q}^\rho \subset R_0$, we then have $R_0 \in \mathcal{D}$ and

$$\begin{aligned} \left| m_{\widetilde{Q}^\rho}(f) - m_{\widetilde{R}^\rho}(f) \right| &\leq \left| m_{\widetilde{Q}^\rho}(f) \right| + |m_{R_0}(f)| + \left| m_{R_0}(f) - m_{\widetilde{R}^\rho}(f) \right| \\ &\lesssim \|f\|_{\mathrm{rbmo}_{\eta,\,\rho}(\mu)}. \end{aligned}$$

Case (ii) $l(\widetilde{R}^\rho) < l(\widetilde{Q}^\rho)$. In this case, $\widetilde{R}^\rho \subset 2\rho\widetilde{Q}^\rho$. Notice that $l(\widetilde{R}^\rho) \geq l(Q)$. Thus, there exists a unique $m \in \mathbb{N}$ such that $l(\rho^{m-1}Q) \leq l(\widetilde{R}^\rho) < l(\rho^m Q)$. Therefore, $\rho^m Q \subset \widetilde{2\rho\widetilde{Q}^\rho}$. Set $Q_0 := \widetilde{2\rho\widetilde{Q}^\rho}$. Then another application of Lemma 2.1 implies that $\delta(\widetilde{Q}^\rho, Q_0) \lesssim 1$ and

$$\delta(\widetilde{R}^\rho, Q_0) \lesssim 1 + \delta(\widetilde{R}^\rho, \rho^m Q) + \delta(\rho^m Q, Q_0) \lesssim 1.$$

Therefore, an argument similar to Case (i) also establishes (3.7) in this case. Thus, (3.7) always holds.

We now establish (3.6) by using (3.7) and considering the following three cases.

Case (1) $\widetilde{Q}^\rho$, $\widetilde{R}^\rho \in \mathcal{D}$ or $\widetilde{Q}^\rho$, $\widetilde{R}^\rho \notin \mathcal{D}$. In this case, (3.6) follows directly from (3.7).

Case (2) $\widetilde{Q}^\rho \notin \mathcal{D}$ and $\widetilde{R}^\rho \in \mathcal{D}$. In this case, the estimate (3.7) together with $\rho \geq \eta$ and the (ρ, β_ρ)-doubling property of $\widetilde{R}^\rho$ yields that

$$|f_Q - f_R| \leq \left|m_{\widetilde{Q}^\rho}(f) - m_{\widetilde{R}^\rho}(f)\right| + \left|m_{\widetilde{R}^\rho}(f)\right| \lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{rbmo}_{\eta,\,\rho}(\mu)}.$$

Case (3) $\widetilde{Q}^\rho \in \mathcal{D}$ and $\widetilde{R}^\rho \notin \mathcal{D}$. In this case, an argument similar to Case (2) also leads to that

$$|f_Q - f_R| \leq \left|m_{\widetilde{Q}^\rho}(f) - m_{\widetilde{R}^\rho}(f)\right| + \left|m_{\widetilde{Q}^\rho}(f)\right| \lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{rbmo}_{\eta,\,\rho}(\mu)}.$$

Thus, (3.6) holds, and hence (3.5) is also true.

Now let us establish the converse of (3.5). For $f \in L^1_{\mathrm{loc}}(\mu)$, assume that there exists a sequence of numbers $\{f_Q\}_Q$ satisfying (3.2), (3.3) and (3.4) with $\widetilde{C}$ replaced by $\|f\|_*$. First we claim that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q |f(x)|\, d\mu(x) \lesssim \|f\|_*. \tag{3.8}$$

For any cube Q and any $x \in \operatorname{supp}(\mu) \cap Q$, let $Q'_{x,2}$ be the biggest (ρ, β_ρ)-doubling cube centered at x with side length $\rho^k l(Q_{x,2})$, $k \leq 0$, and $l(Q'_{x,2}) \leq \frac{\eta-1}{10\eta} l(Q)$ (For the existence of $Q'_{x,2}$, see the proof of Proposition 3.2). From Lemma 2.1, it is easy to see that $\delta(Q'_{x,2}, Q_{x,2}) \lesssim 1$. By Lemma 2.2 (e), we then have $Q'_{x,2} \notin \mathcal{D}$. Applying the Besicovitch covering theorem, we obtain a subsequence of cubes $\{Q'_{x_i,2}\}_i$ covering $Q \cap \operatorname{supp}(\mu)$ with a bounded overlap. From the bounded overlap and (ρ, β_ρ)-doubling property of $\{Q'_{x_i,2}\}_i$, (3.2), (3.4) and the facts that $Q'_{x_i,2} \subset \eta Q$, $Q'_{x_i,2} \notin \mathcal{D}$ and $\rho \geq \eta$, it follows that

$$\begin{aligned}
\frac{1}{\mu(\eta Q)} \int_Q |f(x)|\, d\mu(x) &\leq \sum_i \frac{1}{\mu(\eta Q)} \int_{Q'_{x_i,2}} |f(x) - f_{Q'_{x_i,2}}|\, d\mu(x) \\
&\quad + \sum_i \frac{\mu(Q'_{x_i,2})}{\mu(\eta Q)} \left[|f_{Q'_{x_i,2}} - f_{Q_{x_i,1}}| + |f_{Q_{x_i,1}}|\right] \\
&\lesssim \sum_i \frac{\mu(Q'_{x_i,2})}{\mu(\eta Q)} \left[1 + \delta(Q'_{x_i,2}, Q_{x_i,1})\right] \|f\|_* \lesssim \|f\|_*.
\end{aligned}$$

We now claim that for any cube $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q \left|f(x) - m_{\widetilde{Q}^\rho}(f)\right|\, d\mu(x) \lesssim \|f\|_*. \tag{3.9}$$

Notice that if $Q \notin \mathcal{D}$ and Q is (ρ, β_ρ)-doubling, then using the fact $\rho \geq \eta$, we have

$$|f_Q - m_Q(f)| = \left| \frac{1}{\mu(Q)} \int_Q [f(x) - f_Q] \, d\mu(x) \right| \leq \frac{\mu(\eta Q)}{\mu(Q)} \|f\|_* \lesssim \|f\|_*. \tag{3.10}$$

Therefore, for any cube $Q \notin \mathcal{D}$, if $\widetilde{Q}^\rho \notin \mathcal{D}$, then applying (3.3) and (3.10) implies that

$$\left| f_Q - m_{\widetilde{Q}^\rho}(f) \right| \leq \left| f_Q - f_{\widetilde{Q}^\rho} \right| + \left| f_{\widetilde{Q}^\rho} - m_{\widetilde{Q}^\rho}(f) \right| \lesssim \|f\|_*;$$

if $\widetilde{Q}^\rho \in \mathcal{D}$, then from (3.3), (3.4), (3.8) and $\rho \geq \eta$, it follows that

$$\left| f_Q - m_{\widetilde{Q}^\rho}(f) \right| \leq \left| f_Q - f_{\widetilde{Q}^\rho} \right| + \left| f_{\widetilde{Q}^\rho} \right| + \left| m_{\widetilde{Q}^\rho}(f) \right| \lesssim \|f\|_*.$$

From these estimates and (3.2), we deduce that for any cube $Q \notin \mathcal{D}$,

$$\begin{aligned} \int_Q \left| f(x) - m_{\widetilde{Q}^\rho}(f) \right| d\mu(x) &\leq \int_Q |f(x) - f_Q| \, d\mu(x) + \left| f_Q - m_{\widetilde{Q}^\rho}(f) \right| \mu(Q) \\ &\lesssim \|f\|_* \mu(\eta Q), \end{aligned}$$

which verifies (3.9).

Finally, for any two (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$, if $R \notin \mathcal{D}$, (3.10) together with (3.3) yields that

$$\begin{aligned} |m_Q(f) - m_R(f)| &\leq |m_Q(f) - f_Q| + |f_Q - f_R| + |f_R - m_R(f)| \\ &\lesssim [1 + \delta(Q, R)] \|f\|_*. \end{aligned}$$

If $R \in \mathcal{D}$, (3.10) together with (3.3), (3.4), (3.8) and the (ρ, β_ρ)-doubling property of R leads to that

$$\begin{aligned} |m_Q(f) - m_R(f)| &\leq |m_Q(f) - f_Q| + |f_Q - f_R| + |f_R| + |m_R(f)| \\ &\lesssim [1 + \delta(Q, R)] \|f\|_*. \end{aligned}$$

Thus

$$f \in \mathrm{rbmo}_{\eta,\,\rho}(\mu) \qquad \text{and} \qquad \|f\|_{\mathrm{rbmo}_{\eta,\,\rho}(\mu)} \lesssim \|f\|_*,$$

which completes the proof of Proposition 3.3. ■

Remark 3.1. *Let* $\eta \in (1, \infty)$ *and* $\rho \in [\eta, \infty)$. *From Proposition* 3.2 *and Proposition* 3.3, *it follows that the definition of the space* $\mathrm{rbmo}_{\eta,\,\rho}(\mu)$ *is independent of the choices of* η *and* ρ. *From now on, we will simply write* $\mathrm{rbmo}(\mu)$ *instead of* $\mathrm{rbmo}_{\eta,\,\rho}(\mu)$ *for any* η *and* ρ *as above.*

Proposition 3.4. *Let $\eta \in (1, \infty)$, $\rho \in [\eta, \infty)$ and $\beta_\rho := \rho^{d+1}$. For any $f \in L^1_{\rm loc}(\mu)$, the following are equivalent:*

(i) $f \in {\rm rbmo}_{\eta,\rho}(\mu)$.

(ii) *There exists a nonnegative constant C_b such that for any cube $Q \notin \mathcal{D}$,*

$$\int_Q |f(x) - m_Q(f)| \, d\mu(x) \le C_b \mu(\eta Q),$$

that for any cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|m_Q(f) - m_R(f)| \le C_b[1 + \delta(Q,R)] \left[\frac{\mu(\eta Q)}{\mu(Q)} + \frac{\mu(\eta R)}{\mu(R)}\right], \tag{3.11}$$

and that for any cube $Q \in \mathcal{D}$,

$$\int_Q |f(x)| \, d\mu(x) \le C_b \mu(\eta Q). \tag{3.12}$$

(iii) *There exists a nonnegative constant C_c such that for any (ρ, β_ρ)-doubling cube $Q \notin \mathcal{D}$,*

$$\int_Q |f(x) - m_Q(f)| \, d\mu(x) \le C_c \mu(Q), \tag{3.13}$$

that for any (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|m_Q(f) - m_R(f)| \le C_c[1 + \delta(Q,R)], \tag{3.14}$$

and that for any (ρ, β_ρ)-doubling cube $Q \in \mathcal{D}$,

$$\int_Q |f(x)| \, d\mu(x) \le C_c \mu(Q). \tag{3.15}$$

Moreover, the minimal constants C_b and C_c are equivalent to $\|f\|_{{\rm rbmo}_{\eta,\rho}(\mu)}$.

Proof. By Proposition 3.2 and Proposition 3.3, it suffices to establish Proposition 3.4 with $\eta = \rho = 2$. We write ${\rm rbmo}(\mu)$ instead of ${\rm rbmo}_{\eta,\rho}(\mu)$ for simplicity. Assuming that $f \in {\rm rbmo}(\mu)$, we now show that (ii) holds. For any $Q \notin \mathcal{D}$,

$$\left| m_Q(f) - m_{\widetilde{Q}}(f) \right| \le m_Q\left(\left| f - m_{\widetilde{Q}}(f) \right| \right) \le \frac{\mu(2Q)}{\mu(Q)} \|f\|_{{\rm rbmo}(\mu)}, \tag{3.16}$$

which implies that

$$\begin{aligned}&\int_Q |f(x)-m_Q(f)|\,d\mu(x)\\&\le \int_Q \left|f(x)-m_{\widetilde{Q}}(f)\right|\,d\mu(x)+\left|m_Q(f)-m_{\widetilde{Q}}(f)\right|\mu(Q)\le 2\|f\|_{\mathrm{rbmo}\,(\mu)}\mu(2Q).\end{aligned}$$

To show (3.11), we notice that if $R\in\mathcal{D}$, then $\widetilde{R}\in\mathcal{D}$, and

$$\left|m_R(f)-m_{\widetilde{R}}(f)\right|\lesssim \frac{\mu(2R)}{\mu(R)}\|f\|_{\mathrm{rbmo}\,(\mu)},$$

which together with (3.16) and (3.7) yields that for any $Q\subset R$ with $Q\notin\mathcal{D}$,

$$\begin{aligned}&|m_Q(f)-m_R(f)|\\&\quad\le \left|m_Q(f)-m_{\widetilde{Q}}(f)\right|+\left|m_{\widetilde{Q}}(f)-m_{\widetilde{R}}(f)\right|+\left|m_{\widetilde{R}}(f)-m_R(f)\right|\\&\quad\lesssim [1+\delta(Q,R)]\left[\frac{\mu(2Q)}{\mu(Q)}+\frac{\mu(2R)}{\mu(R)}\right]\|f\|_{\mathrm{rbmo}\,(\mu)}.\end{aligned}$$

This verifies (3.11), and hence (ii) holds.

Since (ii) obviously implies (iii), to finish the proof of Proposition 3.4, we only need to prove that if $f\in L^1_{\mathrm{loc}}(\mu)$ satisfies the assumptions in (iii), then $f\in\mathrm{rbmo}\,(\mu)$.

For any $Q\notin\mathcal{D}$, let $\{Q'_{x_i,\,2}\}_i$ be the sequence of cubes as in the proof of Proposition 3.2 with $\eta_1=\eta_2=2$, which covers $Q\cap\mathrm{supp}\,(\mu)$ with a bounded overlap. We then have that for each i, $Q'_{x_i,\,2}\notin\mathcal{D}$ and $\delta(Q'_{x_i,\,2},Q_{x_i,\,H_Q^{x_i}+2})\lesssim 1$. The last assertion together with Lemma 2.2 and Lemma 2.1 further yields that $\delta(Q'_{x_i,\,2},\widetilde{2\widetilde{Q}})\lesssim 1$. Obviously, by the choice of $\{Q'_{x_i,\,2}\}_i$, we have $Q'_{x_i,\,2}\subset 2Q$. These facts together with (3.14), (3.7) and Lemma 2.1 imply that

$$\left|m_{Q'_{x_i,\,2}}(f)-m_{\widetilde{Q}}(f)\right|\le\left|m_{Q'_{x_i,\,2}}(f)-m_{\widetilde{2\widetilde{Q}}}(f)\right|+\left|m_{\widetilde{Q}}(f)-m_{\widetilde{2\widetilde{Q}}}(f)\right|\lesssim C_c.$$

Then from (3.13), the facts that for each i, $Q'_{x_i,\,2}\subset 2Q$ and that $Q'_{x_i,\,2}$ are almost disjoint, it follows that

$$\begin{aligned}&\int_Q\left|f(x)-m_{\widetilde{Q}}(f)\right|\,d\mu(x)\\&\le\sum_i\int_{Q'_{x_i,\,2}}|f(x)-m_{Q'_{x_i,\,2}}(f)|\,d\mu(x)+\sum_i\left|m_{Q'_{x_i,\,2}}(f)-m_{\widetilde{Q}}(f)\right|\mu(Q'_{x_i,\,2})\\&\lesssim C_c\mu(2Q).\end{aligned}$$

On the other hand, if $Q \in \mathcal{D}$, let $\{Q'_{x_i,2}\}_i$ be the sequence of cubes as in the proof of Proposition 3.3 with $\eta = 2$, which covers $Q \cap \operatorname{supp}(\mu)$ with a bounded overlap. We then have that for each i, $Q'_{x_i,2} \notin \mathcal{D}$ and $\delta(Q'_{x_i,2}, Q_{x_i,2}) \lesssim 1$. The last assertion further implies that for all i,

$$\delta(Q'_{x_i,2}, Q_{x_i,1}) \lesssim 1,$$

which together with (3.14), $Q_{x_i,1} \in \mathcal{D}$ and (3.15) leads to that

$$\left|m_{Q'_{x_i,2}}(f)\right| \le \left|m_{Q'_{x_i,2}}(f) - m_{Q_{x_i,1}}(f)\right| + \left|m_{Q_{x_i,1}}(f)\right| \lesssim C_c.$$

Using the almost disjoint property and the doubling property of $\{Q'_{x_i,2}\}_i$, $Q'_{x_i,2} \notin \mathcal{D}$, (3.13), (3.15) and $Q'_{x_i,2} \subset 2Q$, we obtain

$$\begin{aligned}
&\int_Q |f(x)|\,d\mu(x)\\
&\quad\le \sum_i \left\{\int_{Q'_{x_i,2}} \left|f(x) - m_{Q'_{x_i,2}}(f)\right|\,d\mu(x) + \mu(Q'_{x_i,2})\left|m_{Q'_{x_i,2}}(f)\right|\right\}\\
&\quad\lesssim C_c\mu(2Q),
\end{aligned}$$

which implies (3.1). Thus $f \in \operatorname{rbmo}(\mu)$, and this finishes the proof of Proposition 3.4. ■

The following theorem is a local version of the John-Nirenberg inequality for the space RBMO (μ) in [16]. We prove this by using some ideas from the proof of Theorem 3.1 in [16]; see also [8].

Theorem 3.1. *Let $\eta \in (1,\infty)$ and $f \in \operatorname{rbmo}(\mu)$. If there exists a sequence of numbers $\{f_Q\}_Q$ such that (3.2), (3.3) and (3.4) hold with $\widetilde{C}$ replaced by $C\|f\|_{\operatorname{rbmo}(\mu)}$. Then there exist nonnegative constants C_1 and C_2 such that for any cube $Q \in \mathcal{D}$ and $\lambda > 0$,*

$$\mu(\{x \in Q : |f(x)| > \lambda\}) \le C_1\mu(\eta Q)\exp\left(\frac{-C_2\lambda}{\|f\|_{\operatorname{rbmo}(\mu)}}\right), \tag{3.17}$$

and for any $Q \notin \mathcal{D}$ and $\lambda > 0$,

$$\mu(\{x \in Q : |f(x) - f_Q| > \lambda\}) \le C_1\mu(\eta Q)\exp\left(\frac{-C_2\lambda}{\|f\|_{\operatorname{rbmo}(\mu)}}\right). \tag{3.18}$$

To prove Theorem 3.1, we need the following two technical lemmas.

Lemma 3.1. *Under the assumption of Theorem* 3.1*, if* Q *and* R *are cubes such that* $l(Q) \sim l(R)$ *and* $dist(Q, R) \le Cl(Q)$*, then*

$$|f_Q - f_R| \le C\|f\|_{\mathrm{rbmo}\,(\mu)},$$

where C *is a positive constant independent of* f*,* Q *and* R*.*

Proof. As in Definition 2.1, let R_Q be the smallest cube concentric with R containing Q and R, then $l(R_Q) \lesssim l(Q)$. By Lemma 2.1 (a), $\delta(Q, R_Q) \lesssim 1$ and $\delta(R, R_Q) \lesssim 1$. We then consider the following three cases.

Case (1) $Q \notin \mathcal{D}$ and $R \notin \mathcal{D}$. In this case, an application of (3.3) yields that

$$|f_Q - f_R| \le |f_Q - f_{R_Q}| + |f_{R_Q} - f_R| \lesssim \|f\|_{\mathrm{rbmo}\,(\mu)}.$$

Case (2) $Q \notin \mathcal{D}$ and $R \in \mathcal{D}$, or $Q \in \mathcal{D}$ and $R \notin \mathcal{D}$. By similarity, we only consider the first case. Then (3.3) together with $R_Q \in \mathcal{D}$ and (3.4) implies that

$$|f_Q - f_R| \le |f_Q - f_{R_Q}| + |f_{R_Q}| + |f_R| \lesssim [1 + \delta(Q, R_Q)]\|f\|_{\mathrm{rbmo}\,(\mu)} \lesssim \|f\|_{\mathrm{rbmo}\,(\mu)}.$$

Case (3) Both Q and R are in $\mathcal{D}$. Then (3.4) immediately implies that

$$|f_Q - f_R| \le |f_Q| + |f_R| \lesssim \|f\|_{\mathrm{rbmo}\,(\mu)},$$

which completes the proof of Lemma 3.1. ∎

We also need the following lemma which is an analog of [16, Lemma 3.3]. We omit the details for simplicity.

Lemma 3.2. *Let* $f \in \mathrm{rbmo}\,(\mu)$ *and* f *be a real-valued function. Given* $q > 0$*, let* $f_q(x) := f(x)$ *if* $|f(x)| \le q$*, and let* $f_q(x) := q\frac{f(x)}{|f(x)|}$ *if* $|f(x)| > q$*. Then* $f_q \in \mathrm{rbmo}\,(\mu)$ *with*

$$\|f_q\|_{\mathrm{rbmo}\,(\mu)} \le C\|f\|_{\mathrm{rbmo}\,(\mu)},$$

where C *is a positive constant independent of* q *and* f*.*

Remark 3.2. *Let* $f \in \mathrm{rbmo}\,(\mu)$ *and* $\{f_Q\}_Q$ *satisfy the conditions of Theorem* 3.1*. Assume that* f *and* f_Q *are real-valued, otherwise we consider their real and imaginary parts, respectively. For any given* $q > 0$*, let* $f_{Q,+} := \max(f_Q, 0)$*,* $f_{Q,-} := -\min(f_Q, 0)$*, and* $f_{q,Q} := \min(f_{Q,+}, q) - \min(f_{Q,-}, q)$*. For any given* $\eta \in (1, \infty)$*, it is easy to see that*

$$\sup_{Q \notin \mathcal{D}} \frac{1}{\mu(\eta Q)} \int_Q |f_q(x) - f_{q,Q}|\, d\mu(x) \lesssim \|f\|_{\mathrm{rbmo}\,(\mu)},$$

that for any cubes $Q \subset R$ *with* $Q \notin \mathcal{D}$,

$$|f_{q,Q} - f_{q,R}| \lesssim [1 + \delta(Q,R)]\|f\|_{\mathrm{rbmo}\,(\mu)},$$

and that for any $Q \in \mathcal{D}$, $|f_{q,Q}| \lesssim \|f\|_{\mathrm{rbmo}\,(\mu)}$.

Proof of Theorem 3.1. By Proposition 3.2 and Proposition 3.3, it suffices to establish (3.17) and (3.18) for $\eta = 2$. Assume that f and f_Q are *real-valued*, otherwise we consider their real and imaginary parts, respectively. Let $f \in L^\infty(\mu)$ first and Q be some fixed cube in $\mathcal{D}$. Without loss of generality, we may assume that $\|f\|_{\mathrm{rbmo}\,(\mu)} = 1$. Let $Q' := \frac{3}{2}Q$ and B be a positive constant which will be determined later. By the Lebesgue differentiation theorem, for μ-a. e. $x \in Q \cap \operatorname{supp}(\mu)$ such that $|f(x)| > B$, there exists a doubling cube Q_x centered at x such that

$$m_{Q_x}(|f|) > B. \tag{3.19}$$

Moreover, we assume that Q_x is the biggest doubling cube satisfying (3.19) with side length $2^k l(Q_{x,2})$ for some $k \le 0$ and $l(Q_x) \le \frac{1}{20} l(Q)$. By the Besicovitch covering theorem, there exists an almost disjoint subfamily $\{Q_i\}_i$ of the cubes $\{Q_x\}_x$ such that

$$\{x \in Q : \ |f(x)| > B\} \subset \bigcup_i Q_i. \tag{3.20}$$

By Lemma 2.2 (e), we know that $Q_i \notin \mathcal{D}$. From Proposition 3.2 and Proposition 3.3, it follows that for any $R \in \mathcal{D}$,

$$\frac{1}{\mu(\frac{4}{3}R)} \int_R |f(x)|\, d\mu(x) \lesssim 1.$$

Thus, if we choose B big enough, by (3.19) and the facts that Q_i are almost disjoint and that $Q_i \subset Q'$,

$$\sum_i \mu(Q_i) \le \sum_i \frac{1}{B} \int_{Q_i} |f(x)|\, d\mu(x) \le \frac{C}{B} \int_{Q'} |f(x)|\, d\mu(x) \le \frac{\mu(2Q)}{2^{d+3}}. \tag{3.21}$$

We now prove that for each i,

$$|f_{Q_i}| \lesssim 1. \tag{3.22}$$

Having the fact that $Q_i \notin \mathcal{D}$ in mind, we consider $\widetilde{2Q_i}$ in the following three cases.

Case (1) $l(\widetilde{2Q_i}) > 10 l(Q)$. Then $Q \subset \widetilde{2Q_i}$ and so $\widetilde{2Q_i} \in \mathcal{D}$. By (3.3), (3.4) and Lemma 2.1, $|f_{Q_i}| \le |f_{Q_i} - f_{\widetilde{2Q_i}}| + |f_{\widetilde{2Q_i}}| \lesssim 1$.

Case (2) $\frac{1}{20}l(Q) \le l(\widetilde{2Q_i}) \le 10l(Q)$. In this case, if $\widetilde{2Q_i} \in \mathcal{D}$, as in Case (1), we have $|f_{Q_i}| \lesssim 1$. If $\widetilde{2Q_i} \notin \mathcal{D}$, by (3.3), (3.4), the fact that $\widetilde{2Q_i} \subset 30Q$ and Lemma 2.1, we obtain

$$|f_{Q_i}| \le \left| f_{Q_i} - f_{\widetilde{2Q_i}} \right| + \left| f_{\widetilde{2Q_i}} - f_{30Q} \right| + |f_{30Q}| \lesssim 1.$$

Case (3) $l(\widetilde{2Q_i}) < \frac{1}{20}l(Q)$. By the choice of Q_i, $m_{\widetilde{2Q_i}}(|f|) \le B$, which implies $|m_{\widetilde{2Q_i}}(f)| \le B$. If $\widetilde{2Q_i} \notin \mathcal{D}$, then it follows from (3.2), (3.3) and Lemma 2.1 that

$$|f_{Q_i}| \le \left| f_{Q_i} - f_{\widetilde{2Q_i}} \right| + \left| f_{\widetilde{2Q_i}} - m_{\widetilde{2Q_i}}(f) \right| + \left| m_{\widetilde{2Q_i}}(f) \right| \lesssim 1.$$

If $\widetilde{2Q_i} \in \mathcal{D}$, by (3.3) and (3.4), we then have $|f_{Q_i}| \le |f_{Q_i} - f_{\widetilde{2Q_i}}| + |f_{\widetilde{2Q_i}}| \lesssim 1$. Combining these cases above, we see that (3.22) holds.

For $t > 0$, we define

$$X(t) := \sup_{Q \notin \mathcal{D}} \frac{1}{\mu(2Q)} \int_Q \exp\left(|f(x) - f_Q|t\right)\, d\mu(x) + \sup_{Q \in \mathcal{D}} \frac{1}{\mu(2Q)} \int_Q \exp\left(|f(x)|t\right)\, d\mu(x).$$

It then follows from (3.20) through (3.22) and the doubling property of Q_i that for $Q \in \mathcal{D}$,

$$\begin{aligned}
&\frac{1}{\mu(2Q)} \int_Q \exp\left(|f(x)|t\right)\, d\mu(x) \\
&\quad\le \frac{1}{\mu(2Q)} \int_{Q \setminus \cup_i Q_i} \exp\left(Bt\right)\, d\mu(x) \\
&\qquad + \frac{1}{\mu(2Q)} \sum_i \int_{Q_i} \exp\left(|f(x) - f_{Q_i}|t\right)\, d\mu(x) \exp(Ct) \\
&\quad\le \exp\left(Bt\right) + \frac{1}{4} X(t) \exp(Ct).
\end{aligned}$$

Since $f \in L^\infty(\mu)$, $X(t) < \infty$, which implies that

$$X(t) \left[1 - \frac{1}{4}\exp(Ct)\right] \le \exp(Bt).$$

We then take t_0 small enough and see that $X(t_0) \lesssim 1$. Therefore, for $f \in L^\infty(\mu)$ and $Q \in \mathcal{D}$,

$$\begin{aligned}
\mu\left(\{x \in Q : |f(x)| > \lambda/t_0\}\right) &\le \int_Q \exp\left(|f(x)|t_0\right) \exp(-\lambda)\, d\mu(x) \\
&\lesssim \mu(2Q) \exp(-\lambda).
\end{aligned}$$

In the case that $Q \notin \mathcal{D}$, by a slight modification of the proof for Theorem 3.1 in [16], we also have that

$$\begin{aligned}\mu\left(\{x \in Q : |f(x) - f_Q| > \lambda/t_0\}\right) &\le \int_Q \exp\left(|f(x) - f_Q|t_0 - \lambda\right) d\mu(x)\\ &\lesssim \mu(2Q)\exp(-\lambda).\end{aligned}$$

When f is not bounded, consider the function f_q of Lemma 3.2. From Lemma 3.2 and the subsequent remark, we obtain that if $Q \in \mathcal{D}$,

$$\mu\left(\{x \in Q : |f_q(x)| > \lambda\}\right) \lesssim \mu(2Q)\exp\left(-C_2\lambda\right),$$

and if $Q \notin \mathcal{D}$,

$$\mu\left(\{x \in Q : |f_q(x) - f_{q,Q}| > \lambda\}\right) \lesssim \mu(2Q)\exp\left(-C_2\lambda\right).$$

A limiting argument then completes the proof of Theorem 3.1. ■

From Theorem 3.1, we can easily deduce that the following spaces, $\mathrm{rbmo}^p_{\eta,\rho}(\mu)$, coincide for all $p \in [1,\infty)$.

For any $\eta \in (1,\infty)$, $\rho \in [\eta,\infty)$, $\beta_\rho := \rho^{d+1}$ and $p \in [1,\infty)$, a function $f \in L^1_{\rm loc}(\mu)$ is said to belong to the *space* $\mathrm{rbmo}^p_{\eta,\rho}(\mu)$ if there exists a nonnegative constant $\widetilde{C}$ such that for all $Q \notin \mathcal{D}$,

$$\left\{\frac{1}{\mu(\eta Q)}\int_Q \left|f(x) - m_{\widetilde{Q}^\rho}(f)\right|^p d\mu(x)\right\}^{1/p} \le \widetilde{C},$$

that for any two (ρ,β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|m_Q(f) - m_R(f)| \le \widetilde{C}[1 + \delta(Q,R)],$$

and that for any $Q \in \mathcal{D}$,

$$\left\{\frac{1}{\mu(\eta Q)}\int_Q |f(x)|^p\, d\mu(x)\right\}^{1/p} \le \widetilde{C}.$$

Moreover, we define the minimal constant $\widetilde{C}$ as above to be the $\mathrm{rbmo}^p_{\eta,\rho}(\mu)$ *norm* of f and denote it by $\|f\|_{\mathrm{rbmo}^p_{\eta,\rho}(\mu)}$.

Arguing as for $p = 1$, one can show that another equivalent definition for $\mathrm{rbmo}^p_{\eta,\rho}(\mu)$ can be given in terms of the numbers $\{f_Q\}_Q$ as in (3.2) through (3.4) without depending on the constants $\eta \in (1,\infty)$ and $\rho \in [\eta,\infty)$.

Using Theorem 3.1, by following an argument as in [16, Corollary 3.5], we have the following conclusion, whose details are left to the reader.

Corollary 3.1. *For any $p \in [1, \infty)$, $\eta \in (1, \infty)$ and $\rho \in [\eta, \infty)$, the spaces* $\mathrm{rbmo}^p_{\eta,\,\rho}(\mu)$ *coincide with equivalent norms.*

We have another characterization for $\mathrm{rbmo}\,(\mu)$ which is useful in applications. To be precise, let $f \in L^1_{\mathrm{loc}}(\mu)$. If f is real-valued and for any cube Q, let $\alpha_Q(f)$ be the real number such that $\inf_{\alpha\in\mathbb{R}} m_Q(|f-\alpha|)$ is attained if $\mu(Q) \neq 0$ and $\alpha_Q(f) := 0$ if $\mu(Q) = 0$, then $\alpha_Q(f)$ satisfies that $\mu(\{x \in Q : f(x) > \alpha_Q(f)\}) \le \mu(Q)/2$, and

$$\mu(\{x \in Q : f(x) < \alpha_Q(f)\}) \le \mu(Q)/2.$$

If f is complex-valued, we take

$$\alpha_Q(f) := \mathrm{Re}\,[\alpha_Q(f)] + i\mathrm{Im}\,[\alpha_Q(f)],$$

where $i^2 = -1$. Furthermore, for any $p \in [1, \infty)$, $\eta \in (1, \infty)$, $\rho \in [\eta, \infty)$, $\beta_\rho := \rho^{d+1}$ and $f \in L^1_{\mathrm{loc}}(\mu)$, we denote by $\|f\|_\circ$ the *minimal nonnegative constant* $\widetilde{C}$ such that for any $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q |f(x) - \alpha_{\widetilde{Q}^\rho}(f)|\, d\mu(x) \le \widetilde{C},$$

that for any two (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|\alpha_Q(f) - \alpha_R(f)| \le \widetilde{C}[1 + \delta(Q, R)],$$

and that for any $Q \in \mathcal{D}$, $|\alpha_Q(f)| \le \widetilde{C}\frac{\mu(\eta Q)}{\mu(Q)}$.

Lemma 3.3. *For any $\eta \in (1, \infty)$ and $\rho \in [\eta, \infty)$, $\|\cdot\|_\circ$ is equivalent to* $\|\cdot\|_{\mathrm{rbmo}\,(\mu)}$.

Proof. By Proposition 3.3, it suffices to show $\|\cdot\|_\circ \sim \|\cdot\|_*$. First, we will prove $\|\cdot\|_* \lesssim \|\cdot\|_\circ$. For any $Q \subset \mathbb{R}^d$, set $f_Q := \alpha_{\widetilde{Q}^\rho}(f)$ if $\widetilde{Q}^\rho \notin \mathcal{D}$, and otherwise, set $f_Q := 0$. Arguing as in (3.5), to show that $\|\cdot\|_* \lesssim \|\cdot\|_\circ$, it suffices to verify that for any $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|f_Q - f_R| \lesssim [1 + \delta(Q, R)]\|f\|_\circ.$$

But, as in the proof of (3.5), this can be deduced from the fact

$$|\alpha_{\widetilde{Q}^\rho}(f) - \alpha_{\widetilde{R}^\rho}(f)| \lesssim [1 + \delta(Q, R)]\|f\|_\circ,$$

which can be proved by repeating the proof of (3.7).

Now we prove the converse. For any cube $Q \in \mathcal{D}$, by the definition of $\alpha_Q(f)$, we have

$$\left|\alpha_Q(f)\mu(Q) - \int_Q f(x)\,d\mu(x)\right| \le \int_Q |f(x) - \alpha_Q(f)|\,d\mu(x) \le \int_Q |f(x)|\,d\mu(x),$$

which implies in turn that

$$|\alpha_Q(f)|\mu(Q) \lesssim \int_Q |f(x)|\,d\mu(x). \tag{3.23}$$

Therefore, by Proposition 3.3,

$$|\alpha_Q(f)| \lesssim \frac{\mu(\eta Q)}{\mu(Q)}\|f\|_*. \tag{3.24}$$

On the other hand, for any (ρ, β_ρ)-doubling cube $Q \notin \mathcal{D}$, by the definition of $\alpha_Q(f)$ again,

$$\begin{aligned}\left|\alpha_Q(f) - f_Q\right| &\le \frac{1}{\mu(Q)}\int_Q [|f(x) - f_Q| + |f(x) - \alpha_Q(f)|]\,d\mu(x)\\ &\lesssim \|f\|_*.\end{aligned} \tag{3.25}$$

This fact together with (3.3) implies that for any two (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$ and $R \notin \mathcal{D}$,

$$|\alpha_Q(f) - \alpha_R(f)| \le |\alpha_Q(f) - f_Q| + |f_Q - f_R| + |f_R - \alpha_R(f)| \lesssim [1 + \delta(Q, R)]\|f\|_*.$$

Moreover, (3.25) together with (3.3), (3.4) and (3.24) yields that for any two (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$ and $R \in \mathcal{D}$,

$$|\alpha_Q(f) - \alpha_R(f)| \le |\alpha_Q(f) - f_Q| + |f_Q - f_R| + |f_R| + |\alpha_R(f)| \lesssim [1 + \delta(Q, R)]\|f\|_*.$$

Combining these estimates above, we see that for any two (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|\alpha_Q(f) - \alpha_R(f)| \lesssim [1 + \delta(Q, R)]\|f\|_*.$$

Finally, for any cube $Q \notin \mathcal{D}$,

$$\begin{aligned}&\frac{1}{\mu(\eta Q)}\int_Q \left|f(x) - \alpha_{\widetilde{Q}^\rho}(f)\right|\,d\mu(x)\\ &\quad\le \frac{1}{\mu(\eta Q)}\int_Q |f(x) - f_Q|\,d\mu(x) + \frac{\mu(Q)}{\mu(\eta Q)}\left[\left|f_Q - f_{\widetilde{Q}^\rho}\right| + \left|f_{\widetilde{Q}^\rho} - \alpha_{\widetilde{Q}^\rho}(f)\right|\right]\\ &\quad\lesssim \|f\|_*,\end{aligned}$$

where in the last inequality, we used (3.2) through (3.4) and (3.24) when $\widetilde{Q}^\rho \in \mathcal{D}$, and (3.2), (3.3) and (3.25) when $\widetilde{Q}^\rho \notin \mathcal{D}$. This completes the proof of Lemma 3.3. ∎

Remark 3.3. *By Lemma* 3.3*, Proposition* 3.2 *and Proposition* 3.3*, in the rest of this paper, unless otherwise stated, we will always assume that both constants* ρ *and* η *in the definition of* $\|\cdot\|_\circ$ *as well as that of* $\|\cdot\|_{\mathrm{rbmo}\,(\mu)}$ *are equal to* 2*.*

We now introduce local Hardy spaces.

Definition 3.2. *Let* $\eta\in(1,\infty)$*. A function* $b\in L^1_{\mathrm{loc}}(\mu)$ *is called an* ∞*-block if only (i) and (iii) with* $p=\infty$ *of Definition* 2.7 *hold. Moreover, we let* $|b|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}:=\sum_{j=1}^2|\lambda_j|$*.*

A function $f\in L^1(\mu)$ *is said to belong to the space* $h^{1,\infty}_{\mathrm{atb}}(\mu)$ *if there exist* ∞*-atomic blocks or* ∞*-blocks* $\{b_i\}_i$ *such that* $f=\sum_i b_i$ *and* $\sum_i|b_i|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}<\infty$*, where* b_i *is an* ∞*-atomic block as in Definition 2.7 if* $\mathrm{supp}\,(b_i)\subset R_i$ *and* $R_i\notin\mathcal{D}$*, while* b_i *is an* ∞*-block if* $\mathrm{supp}\,(b_i)\subset R_i$ *and* $R_i\in\mathcal{D}$*. Moreover, the* $h^{1,\infty}_{\mathrm{atb}}(\mu)$ *norm of* f *is defined by*

$$\|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}:=\inf\left\{\sum_i|b_i|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}\right\},$$

where the infimum is taken over all decompositions of f *in* ∞*-atomic blocks or* ∞*-blocks as above.*

Remark 3.4. *Let* $\rho\in(1,\infty)$ *and* $\beta_\rho:=\rho^{d+1}$*. Due to the fact that for any cubes* $Q\subset R$*,*

$$1+\delta(Q,\widetilde{R}^\rho)\sim 1+\delta(Q,R),$$

if necessary, we may assume that the cube R *in Definition* 3.2 *is* (ρ,β_ρ)*-doubling.*

It is easy to see that $H^1(\mu)\subsetneq h^{1,\infty}_{\mathrm{atb}}(\mu)\subsetneq L^1(\mu)$. Moreover, we have the following basic properties on the space $h^{1,\infty}_{\mathrm{atb}}(\mu)$.

Proposition 3.5. *The following three properties hold:*

(i) $h^{1,\infty}_{\mathrm{atb}}(\mu)\subset L^1(\mu)$ *with* $\|f\|_{L^1(\mu)}\le\|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}$*.*

(ii) *The space* $h^{1,\infty}_{\mathrm{atb}}(\mu)$ *is a Banach space.*

(iii) *The definition of* $h^{1,\infty}_{\mathrm{atb}}(\mu)$ *is independent of the choice of the constant* $\eta\in(1,\,\infty)$*.*

Proof. The proofs of the first two properties are similar to the usual proofs for the classical atomic Hardy spaces with μ being the d-dimensional Lebesgue measure, thus we omit the details.

To prove Property (iii), let $\eta_1>\eta_2>1$. It is obvious that for any $b\in h^{1,\infty}_{\mathrm{atb},\,\eta_1}(\mu)$, we have $b\in h^{1,\infty}_{\mathrm{atb},\,\eta_2}(\mu)$ and $\|b\|_{h^{1,\infty}_{\mathrm{atb},\,\eta_2}(\mu)}\le\|b\|_{h^{1,\infty}_{\mathrm{atb},\,\eta_1}(\mu)}$.

To prove the converse, let $b := \sum_{j=1}^{2} \lambda_j a_j \in h^{1,\infty}_{\mathrm{atb},\,\eta_2}(\mu)$ be an ∞-atomic block with $\operatorname{supp}(b) \subset R \notin \mathcal{D}$ or an ∞-block with $\operatorname{supp}(b) \subset R \in \mathcal{D}$ as in Definition 3.2. By Remark 3.4, we may assume that R is (ρ, β_ρ)-doubling with $\rho \geq \eta_1$. Then for each $j = 1,\, 2$,

$$\|a_j\|_{L^\infty(\mu)} \leq \{\mu(\eta_2 Q_j)[1 + \delta(Q_j, R)]\}^{-1}.$$

For any $x \in Q_j \cap \operatorname{supp}(\mu)$, let Q_x be the cube centered at x with side length $\frac{\eta_2 - 1}{10\eta_1} l(Q_j)$. It then follows that $\eta_1 Q_x \subset \eta_2 Q_j$. By the Besicovitch covering theorem, there exists an almost disjoint subfamily $\{Q_{j,\,k}\}_k$ of the cubes $\{Q_x\}_x$ covering $Q_j \cap \operatorname{supp}(\mu)$. Moreover, for each $j = 1,\, 2$, the number of cubes $\{Q_{j,\,k}\}_k$ of the Besicovitch covering is bounded by some constant $N_B \in \mathbb{N}$ depending only on η_1, η_2 and d; see [16, p. 99]. Since $l(Q_{j,\,k}) \sim l(Q_j)$ for all k, by Lemma 2.1, we have $\delta(Q_{j,\,k}, Q_j) \lesssim 1$. Moreover, it follows from Lemma 2.1 again that $\delta(Q_{j,\,k}, \eta_2 R) \lesssim 1 + \delta(Q_j, R)$. Therefore, by letting

$$a_{j,\,k} := a_j \frac{\chi_{Q_{j,\,k}}}{\sum_{k=1}^{N_B} \chi_{Q_{j,\,k}}}$$

and $\lambda_{j,\,k} := \lambda_j$, $k = 1, \cdots, N_B$, we see that

$$b = \sum_{j=1}^{2} \lambda_j a_j = \sum_{j=1}^{2} \sum_{k=1}^{N_B} \lambda_{j,\,k} a_{j,\,k}$$

and

$$\|a_{j,\,k}\|_{L^\infty(\mu)} \leq \|a_j\|_{L^\infty(\mu)} \lesssim \{\mu(\eta_1 Q_{j,\,k})[1 + \delta(Q_{j,\,k}, \eta_2 R)]\}^{-1}.$$

If b is an ∞-atomic block and $R \notin \mathcal{D}$, then for each $j,\, k$, let

$$c_{j,\,k} := \lambda_{j,\,k} a_{j,\,k} + \gamma_{j,\,k} \chi_R,$$

where $\gamma_{j,\,k}$ is the constant such that $c_{j,\,k}$ has zero mean. Then we see that

$$\gamma_{j,\,k} = -\frac{\lambda_{j,\,k}}{\mu(R)} \int_{Q_{j,\,k}} a_{j,\,k}(x)\, d\mu(x).$$

Since R is (ρ, β_ρ)-doubling, it is obvious that for each $j,\, k$, $c_{j,\,k}$ is an ∞-atomic block with $|c_{j,\,k}|_{h^{1,\infty}_{\mathrm{atb},\,\eta_1}(\mu)} \lesssim |\lambda_{j,\,k}|$ and $\operatorname{supp}(c_{j,\,k}) \subset \eta_2 R$. Therefore,

$$b = \sum_{j=1}^{2} \sum_{k=1}^{N_B} c_{j,\,k}$$

and

$$\|b\|_{h^{1,\infty}_{\mathrm{atb},\,\eta_1}(\mu)} \leq \sum_{j=1}^{2} \sum_{k=1}^{N_B} |c_{j,\,k}|_{h^{1,\infty}_{\mathrm{atb},\,\eta_1}(\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|,$$

which implies that $\|b\|_{h^{1,\infty}_{\mathrm{atb},\,\eta_1}(\mu)} \lesssim \|b\|_{h^{1,\infty}_{\mathrm{atb},\,\eta_2}(\mu)}$.

If b is an ∞-block and $R \in \mathcal{D}$, then for each j, k, let $c_{j,k} := \lambda_{j,k} a_{j,k}$. It is obvious that for each j, k, $c_{j,k}$ is an ∞-block with $|c_{j,k}|_{h^{1,\infty}_{\mathrm{atb},\eta_1}(\mu} \lesssim |\lambda_{j,k}|$ and $\operatorname{supp}(c_{j,k}) \subset \eta_2 R$. Moreover, $b = \sum_{j=1}^2 \sum_{k=1}^{N_B} c_{j,k}$ and

$$\|b\|_{h^{1,\infty}_{\mathrm{atb},\eta_1}(\mu)} \lesssim \sum_{j=1}^{2} \sum_{k=1}^{N_B} |\lambda_{j,k}| \lesssim \sum_{j=1}^{2} |\lambda_j|,$$

which implies that $\|b\|_{h^{1,\infty}_{\mathrm{atb},\eta_1}(\mu)} \lesssim \|b\|_{h^{1,\infty}_{\mathrm{atb},\eta_2}(\mu)}$, and hence completes the proof of Proposition 3.5. ■

Remark 3.5. *By Proposition* 3.5*, unless otherwise stated, we will always assume that the constant* η *in Definition* 3.2 *is equal to* 2*.*

Inspired by the duality between $H^1(\mu)$ and $\mathrm{RBMO}(\mu)$, we will show that the space $h^{1,\infty}_{\mathrm{atb}}(\mu)$ is the predual space of the space $\mathrm{rbmo}(\mu)$. To this end, we will follow the scheme of [8, pp. 34-40] (see also [16]).

Lemma 3.4. $\mathrm{rbmo}(\mu) \subset [h^{1,\infty}_{\mathrm{atb}}(\mu)]^*$. *That is, for any* $g \in \mathrm{rbmo}(\mu)$*, the linear functional*

$$L_g(f) := \int_{\mathbb{R}^d} f g \, d\mu$$

defined on bounded functions f *with compact support extends to a continuous linear functional* L_g *over* $h^{1,\infty}_{\mathrm{atb}}(\mu)$ *with* $\|L_g\|_{[h^{1,\infty}_{\mathrm{atb}}(\mu)]^*} \le C\|g\|_{\mathrm{rbmo}(\mu)}$*, where* C *is a positive constant independent of* g*.*

Proof. By Remark 3.3 and Remark 3.5, we take $\rho = \eta = 2$ in Definition 3.1 and Definition 3.2. Following some standard arguments (see, for example, [5, pp. 294-296]), we only need to show that if $b := \sum_{j=1}^2 \lambda_j a_j$ is an ∞-atomic block with $\operatorname{supp}(b) \subset R \notin \mathcal{D}$ as in Definition 2.7 or an ∞-block with $\operatorname{supp}(b) \subset R \in \mathcal{D}$ as in Definition 3.2, then for any $g \in \mathrm{rbmo}(\mu)$,

$$\left| \int_{\mathbb{R}^d} b(x) g(x) \, d\mu(x) \right| \lesssim |b|_{h^{1,\infty}_{\mathrm{atb}}(\mu)} \|g\|_{\mathrm{rbmo}(\mu)}.$$

If b is an ∞-atomic block with $\operatorname{supp}(b) \subset R \notin \mathcal{D}$, an argument similar to that in [16, p. 115] yields that

$$\left| \int_{\mathbb{R}^d} b(x) g(x) \, d\mu(x) \right| \lesssim \sum_{i=1}^{2} |\lambda_i| \|g\|_{\mathrm{rbmo}(\mu)}.$$

If b is an ∞-block with $\operatorname{supp}(b) \subset R \in \mathcal{D}$, we have

$$\left| \int_{\mathbb{R}^d} b(x) g(x) \, d\mu(x) \right| \le \sum_{i=1}^{2} |\lambda_i| \int_{Q_i} |a_i(x)| |g(x)| \, d\mu(x).$$

Now for $i = 1, 2$, if $Q_i \in \mathcal{D}$, it follows from (2.3) that

$$\int_{Q_i} |a_i(x)||g(x)|\,d\mu(x) \le \{\mu(2Q_i)[1+\delta(Q_i,R)]\}^{-1}\int_{Q_i}|g(x)|\,d\mu(x) \le \|g\|_{\mathrm{rbmo}\,(\mu)}.$$

If $Q_i \notin \mathcal{D}$, then (2.3) together with Definition 3.1 and (3.7) yields that

$$\begin{aligned}\int_{Q_i} &|a_i(x)||g(x)|\,d\mu(x)\\ &\le \int_{Q_i}|a_i(x)|\left|g(x)-m_{\widetilde{Q_i}}(g)\right|\,d\mu(x)+\left|m_{\widetilde{Q_i}}(g)\right|\int_{Q_i}|a_i(x)|\,d\mu(x)\\ &\le \|a_i\|_{L^\infty(\mu)}\left[\int_{Q_i}\left|g(x)-m_{\widetilde{Q_i}}(g)\right|\,d\mu(x)\right.\\ &\quad\left.+\mu(Q_i)\left|m_{\widetilde{Q_i}}(g)-m_{\widetilde{R}}(g)\right|+\mu(Q_i)\left|m_{\widetilde{R}}(g)\right|\right]\lesssim \|g\|_{\mathrm{rbmo}\,(\mu)}.\end{aligned}$$

Therefore, we have

$$\left|\int_{\mathbb{R}^d} b(x)g(x)\,d\mu(x)\right| \lesssim \sum_{i=1}^{2}|\lambda_i|\|g\|_{\mathrm{rbmo}\,(\mu)} = |b|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}\|g\|_{\mathrm{rbmo}\,(\mu)},$$

which completes the proof of Lemma 3.4. ■

Lemma 3.5. *If $g \in \mathrm{rbmo}\,(\mu)$, then $\|L_g\|_{[h^{1,\infty}_{\mathrm{atb}}(\mu)]^*} \sim \|g\|_{\mathrm{rbmo}\,(\mu)}$.*

Proof. By Lemma 3.4, it suffices to show $\|L_g\|_{[h^{1,\infty}_{\mathrm{atb}}(\mu)]^*} \gtrsim \|g\|_{\mathrm{rbmo}\,(\mu)}$. Without loss of generality, we may assume that g is real-valued. With the aid of Lemma 3.3, we only need to prove that there exists some function $f \in h^{1,\infty}_{\mathrm{atb}}(\mu)$ such that

$$|L_g(f)| \gtrsim \|g\|_\circ \|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}. \tag{3.26}$$

By Remark 3.3 and Remark 3.5, we take $\rho = \eta = 2$ in the definition of $\|\cdot\|_\circ$ and Definition 3.2. Let $\epsilon \in (0, 1/8]$. There exist two possibilities.

Case (1) There exists some doubling cube $Q \subset \mathbb{R}^d$ with $Q \notin \mathcal{D}$ such that

$$\int_Q |g(x)-\alpha_Q(g)|\,d\mu(x) \ge \epsilon\|g\|_\circ \mu(Q), \tag{3.27}$$

or there exists some doubling cube $Q \subset \mathbb{R}^d$ with $Q \in \mathcal{D}$ such that

$$|\alpha_Q(g)| \ge \epsilon\|g\|_\circ. \tag{3.28}$$

If (3.27) holds, then for such a cube $Q \notin \mathcal{D}$ satisfying (3.27), by an argument similar to that in [16, p. 116], we find an $f \in h^{1,\infty}_{\mathrm{atb}}(\mu)$ such that (3.26) holds.

If (3.28) holds, for such a cube $Q \in \mathcal{D}$ satisfying (3.28), we take $f := \operatorname{sgn}(g)\chi_Q$, where and in what follows, $\operatorname{sgn}(g)$ denotes the *sign function of the function* g. It immediately follows that f is an ∞-block with $\operatorname{supp}(f) \subset Q$ and $|f|_{h^{1,\infty}_{\mathrm{atb}}(\mu)} \lesssim \mu(Q)$. By this fact, (3.23) and (3.28), we see that

$$|L_g(f)| = \left|\int_Q g(x)f(x)\,d\mu(x)\right| = \int_Q |g(x)|\;d\mu(x) \gtrsim \epsilon\|g\|_\circ\|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}.$$

Thus, in Case (1), (3.26) holds.

Case (2) For any doubling $Q \subset \mathbb{R}^d$ with $Q \notin \mathcal{D}$, (3.27) fails, and for any doubling cube $Q \subset \mathbb{R}^d$ with $Q \in \mathcal{D}$, (3.28) fails. In this case, we further consider the following two subcases.

Subcase (i) For any two doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|\alpha_Q(g) - \alpha_R(g)| \le \frac{1}{2}[1 + \delta(Q,R)]\|g\|_\circ.$$

In this subcase, from the definition of $\|g\|_\circ$, there exists some cube $Q \notin \mathcal{D}$ such that

$$\int_Q \left|g(x) - \alpha_{\widetilde{Q}}(g)\right|\;d\mu(x) \ge \frac{1}{2}\|g\|_\circ\mu(2Q). \tag{3.29}$$

If $\widetilde{Q} \notin \mathcal{D}$, then by the argument in [16, p. 117], we obtain that (3.26) holds. If $\widetilde{Q} \in \mathcal{D}$, we then let $f := \chi_{Q\cap\{g>\alpha_{\widetilde{Q}}(g)\}} - \chi_{Q\cap\{g<\alpha_{\widetilde{Q}}(g)\}}$. It is easy to see that f is an ∞-block with $\operatorname{supp}(f) \subset \widetilde{Q}$ and $\|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)} \lesssim \mu(2Q)$. Moreover, since (3.28) fails for $\widetilde{Q}$, using (3.29), we have

$$\begin{aligned}|L_g(f)| &\ge \left|\int_Q \left[g(x) - \alpha_{\widetilde{Q}}(g)\right] f(x)\,d\mu(x)\right| - \left|\alpha_{\widetilde{Q}}(g)\right|\left|\int_Q f(x)\,d\mu(x)\right|\\ &\ge \frac{1}{2}\|g\|_\circ\mu(2Q) - \epsilon\|g\|_\circ\mu(Q) \gtrsim \|g\|_\circ\|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}.\end{aligned}$$

Subcase (ii) There exists some doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|\alpha_Q(g) - \alpha_R(g)| > \frac{1}{2}[1 + \delta(Q,R)]\|g\|_\circ.$$

In this subcase, we also only need to consider the case that $R \in \mathcal{D}$, because if $R \notin \mathcal{D}$, the argument in [16, p. 118] works here as well. Assume $R \in \mathcal{D}$ and take $f := \chi_Q$. Then f is an ∞-block with $\operatorname{supp}(f) \subset R$ and

$$\|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)} \lesssim [1 + \delta(Q,R)]\mu(Q).$$

Since (3.27) fails for Q and (3.28) fails for R, it follows from the assumption of this subcase and the fact that $\epsilon \le \frac{1}{8}$ that

$$\begin{aligned}|L_g(f)| &= \Big| \int_Q [g(x) - \alpha_Q(g)] f(x)\, d\mu(x) + \alpha_Q(g)\mu(Q) \Big| \\ &\ge |\alpha_Q(g) - \alpha_R(g)|\mu(Q) - \int_Q |g(x) - \alpha_Q(g)|\, d\mu(x) - |\alpha_R(g)|\mu(Q) \\ &> \frac{1}{2}[1 + \delta(Q,R)]\|g\|_\circ \mu(Q) - 2\epsilon \|g\|_\circ \mu(Q) \\ &\ge \frac{1}{4}[1 + \delta(Q,R)]\|g\|_\circ \mu(Q).\end{aligned}$$

Therefore (3.26) also holds in this case, which completes the proof of Lemma 3.5. ■

We now introduce the spaces $h^{1,p}_{\rm atb}(\mu)$ for $p \in (1,\infty)$ and prove that they coincide with $h^{1,\infty}_{\rm atb}(\mu)$ and the dual of the space $h^{1,\infty}_{\rm atb}(\mu)$ is $\mathrm{rbmo}\,(\mu)$ simultaneously.

Definition 3.3. *Let $\eta \in (1,\infty)$ and $p \in (1,\infty)$. A function $b \in L^1_{\rm loc}(\mu)$ is called a p-block if only (i) and (iii) with $p \in (1,\infty)$ of Definition 2.7 hold. Moreover, we let $|b|_{h^{1,p}_{\rm atb}(\mu)} := \sum_{j=1}^2 |\lambda_j|$.*

A function $f \in L^1(\mu)$ is said to belong to the space $h^{1,p}_{\rm atb}(\mu)$ if there exist p-atomic blocks or p-blocks $\{b_i\}_i$ such that $f = \sum_i b_i$ and $\sum_i |b_i|_{h^{1,p}_{\rm atb}(\mu)} < \infty$, where b_i is a p-atomic block as in Definition 2.7 if $\mathrm{supp}\,(b_i) \subset R_i$ and $R_i \notin \mathcal{D}$, while b_i is a p-block if $\mathrm{supp}\,(b_i) \subset R_i$ and $R_i \in \mathcal{D}$. We define the $h^{1,p}_{\rm atb}(\mu)$ norm of f by letting

$$\|f\|_{h^{1,p}_{\rm atb}(\mu)} := \inf \left\{ \sum_i |b_i|_{h^{1,p}_{\rm atb}(\mu)} \right\},$$

where the infimum is taken over all possible decompositions of f in p-atomic blocks or p-blocks as above.

Remark 3.6. *Let $\rho \in (1,\infty)$ and $\beta_\rho := \rho^{d+1}$. Due to the fact that for any cubes $Q \subset R$,*

$$1 + \delta(Q, \widetilde{R}^\rho) \sim 1 + \delta(Q,R).$$

We may assume that the cube R as in Definition 3.3 is (ρ,β_ρ)-doubling if necessary.

It is easy to see that $H^{1,p}_{\rm atb}(\mu) \subsetneq h^{1,p}_{\rm atb}(\mu) \subsetneq L^1(\mu)$. In fact, the spaces $h^{1,p}_{\rm atb}(\mu)$ have the following properties similar to $h^{1,\infty}_{\rm atb}(\mu)$. We omit the details; see [16, Proposition 5.1].

Proposition 3.6. *Let $p \in (1, \infty)$. The following four properties hold:*

(i) $h^{1,\,p}_{\rm atb}(\mu) \subset L^1(\mu)$ *with* $\|f\|_{L^1(\mu)} \le \|f\|_{h^{1,\,p}_{\rm atb}(\mu)}$.

(ii) *The space* $h^{1,\,p}_{\rm atb}(\mu)$ *is a Banach space.*

(iii) *For any* $p_1,\, p_2 \in (1, \infty]$ *with* $p_1 \le p_2$, *we have the continuous inclusion* $h^{1,\,p_2}_{\rm atb}(\mu) \subset h^{1,\,p_1}_{\rm atb}(\mu)$.

(iv) *The definition of* $h^{1,\,p}_{\rm atb}(\mu)$ *is independent of the choice of the constant* $\eta \in (1,\, \infty)$.

Remark 3.7. *By Proposition* 3.6 (iv), *unless otherwise stated, we will always assume that the constant* η *in Definition* 3.3 *is equal to* 2.

Lemma 3.6. *For any* $p \in (1, \infty)$, $\mathrm{rbmo}\,(\mu) \subset [h^{1,\,p}_{\rm atb}(\mu)]^*$. *That is, for any* $g \in \mathrm{rbmo}\,(\mu)$, *the linear functional*

$$L_g(f) := \int_{\mathbb{R}^d} f(x)g(x)\,d\mu(x)$$

defined over $f \in L^\infty(\mu)$ *with compact support extends to a unique continuous linear functional* L_g *over* $h^{1,\,p}_{\rm atb}(\mu)$ *with* $\|L_g\|_{[h^{1,\,p}_{\rm atb}(\mu)]^*} \le C\|g\|_{\mathrm{rbmo}\,(\mu)}$, *where* C *is a positive constant independent of* g.

Proof. By Remark 3.3 and Remark 3.7, we take $\rho = \eta = 2$ in Definition 3.1 and Definition 3.3. Similar to the proof of Lemma 3.4, it suffices to show that if $b := \sum_{i=1}^2 \lambda_i a_i$ is a p-atomic block with $\mathrm{supp}\,(b) \subset R \notin \mathcal{D}$ as in Definition 2.7 or a p-block with $\mathrm{supp}\,(b) \subset R \in \mathcal{D}$ as in Definition 3.3, then for any $g \in \mathrm{rbmo}\,(\mu)$,

$$\left|\int_{\mathbb{R}^d} b(x)g(x)\,d\mu(x)\right| \lesssim |b|_{h^{1,\,p}_{\rm atb}(\mu)}\|g\|_{\mathrm{rbmo}\,(\mu)}.$$

If b is a p-atomic block with $\mathrm{supp}\,(b) \subset R \notin \mathcal{D}$, then an argument similar to that in [16, p. 120] gives us the desired estimate.

Now suppose b is a p-block with $\mathrm{supp}\,(b) \subset R \in \mathcal{D}$. In this case, we also have that

$$\left|\int_{\mathbb{R}^d} b(x)g(x)\,d\mu(x)\right| \le \sum_{i=1}^2 |\lambda_i| \int_{Q_i} |a_i(x)||g(x)|\,d\mu(x).$$

For each i, if $Q_i \in \mathcal{D}$, then it follows from the Hölder inequality, (2.3) and Corollary 3.1 that

$$\begin{aligned}\int_{Q_i} |a_i(x)||g(x)|\,d\mu(x) &\le \left[\int_{Q_i} |a_i(x)|^p\,d\mu(x)\right]^{1/p}\left[\int_{Q_i} |g(x)|^{p'}\,d\mu(x)\right]^{1/p'}\\ &\lesssim \|g\|_{\mathrm{rbmo}\,(\mu)}.\end{aligned}$$

If $Q_i \notin \mathcal{D}$, then using the Hölder inequality, (3.7), (2.3), Definition 3.1 and Corollary 3.1 again, we have

$$\begin{aligned}
&\int_{Q_i} |a_i(x)||g(x)|\,d\mu(x)\\
&\quad\le \int_{Q_i} |a_i(x)|\left|g(x) - m_{\widetilde{Q_i}}(g)\right|\,d\mu(x) + \left|m_{\widetilde{Q_i}}(g)\right| \int_{Q_i} |a_i(x)|\,d\mu(x)\\
&\quad\le \|a_i\|_{L^p(\mu)}\left[\left\|(g - m_{\widetilde{Q_i}}(g))\chi_{Q_i}\right\|_{L^{p'}(\mu)} + \left|m_{\widetilde{Q_i}}(g) - m_{\widetilde{R}}(g)\right|[\mu(Q_i)]^{1/p'}\right.\\
&\qquad\left. + \left|m_{\widetilde{R}}(g)\right|[\mu(Q_i)]^{1/p'}\right] \lesssim \|g\|_{\mathrm{rbmo}\,(\mu)}.
\end{aligned}$$

Therefore,

$$\left|\int_{\mathbb{R}^d} b(x)g(x)\,d\mu(x)\right| \lesssim \sum_{i=1}^{2} |\lambda_i|\|g\|_{\mathrm{rbmo}\,(\mu)} = |b|_{h^{1,p}_{\mathrm{atb}}(\mu)}\|g\|_{\mathrm{rbmo}\,(\mu)}.$$

This completes the proof of Lemma 3.6. ■

Lemma 3.7. *Let* $p \in (1,\infty)$. *Then* $[h^{1,p}_{\mathrm{atb}}(\mu)]^* \subset L^{p'}_{\mathrm{loc}}(\mu)$, *where* $1/p + 1/p' = 1$.

Proof. This lemma is an easy consequence of the Riesz representation theorem, and it can be proved by a slight modification of the argument in [8, pp. 39-40]; see also Lemma 5.4 in [16]. We omit the details. ■

Lemma 3.8. *For any* $p \in (1,\infty)$, $[h^{1,p}_{\mathrm{atb}}(\mu)]^* = \mathrm{rbmo}\,(\mu)$.

Proof. By Lemma 3.6, to prove the lemma, it suffices to show that for any $p \in (1,\infty)$, $[h^{1,p}_{\mathrm{atb}}(\mu)]^* \subset \mathrm{rbmo}\,(\mu)$. Based on Lemma 3.7, we let $g \in L^{p'}_{\mathrm{loc}}(\mu)$ such that $L_g \in [h^{1,p}_{\mathrm{atb}}(\mu)]^*$. We will prove that $g \in \mathrm{rbmo}\,(\mu)$ by verifying that for any $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(2Q)}\int_Q \left|g(x) - \alpha_{\widetilde{Q}}(g)\right|\,d\mu(x) \lesssim \|L_g\|_{[h^{1,p}_{\mathrm{atb}}(\mu)]^*}, \tag{3.30}$$

that for any two doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|\alpha_Q(g) - \alpha_R(g)| \lesssim [1 + \delta(Q,R)]\|L_g\|_{[h^{1,p}_{\mathrm{atb}}(\mu)]^*}, \tag{3.31}$$

and that for any $Q \in \mathcal{D}$,

$$|\alpha_Q(g)| \lesssim \frac{\mu(2Q)}{\mu(Q)}\|L_g\|_{[h^{1,p}_{\mathrm{atb}}(\mu)]^*}. \tag{3.32}$$

We first verify (3.32). Let $Q \in \mathcal{D}$ and $f := \operatorname{sgn}(g)\chi_Q$. Then f is a p-block with $\operatorname{supp}(f) \subset Q$ and $|f|_{h^{1,p}_{\rm atb}(\mu)} \lesssim \mu(2Q)$. By the definition of $\alpha_Q(g)$, we see that

$$\begin{aligned}|\alpha_Q(g)\mu(Q)| &\le \int_Q |g(x) - \alpha_Q(g)|\, d\mu(x) + \left|\int_Q g(x)\, d\mu(x)\right| \lesssim \int_Q |g(x)|\, d\mu(x)\\ &= |L_g(f)| \lesssim \|L_g\|_{[h^{1,p}_{\rm atb}(\mu)]^*}\mu(2Q),\end{aligned}$$

which implies (3.32).

If Q is doubling and $Q \notin \mathcal{D}$, then (3.30) is true by following the argument for (5.2) in [16], therefore we only need to show that (3.30) holds when Q is not doubling and $Q \notin \mathcal{D}$. Moreover, we may assume $\widetilde{Q} \in \mathcal{D}$, since the proof of (5.2) in [16] also works here for $\widetilde{Q} \notin \mathcal{D}$. Let

$$f := \frac{|g - \alpha_{\widetilde{Q}}(g)|^{p'}}{g - \alpha_{\widetilde{Q}}(g)}\chi_{Q\cap\{g\ne\alpha_{\widetilde{Q}}(g)\}}.$$

Then f is a p-block with $\operatorname{supp}(f) \subset \widetilde{Q}$ and

$$|f|_{h^{1,p}_{\rm atb}(\mu)} \lesssim \left[\int_Q \left|g(x) - \alpha_{\widetilde{Q}}(g)\right|^{p'} d\mu(x)\right]^{1/p} [\mu(2Q)]^{1/p'}. \tag{3.33}$$

On the other hand, by (3.32) together with the doubling property of $\widetilde{Q}$ and Proposition 3.6 (i), we have

$$\begin{aligned}\int_Q \left|g(x) - \alpha_{\widetilde{Q}}(g)\right|^{p'} d\mu(x) &= \int_Q \left[g(x) - \alpha_{\widetilde{Q}}(g)\right] f(x)\, d\mu(x)\\ &\le \left|\int_Q g(x)f(x)\, d\mu(x)\right| + \left|\alpha_{\widetilde{Q}}(g)\right| \int_Q |f(x)|\, d\mu(x)\\ &\lesssim \|L_g\|_{[h^{1,p}_{\rm atb}(\mu)]^*}\|f\|_{h^{1,p}_{\rm atb}(\mu)},\end{aligned}$$

which together with (3.33) and the Hölder inequality implies (3.30) in this case.

To prove (3.31), let $Q \subset R$ with $Q \notin \mathcal{D}$ be any two doubling cubes. If $R \notin \mathcal{D}$, then by the proof of (5.3) in [16], we obtain (3.31). Now suppose that $R \in \mathcal{D}$. We choose

$$f := \frac{|g - \alpha_R(g)|^{p'}}{g - \alpha_R(g)}\chi_{Q\cap\{g\ne\alpha_R(g)\}}.$$

Then f is a p-block with $\operatorname{supp}(f) \subset R$ and

$$|f|_{h^{1,p}_{\rm atb}(\mu)} \lesssim [1 + \delta(Q, R)] \left[\int_Q |g(x) - \alpha_R(g)|^{p'} d\mu(x)\right]^{1/p} [\mu(2Q)]^{1/p'}.$$

Consequently, by applying (3.32), Proposition 3.6 (i) and the doubling property of R, we see that

$$\begin{aligned}
&\int_Q |g(x) - \alpha_R(g)|^{p'}\, d\mu(x) \\
&\quad = \int_Q [g(x) - \alpha_R(g)] f(x)\, d\mu(x) \\
&\quad \le \left| \int_Q g(x) f(x)\, d\mu(x) \right| + |\alpha_R(g)| \int_Q |f(x)|\, d\mu(x) \lesssim \|L_g\|_{[h^{1,p}_{\rm atb}(\mu)]^*} \|f\|_{h^{1,p}_{\rm atb}(\mu)} \\
&\quad \lesssim \|L_g\|_{[h^{1,p}_{\rm atb}(\mu)]^*} [1 + \delta(Q,R)] \left[\int_Q |g(x) - \alpha_R(g)|^{p'}\, d\mu(x) \right]^{1/p} [\mu(2Q)]^{1/p'}.
\end{aligned}$$

Therefore,

$$\left[\frac{1}{\mu(2Q)} \int_Q |g(x) - \alpha_R(g)|^{p'}\, d\mu(x) \right]^{1/p'} \lesssim [1 + \delta(Q,R)] \|L_g\|_{[h^{1,p}_{\rm atb}(\mu)]^*}.$$

Recall that Q is doubling. From this fact, the last estimate as above, (3.30) and the Hölder inequality, it follows that

$$\begin{aligned}
&|\alpha_Q(g) - \alpha_R(g)| \\
&\qquad \le \frac{1}{\mu(Q)} \int_Q |g(x) - \alpha_Q(g)|\, d\mu(x) + \frac{1}{\mu(Q)} \int_Q |g(x) - \alpha_R(g)|\, d\mu(x) \\
&\qquad \lesssim [1 + \delta(Q,R)] \|L_g\|_{[h^{1,p}_{\rm atb}(\mu)]^*},
\end{aligned}$$

which verifies (3.31) and hence completes the proof of Lemma 3.8. ■

Lemma 3.9. *Let $p \in (1,\infty)$. The local atomic Hardy space $h^{1,\infty}_{\rm atb}(\mu)$ is dense in the local Hardy space $h^{1,p}_{\rm atb}(\mu)$.*

Proof. By Definition 3.3, for every $f \in h^{1,p}_{\rm atb}(\mu)$ and $\epsilon > 0$, there exists $m \in \mathbb{N}$ and $g := \sum_{j=1}^m b_j$ such that $\|f - g\|_{h^{1,p}_{\rm atb}(\mu)} < \frac{\epsilon}{2}$, where for $j = 1, \ldots, m$, b_j is a p-atomic block if $\operatorname{supp}(b_j) \subset R_j \notin \mathcal{D}$ or a p-block if $\operatorname{supp}(b_j) \subset R_j \in \mathcal{D}$. Moreover, for any $j = 1, \ldots, m$, it is easy to see that there exists an $h_j \in L^\infty(\mu)$ such that $\operatorname{supp}(h_j) \subset R_j$ and

$$\|b_j - h_j\|_{L^p(\mu)} < \frac{\epsilon}{2^{m+1}[\mu(2R_j)]^{1-1/p}}.$$

For each j, if b_j is a p-atomic block with $\operatorname{supp}(b_j) \subset R_j \notin \mathcal{D}$, then take

$$\widetilde{b_j} := h_j - \frac{\chi_{R_j}}{\mu(R_j)} \int_{\mathbb{R}^d} h_j\, d\mu.$$

By an argument similar to that in the proof for Lemma 2.1 of [6], we see that $\widetilde{b_j}$ is an ∞-atomic block with $\operatorname{supp}(b_j) \subset R_j$ and $\|b_j - \widetilde{b_j}\|_{h^{1,p}_{\rm atb}(\mu)} < \frac{\epsilon}{2^m}$. If b_j is a p-block with $\operatorname{supp}(b_j) \subset R_j \in \mathcal{D}$, then take $\widetilde{b_j} := h_j$. It is easy to see that $\widetilde{b_j}$ is an ∞-block with $\operatorname{supp}(\widetilde{b_j}) \subset R_j \in \mathcal{D}$ and $\|b_j - \widetilde{b_j}\|_{h^{1,p}_{\rm atb}(\mu)} < \frac{\epsilon}{2^m}$.

Now let $\widetilde{g} := \sum_{j=1}^m \widetilde{b_j}$. From Definition 3.2, it further follows that $\widetilde{g} \in h^{1,\infty}_{\rm atb}(\mu)$ and

$$\|f - \widetilde{g}\|_{h^{1,p}_{\rm atb}(\mu)} \le \|f - g\|_{h^{1,p}_{\rm atb}(\mu)} + \|g - \widetilde{g}\|_{h^{1,p}_{\rm atb}(\mu)} < \epsilon,$$

which completes the proof of Lemma 3.9. ■

Theorem 3.2. *For any fixed* $p \in (1, \infty)$, $h^{1,p}_{\rm atb}(\mu) = h^{1,\infty}_{\rm atb}(\mu)$ *and*

$$[h^{1,\infty}_{\rm atb}(\mu)]^* = \operatorname{rbmo}(\mu).$$

Proof. By Lemma 3.9, we see that if $f \in [h^{1,p}_{\rm atb}(\mu)]^*$, then $f \in [h^{1,\infty}_{\rm atb}(\mu)]^*$. With the aid of Lemma 3.8, we consider the maps $i:\ h^{1,\infty}_{\rm atb}(\mu) \to h^{1,p}_{\rm atb}(\mu)$ and

$$i^*:\ \operatorname{rbmo}(\mu) = [h^{1,p}_{\rm atb}(\mu)]^* \to [h^{1,\infty}_{\rm atb}(\mu)]^*.$$

Notice that the map i is an inclusion and i^* is the canonical injection of $\operatorname{rbmo}(\mu)$ in $[h^{1,\infty}_{\rm atb}(\mu)]^*$ (with the identification $g \equiv L_g$ for $g \in \operatorname{rbmo}(\mu)$). By Lemma 3.5, $i^*(\operatorname{rbmo}(\mu))$ is closed in $[h^{1,\infty}_{\rm atb}(\mu)]^*$. An application of the Banach closed range theorem (see [29, p. 205]) shows that $h^{1,\infty}_{\rm atb}(\mu)$ is closed in $h^{1,p}_{\rm atb}(\mu)$, which together with Lemma 3.9 implies that $h^{1,\infty}_{\rm atb}(\mu) = h^{1,p}_{\rm atb}(\mu)$ as a set. Thus i maps $h^{1,\infty}_{\rm atb}(\mu)$ onto $h^{1,p}_{\rm atb}(\mu)$. Observing that both $h^{1,\infty}_{\rm atb}(\mu)$ and $h^{1,p}_{\rm atb}(\mu)$ are Banach spaces, by the corollary of the open mapping theorem (see [29, p. 77]), we obtain that $h^{1,\infty}_{\rm atb}(\mu) = h^{1,p}_{\rm atb}(\mu)$ with an equivalent norm, which completes the proof of Theorem 3.2. ■

We next come to establish relations between spaces $H^1(\mu)$ and $h^{1,\infty}_{\rm atb}(\mu)$, and between spaces $\operatorname{RBMO}(\mu)$ and $\operatorname{rbmo}(\mu)$, respectively. If μ is the d-dimensional Lebesgue measure, Proposition 3.7 and Corollary 3.2 below are obtained by Goldberg [4].

Proposition 3.7. *Let* $k \in \mathbb{N}$ *and* S_k *be as in Section 2. If* $f \in h^{1,\infty}_{\rm atb}(\mu)$, *then* $f - S_k(f) \in H^1(\mu)$ *and*

$$\|f - S_k(f)\|_{H^1(\mu)} \le C\|f\|_{h^{1,\infty}_{\rm atb}(\mu)},$$

where C *is a positive constant independent of* f *and* k.

Proof. Notice that for any ∞-block or ∞-atomic block $b := \sum_{j=1}^{2} \lambda_j a_j$ as in Definition 3.2 or Definition 2.7, and any $k \in \mathbb{N}$, by (A-2) in Section 2 and the Tonelli theorem, we have

$$\|S_k(b)\|_{L^1(\mu)} \le \|b\|_{L^1(\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|. \tag{3.34}$$

From this and Definition 2.6, to prove Proposition 3.7, it suffices to show that

$$\|\mathcal{M}_\Phi(b - S_k(b))\|_{L^1(\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|, \tag{3.35}$$

where $\mathcal{M}_\Phi$ is as in Definition 2.5.

Let b be an ∞-atomic block with $\operatorname{supp}(b) \subset R \notin \mathcal{D}$. Then by the fact that $\{S_k\}_{k\in\mathbb{N}}$ are uniformly bounded on $H^1(\mu)$ (see Theorem 3.1 in [28]), we have

$$\|\mathcal{M}_\Phi(S_k(b))\|_{L^1(\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|.$$

This together with the sublinear property of $\mathcal{M}_\Phi$ and Definition 2.6 yields (3.35) in this case.

Now assume that b is an ∞-block with $\operatorname{supp}(b) \subset R \in \mathcal{D}$. Fix any $x_0 \in R \cap \operatorname{supp}(\mu)$. We consider the following two cases: (1) $k \le H_R^{x_0}$; (2) $k \ge H_R^{x_0} + 1$.

In Case (1), write

$$\begin{aligned}\|\mathcal{M}_\Phi(b - S_k(b))\|_{L^1(\mu)} &= \int_{8R} \mathcal{M}_\Phi(b - S_k(b))(x)\, d\mu(x) + \int_{\mathbb{R}^d\setminus 8R} \cdots \\ &=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

Since $\mathcal{M}_\Phi$ is sublinear, we have that

$$\begin{aligned}\mathrm{I}_1 &\le \sum_{j=1}^{2} |\lambda_j| \int_{2Q_j} \mathcal{M}_\Phi(a_j)(x)\, d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{8R\setminus 2Q_j} \cdots \\ &\quad + \int_{8R} \mathcal{M}_\Phi(S_k(b))(x)\, d\mu(x) \\ &=: \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3.\end{aligned}$$

By an argument similar to that in the proof for I in Theorem 3.1 of [28], we have $\mathrm{J}_3 \lesssim \sum_{j=1}^{2} |\lambda_j|$. Thus I_1 is reduced to showing that $\mathrm{J}_1 + \mathrm{J}_2 \lesssim \sum_{j=1}^{2} |\lambda_j|$.

For each $j=1$, 2, by Definition 3.2, for any $x\in 2Q_j$ and $\varphi\sim x$,

$$\left|\int_{\mathbb{R}^d}\varphi(y)a_j(y)\,d\mu(y)\right|\le \|a_j\|_{L^\infty(\mu)}\|\varphi\|_{L^1(\mu)}\le \|a_j\|_{L^\infty(\mu)},$$

which implies that $\mathcal{M}_\Phi(a_j)(x)\le \|a_j\|_{L^\infty(\mu)}$ for any $x\in 2Q_j$. This fact together with Definition 3.2 further yields that

$$\mathrm{J}_1\le\sum_{j=1}^2|\lambda_j|\|a_j\|_{L^\infty(\mu)}\mu(2Q_j)\lesssim\sum_{j=1}^2|\lambda_j|.$$

On the other hand, for any $j=1$, 2, and any $x\in 8R\setminus 2Q_j$, we obtain that for any $y\in Q_j$, $|x-y|\sim|x-x_j|$, where x_j is the center of Q_j. From this, it follows that for any $x\in 8R\setminus 2Q_j$,

$$\mathcal{M}_\Phi(a_j)(x)\le\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d}\varphi(y)a_j(y)\,d\mu(y)\right|\lesssim\frac{\|a_j\|_{L^1(\mu)}}{|x-x_j|^n},$$

which together with Lemma 2.1 and Definition 3.2 yields that

$$\begin{aligned}\int_{8R\setminus 2Q_j}\mathcal{M}_\Phi(a_j)(x)\,d\mu(x)&\lesssim\|a_j\|_{L^1(\mu)}\delta(2Q_j,8R)\\&\lesssim\|a\|_{L^\infty(\mu)}\mu(Q_j)[1+\delta(Q_j,R)]\lesssim 1.\end{aligned}$$

Therefore $\mathrm{J}_2\lesssim\sum_{j=1}^2|\lambda_j|$, which together with estimates for J_1 and J_3 implies that $\mathrm{I}_1\lesssim\sum_{j=1}^2|\lambda_j|$.

Now we estimate I_2. By the facts that $\int_{\mathbb{R}^d}[b(x)-S_k(b)(x)]\,d\mu(x)=0$ and that $\operatorname{supp}(b)\subset R$, we write

$$\begin{aligned}\mathrm{I}_2\le&\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\int_R|b(y)||\varphi(y)-\varphi(x_0)|\,d\mu(y)\,d\mu(x)\\&+\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\int_{2R}|S_k(b)(y)||\varphi(y)-\varphi(x_0)|\,d\mu(y)\,d\mu(x)\\&+\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d\setminus 2R}S_k(b)(y)[\varphi(y)-\varphi(x_0)]\,d\mu(y)\right|\,d\mu(x)\\=:&\ \mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3.\end{aligned}$$

Observing that by arguing as in the proofs for II_1 and II_2 in Theorem 3.1 of [28], we obtain $\mathrm{L}_2+\mathrm{L}_3\lesssim\sum_{j=1}^2|\lambda_j|$. Thus, we only need to verify that $\mathrm{L}_1\lesssim\sum_{j=1}^2|\lambda_j|$.

For any $y\in R$ and $x\in 2^{m+1}R\setminus 2^mR$ with $m\ge 3$, it is obvious that $|x-x_0|\ge l(2^{m-2}R)$ and $|y-x_0|\le\sqrt{d}\,l(R)$, which implies that $|y-x_0|\lesssim$

$|x-x_0|$. This fact together with the mean value theorem yields that for any $\varphi\sim x$,

$$|\varphi(y)-\varphi(x_0)|\lesssim\frac{|y-x_0|}{|x-x_0|^{n+1}}.$$

From this, Definition 3.2 and (1.1), it follows that

$$\begin{aligned}\mathrm{L}_1&\le\sum_{j=1}^2|\lambda_j|\sum_{m=3}^\infty\int_{2^{m+1}R\setminus 2^mR}\sup_{\varphi\sim x}\int_{Q_j}|a_j(y)||\varphi(y)-\varphi(x_0)|\,d\mu(y)\,d\mu(x)\\&\lesssim\sum_{j=1}^2|\lambda_j|\sum_{m=3}^\infty\int_{2^{m+1}R\setminus 2^mR}\frac{l(R)}{\left[l(2^mR)\right]^{n+1}}\|a_j\|_{L^\infty(\mu)}\mu(Q_j)\,d\mu(x)\lesssim\sum_{j=1}^2|\lambda_j|.\end{aligned}$$

Therefore, we obtain (3.35) in Case (1).

In Case (2), we further consider the following two subcases. *Subcase* (i) $k\ge H_R^{x_0}+1$ and for all $y\in R\cap\operatorname{supp}(\mu)$, $R\not\subset Q_{y,\,k-1}$. In this subcase, it is easy to see that for any $y\in R$, $Q_{y,\,k-1}\subset 4R$, which together with $\operatorname{supp}(S_k(b))\subset\cup_{y\in R}Q_{y,\,k-1}$ implies that $\operatorname{supp}(S_k(b))\subset 4R$. Let I_1 and I_2 be as in Case (1). We also have $\|\mathcal{M}_\Phi(b-S_k(b))\|_{L^1(\mu)}\le\mathrm{I}_1+\mathrm{I}_2$ and $\mathrm{I}_1\lesssim\sum_{j=1}^2|\lambda_j|$. On the other hand, since $\operatorname{supp}(S_k(b))\subset 4R$, by

$$\int_{\mathbb{R}^d}[b(x)-S_k(b)(x)]\,d\mu(x)=0,$$

we have

$$\begin{aligned}\mathrm{I}_2&\le\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d}[b(y)-S_k(b)(y)][\varphi(y)-\varphi(x_0)]\,d\mu(y)\right|d\mu(x)\\&\le\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\int_R|b(y)||\varphi(y)-\varphi(x_0)|\,d\mu(y)\,d\mu(x)\\&\quad+\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\int_{4R}|S_k(b)(y)||\varphi(y)-\varphi(x_0)|\,d\mu(y)\,d\mu(x).\end{aligned}$$

Moreover, using estimates similar to those for L_1 and L_2 in Case (1) with $2R$ in L_2 replaced by $4R$, we obtain $\mathrm{I}_2\lesssim\sum_{j=1}^2|\lambda_j|$.

Subcase (ii) $k\ge H_R^{x_0}+1$ and there exists some $y_0\in R\cap\operatorname{supp}(\mu)$ such that $R\subset Q_{y_0,\,k-1}$. In this subcase, by applying Lemma 4.2 in [17], we see that $\operatorname{supp}(S_k(b))\subset\cup_{y\in R}Q_{y,\,k-1}\subset Q_{y_0,\,k-2}\subset Q_{x_0,\,k-3}$. Then

$$\begin{aligned}&\|\mathcal{M}_\Phi(b-S_k(b))\|_{L^1(\mu)}\\&\quad=\int_{4Q_{x_0,\,k-3}}\mathcal{M}_\Phi(b-S_k(b))(x)\,d\mu(x)+\int_{\mathbb{R}^d\setminus 4Q_{x_0,\,k-3}}\cdots=:\mathrm{E}_1+\mathrm{E}_2.\end{aligned}$$

Arguing as in estimates for L_1 and L_2 in Case (1) with $2R$ in L_2 replaced by $Q_{x_0,k-3}$, we have $\mathrm{E}_2 \lesssim \sum_{j=1}^2 |\lambda_j|$. On the other hand, by the fact that $\mathcal{M}_\Phi$ is sublinear, we obtain

$$\begin{aligned}\mathrm{E}_1 \le \sum_{j=1}^2 |\lambda_j| \int_{2Q_j} \mathcal{M}_\Phi(a_j)(x)\,d\mu(x) + \sum_{j=1}^2 |\lambda_j| \int_{4Q_{x_0,k-3}\setminus 2Q_j} \cdots \\ + \int_{4Q_{x_0,k-3}} \mathcal{M}_\Phi(S_k(b))(x)\,d\mu(x) =: \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3.\end{aligned}$$

By using an argument similar to that in the proof of J_1 in Case (1), we obtain $\mathrm{F}_1 \lesssim \sum_{j=1}^2 |\lambda_j|$. On the other hand, because $R \subset Q_{y_0,k-1}$, we obtain that $k \le H_R^{y_0} + 1$. This fact together with Lemma 2.2 (c) yields that $k \le H_R^{x_0} + 2$. Then the assumption that $H_R^{x_0} + 1 \le k$ together with Lemma 2.1 and Lemma 2.2 (g) implies $\delta(R, Q_{x_0,k-3}) \lesssim 1$. Moreover, another application of Lemma 2.1 implies that $\delta(2Q_j, 4Q_{x_0,k-3}) \lesssim 1 + \delta(Q_j, R)$. Therefore, arguing as in Case (1), we have that for any $x \in 4Q_{x_0,k-3} \setminus 2Q_j$,

$$\mathcal{M}_\Phi(a_j)(x) \lesssim \frac{\|a_j\|_{L^\infty(\mu)}\mu(Q_j)}{|x - x_j|^n}.$$

This together with Definition 3.2 implies that

$$\mathrm{F}_2 \lesssim \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^\infty(\mu)} \mu(Q_j) \delta(2Q_j, 4Q_{x_0,k-3}) \lesssim \sum_{j=1}^2 |\lambda_j|.$$

Similarly, by (A-2) and (A-4), we have

$$\begin{aligned}\mathrm{F}_3 &\le \sum_{j=1}^2 |\lambda_j| \left[\int_{2Q_j} \mathcal{M}_\Phi(S_k(a_j))(x)\,d\mu(x) + \int_{4Q_{x_0,k-3}\setminus 2Q_j} \cdots\right] \\ &\lesssim \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^\infty(\mu)} \mu(2Q_j) \left[1 + \delta(2Q_j, 4Q_{x_0,k-3})\right] \lesssim \sum_{j=1}^2 |\lambda_j|,\end{aligned}$$

which completes the proof of Proposition 3.7. ■

Remark 3.8. *In fact, from Theorem* 3.1 *in* [28], *we see that Proposition* 3.7 *and Corollary* 3.2 *below also hold for* S_k *with* $k \le 0$ *when* $\mathbb{R}^d$ *is not an initial cube.*

To establish the relation between RBMO(μ) and rbmo(μ), we need the following estimate, which is a simple corollary of Lemma 3.1 in [27] and the fact that rbmo$(\mu) \subset$ RBMO(μ) (see Proposition 3.1). We only point out that the proof of Lemma 3.1 in [27] still works, even when $\mathbb{R}^d$ is an initial cube.

Lemma 3.10. *There exists a positive constant C such that for any two cubes $Q \subset R$ and $f \in \mathrm{rbmo}\,(\mu)$,*

$$\int_R \frac{|f(y) - m_{\widetilde{Q}}(f)|}{[|y - x_Q| + l(Q)]^n}\, d\mu(y) \le C[1 + \delta(Q, R)]^2 \|f\|_{\mathrm{rbmo}\,(\mu)}.$$

Corollary 3.2. *Let $k \in \mathbb{N}$ and S_k be as in Section 2. Then*

(i) $\mathrm{rbmo}\,(\mu) = \{b \in \mathrm{RBMO}\,(\mu) : S_k(b) \in L^\infty(\mu)\}$*; moreover, for any* $b \in \mathrm{rbmo}\,(\mu)$*,* $k\|b\|_{\mathrm{rbmo}\,(\mu)} \sim \|S_k(b)\|_{L^\infty(\mu)} + \|b\|_{\mathrm{RBMO}\,(\mu)}$.

(ii) *If $f \in \mathrm{RBMO}\,(\mu)$, then $f - S_k(f) \in \mathrm{rbmo}\,(\mu)$; moreover, there exists a positive constant C independent of k and f such that*

$$\|f - S_k(f)\|_{\mathrm{rbmo}\,(\mu)} \le C\|f\|_{\mathrm{RBMO}\,(\mu)}.$$

Proof. To prove (i), assume that $b \in \mathrm{RBMO}\,(\mu)$ with $S_k(b) \in L^\infty(\mu)$ first. For any $f \in h^{1,\infty}_{\mathrm{atb}}(\mu)$, Proposition 3.7 together with (A-1) and Theorem 5.5 in [16] implies

$$\begin{aligned}\left|\int_{\mathbb{R}^d} b(x)f(x)\, d\mu(x)\right| &\le \left|\int_{\mathbb{R}^d} b(x)\,[f(x) - S_k(f)(x)]\ d\mu(x)\right| + \left|\int_{\mathbb{R}^d} b(x)S_k(f)(x)\, d\mu(x)\right| \\ &\lesssim \|f\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}[\|b\|_{\mathrm{RBMO}\,(\mu)} + \|S_k(b)\|_{L^\infty(\mu)}].\end{aligned}$$

Thus by Theorem 3.2, $b \in \mathrm{rbmo}\,(\mu)$ and

$$\|b\|_{\mathrm{rbmo}\,(\mu)} \lesssim \|b\|_{\mathrm{RBMO}\,(\mu)} + \|S_k(b)\|_{L^\infty(\mu)}.$$

Conversely, assume that $b \in \mathrm{rbmo}\,(\mu)$. If $k \ge 2$, then for any $x \in \mathrm{supp}\,(\mu)$, $Q_{x,k} \notin \mathcal{D}$ by (e) of Lemma 2.2. Therefore, from (A-2) through (A-4), the fact that $Q_{x,1} \in \mathcal{D}$, Definition 3.1 and Lemma 3.10, it follows that for any $x \in \mathrm{supp}\,(\mu)$,

$$\begin{aligned}|S_k(b)(x)| \le &\int_{Q_{x,\,k-1}} S_k(x,y)\left|b(y) - m_{Q_{x,\,k}}(b)\right|\, d\mu(y) \\ &+ \left|m_{Q_{x,\,k}}(b) - m_{Q_{x,\,1}}(b)\right| + \left|m_{Q_{x,\,1}}(b)\right| \lesssim k\|b\|_{\mathrm{rbmo}\,(\mu)}.\end{aligned}$$

Let $k = 1$. If $\mathbb{R}^d$ is an initial cube, we first claim that for any $x \in \mathrm{supp}\,(\mu)$,

$$\int_{\mathbb{R}^d \setminus Q_{x,\,1}} \frac{|b(y)|}{|x - y|^n}\, d\mu(y) \lesssim \|b\|_{\mathrm{rbmo}\,(\mu)}.$$

In fact, by the fact that $2^{j+1}Q_{x,1} \in \mathcal{D}$ for all $j \ge 0$ together with Definition 3.1 and the fact that $\delta(Q_{x,1}, \mathbb{R}^d) \lesssim 1$, for any $j_0 \in \mathbb{N}$,

$$\int_{2^{j_0}Q_{x,1}\setminus Q_{x,1}} \frac{|b(y)|}{|y-x|^n}\,d\mu(y) \lesssim \sum_{j=0}^{j_0-1} \frac{1}{[l(2^{j+2}Q_{x,1})]^n} \int_{2^{j+1}Q_{x,1}\setminus 2^jQ_{x,1}} |b(y)|\,d\mu(y)$$
$$\lesssim [1+\delta(Q_{x,1},\,\mathbb{R}^d)]\|b\|_{\mathrm{rbmo}\,(\mu)} \lesssim \|b\|_{\mathrm{rbmo}\,(\mu)}.$$

By letting $j_0 \to \infty$, we know that the above claim holds.

By this claim, (A-4) and Definition 3.1 together with $Q_{x,1} \in \mathcal{D}$, we see that for any $x \in \mathrm{supp}\,(\mu)$,

$$\begin{aligned}(3.36)\qquad |S_1(b)(x)| &\lesssim \int_{\mathbb{R}^d} \frac{|b(z)|}{[|x-z|+l(Q_{x,1})]^n}\,d\mu(z)\\ &\le \int_{\mathbb{R}^d\setminus Q_{x,1}} \frac{|b(z)|}{|x-z|^n}\,d\mu(z) + \int_{Q_{x,1}} \frac{|b(z)|}{[l(Q_{x,1})]^n}\,d\mu(z)\\ &\lesssim \|b\|_{\mathrm{rbmo}\,(\mu)}.\end{aligned}$$

If $\mathbb{R}^d$ is not an initial cube, then by (A-3), (A-4) and Definition 3.1 together with $Q_{x,0},\, Q_{x,1} \in \mathcal{D}$, for any $x \in \mathrm{supp}\,(\mu)$,

$$\begin{aligned}|S_1(b)(x)| &\lesssim \int_{Q_{x,0}} \frac{|b(z)|}{[|x-z|+l(Q_{x,1})]^n}\,d\mu(z)\\ &\le \int_{Q_{x,0}\setminus Q_{x,1}} \frac{|b(z)|}{|x-z|^n}\,d\mu(z) + \int_{Q_{x,1}} \frac{|b(z)|}{[l(Q_{x,1})]^n}\,d\mu(z) \lesssim \|b\|_{\mathrm{rbmo}\,(\mu)}.\end{aligned}$$

Combining these estimates above, we have that for each $k \in \mathbb{N}$, $S_k(b) \in L^\infty(\mu)$ and $\|S_k(b)\|_{L^\infty(\mu)} \lesssim k\|b\|_{\mathrm{rbmo}\,(\mu)}$, which implies that

$$\|S_k(b)\|_{L^\infty(\mu)} + \|b\|_{\mathrm{RBMO}\,(\mu)} \lesssim k\|b\|_{\mathrm{rbmo}\,(\mu)}.$$

This establishes (i).

For any $b \in h^{1,\infty}_{\mathrm{atb}}(\mu)$, it follows from Proposition 3.7 that

$$\|b - S_k(b)\|_{H^1(\mu)} \lesssim \|b\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)}.$$

By this fact and the duality between $H^1(\mu)$ and $\mathrm{RBMO}\,(\mu)$, for any $f \in \mathrm{RBMO}\,(\mu)$,

$$\left|\int_{\mathbb{R}^d} [f(x) - S_k(f)(x)]b(x)\,d\mu(x)\right| = \left|\int_{\mathbb{R}^d} f(x)[b(x) - S_k(b)(x)]\,d\mu(x)\right|$$
$$\lesssim \|f\|_{\mathrm{RBMO}\,(\mu)}\|b\|_{h^{1,\infty}_{\mathrm{atb}}(\mu)},$$

which via Theorem 3.2 implies that $f - S_k(f) \in \mathrm{rbmo}\,(\mu)$. This establishes (ii), and hence completes the proof of Corollary 3.2. ■

3.2. The spaces $\mathrm{RBLO}\,(\mu)$ and $\mathrm{rblo}\,(\mu)$

To begin with, we prove that the definition of the space $\mathrm{RBLO}\,(\mu)$ is independent of the chosen constants $\eta \in (1, \infty)$ and $\rho \in (1, \infty)$.

Let $\eta \in (1, \infty)$. Suppose that for a given $f \in L^1_{\mathrm{loc}}(\mu)$, there exists a nonnegative constant $\widetilde{C}$ and a collection of numbers $\{f_Q\}_Q$ such that

$$\sup_Q \frac{1}{\mu(\eta Q)} \int_Q [f(y) - f_Q]\, d\mu(y) \le \widetilde{C}, \tag{3.37}$$

that for any two cubes $Q \subset R$,

$$|f_Q - f_R| \le \widetilde{C}[1 + \delta(Q, R)], \tag{3.38}$$

and that for any cube Q,

$$f_Q \le \operatorname*{essinf}_Q f. \tag{3.39}$$

We then define the *norm* $\|f\|_{**,\,\eta} := \inf\{\widetilde{C}\}$, where the infimum is taken over all the constants $\widetilde{C}$ as above and all the numbers $\{f_Q\}_Q$ satisfying (3.37) through (3.39).

With a minor modification of the proof for Lemma 2.6 in [16], we have the following proposition and we leave the details to the reader.

Proposition 3.8. *The norm $\|\cdot\|_{**,\,\eta}$ is independent of the choice of the constant $\eta \in (1, \infty)$.*

Based on Proposition 3.8, from now on, we write $\|\cdot\|_{**}$ *instead of* $\|\cdot\|_{**,\,\eta}$.

Proposition 3.9. *Let $\eta \in (1, \infty)$, $\rho \in (1, \infty)$ and $\beta_\rho := \rho^{d+1}$. Then the norms $\|\cdot\|_{**}$ and $\|\cdot\|_{\mathrm{RBLO}\,(\mu)}$ are equivalent.*

Proof. Suppose that $f \in L^1_{\mathrm{loc}}(\mu)$. We first show that

$$\|f\|_{**} \lesssim \|f\|_{\mathrm{RBLO}\,(\mu)}. \tag{3.40}$$

For any cube Q, let $f_Q := \operatorname{essinf}_{\widetilde{Q}^\rho} f$. Then (3.37) and (3.39) hold with $\widetilde{C} := \|f\|_{\mathrm{RBLO}\,(\mu)}$. To verify that (3.38) also holds, let $R_0 := \widetilde{2\widetilde{R}^\rho}$ if $l(\widetilde{R}^\rho) \ge l(\widetilde{Q}^\rho)$ and $R_0 := \widetilde{2\rho\widetilde{Q}^\rho}$ if $l(\widetilde{R}^\rho) < l(\widetilde{Q}^\rho)$. Then arguing as in the proof of (3.7), we obtain that for any two cubes $Q \subset R$,

$$\left| \operatorname*{essinf}_{\widetilde{Q}^\rho} f - \operatorname*{essinf}_{\widetilde{R}^\rho} f \right| \lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{RBLO}\,(\mu)}. \tag{3.41}$$

Now let us establish the converse of (3.40). For $f \in L^1_{\rm loc}(\mu)$, assume that there exists a sequence of numbers $\{f_Q\}_Q$ satisfying (3.37) through (3.39) with $\widetilde{C}$ replaced by $\|f\|_{**}$. For any cube Q, by (3.38), (3.39) and Lemma 2.1,

$$f_Q - \operatorname*{essinf}_{\widetilde{Q}^\rho} f = f_Q - f_{\widetilde{Q}^\rho} + f_{\widetilde{Q}^\rho} - \operatorname*{essinf}_{\widetilde{Q}^\rho} f \le [1+\delta(Q, \widetilde{Q}^\rho)]\|f\|_{**} \lesssim \|f\|_{**}.$$

This fact together with (3.37) yields that for any cube Q,

$$\begin{aligned}&\frac{1}{\mu(\eta Q)}\int_Q \left[f(y) - \operatorname*{essinf}_{\widetilde{Q}^\rho} f\right] d\mu(y)\\ &\quad= \frac{1}{\mu(\eta Q)}\int_Q [f(y)-f_Q]\, d\mu(y) + \frac{\mu(Q)}{\mu(\eta Q)}\left[f_Q - \operatorname*{essinf}_{\widetilde{Q}^\rho} f\right] \lesssim \|f\|_{**}.\end{aligned}$$

On the other hand, for any (ρ, β_ρ)-doubling cube Q, since (3.37) holds with $\eta = \rho$ by Proposition 3.8, (3.39) implies that

$$m_Q(f) - f_Q = \frac{1}{\mu(Q)}\int_Q [f(x)-f_Q]\, d\mu(x) \le \frac{\mu(\rho Q)}{\mu(Q}\|f\|_{**} \lesssim \|f\|_{**}.$$

Then from (3.38) and (3.39), it follows that for any two (ρ, β_ρ)-doubling cubes $Q \subset R$,

$$\begin{aligned}\operatorname*{essinf}_{Q} f - \operatorname*{essinf}_{R} f &\le \operatorname*{essinf}_{Q} f - f_Q + f_Q - f_R\\ &\le m_Q(f) - f_Q + [1+\delta(Q,R)]\|f\|_{**} \lesssim [1+\delta(Q,R)]\|f\|_{**}.\end{aligned}$$

This establishes the converse of (3.40), and hence completes the proof of Proposition 3.9. ■

Proposition 3.10. *Let $\eta \in (1,\infty)$, $\rho \in (1,\infty)$ and $\beta_\rho := \rho^{d+1}$. For $f \in L^1_{\rm loc}(\mu)$, the following statements are equivalent:*

(i) $f \in \mathrm{RBLO}\,(\mu)$.

(ii) *There exists a nonnegative constant C_3 satisfying* (2.5) *and that for any (ρ, β_ρ)-doubling cube Q,*

$$\frac{1}{\mu(Q)}\int_Q \left[f(x) - \operatorname*{essinf}_{Q} f\right] d\mu(x) \le C_3. \tag{3.42}$$

(iii) *There exists a nonnegative constant C_4 satisfying* (3.42) *and that for any (ρ, β_ρ)-doubling cubes $Q \subset R$,*

$$m_Q(f) - m_R(f) \le C_4[1+\delta(Q,R)]. \tag{3.43}$$

Moreover, the minimal constants C_3 and C_4 are equivalent to $\|f\|_{\mathrm{RBLO}\,(\mu)}$.

Proof. By Proposition 3.8 and Proposition 3.9, it suffices to establish Proposition 3.10 with $\eta = \rho = 2$. Notice that the fact (i) implies (ii) automatically. We now prove that (ii) implies (iii). From (2.5) together with (3.42), it follows that for any doubling cubes $Q \subset R$,

$$m_Q(f) - m_R(f) \le m_Q(f) - \operatorname*{essinf}_Q f + \operatorname*{essinf}_Q f - \operatorname*{essinf}_R f \lesssim C_3[1 + \delta(Q, R)],$$

which implies (iii).

Finally, assume that (iii) holds. For any cube $Q \subset \mathbb{R}^d$ and any $x \in Q \cap \operatorname{supp}(\mu)$, let Q_x be the biggest doubling cube centered at x with side length $2^k l(Q)$, $k \le 0$, and $l(Q_x) \le \frac{1}{20} l(Q)$. Then Lemma 2.1 yields that $\delta(Q_x, Q) \lesssim 1$. By the Besicovitch covering theorem, there exists a subsequence of cubes $\{Q_{x_i}\}_i$ which covers $Q \cap \operatorname{supp}(\mu)$ and has a bounded overlap. Moreover, from (3.42), (3.43) and the fact that $Q_{x_i} \subset 2Q$, it follows that

$$\begin{aligned}\operatorname*{essinf}_{Q_{x_i}} f - \operatorname*{essinf}_{\widetilde{2Q}} f &\le m_{Q_{x_i}}(f) - m_{\widetilde{2Q}}(f) + m_{\widetilde{2Q}}(f) - \operatorname*{essinf}_{\widetilde{2Q}} f \\ &\lesssim C_4[1 + \delta(Q_{x_i}, \widetilde{\widetilde{2Q}})] \lesssim C_4.\end{aligned}$$

This fact together with the facts that $\{Q_{x_i}\}_i$ covers $Q \cap \operatorname{supp}(\mu)$ with a bounded overlap, that $Q_{x_i} \subset 2Q$, that Q_{x_i} is doubling and (3.42) implies that

$$\begin{aligned}&\int_Q \left[f(x) - \operatorname*{essinf}_{\widetilde{Q}} f\right] d\mu(x) \\ &\le \sum_i \int_{Q_{x_i}} \left[f(x) - \operatorname*{essinf}_{Q_{x_i}} f\right] d\mu(x) + \sum_i \mu(Q_{x_i}) \left[\operatorname*{essinf}_{Q_{x_i}} f - \operatorname*{essinf}_{\widetilde{2Q}} f\right] \\ &\lesssim C_4 \sum_i \mu(Q_{x_i}) \lesssim C_4 \mu(2Q).\end{aligned}$$

On the other hand, from (3.43) and (3.42), it follows that for any doubling cubes $Q \subset R$,

$$\operatorname*{essinf}_Q f - \operatorname*{essinf}_R f \le m_Q(f) - m_R(f) + m_R(f) - \operatorname*{essinf}_R f \lesssim C_4[1 + \delta(Q, R)].$$

Therefore, we see $f \in \mathrm{RBLO}\,(\mu)$, which implies (i). This completes the proof of Proposition 3.10. ■

Remark 3.9. *(i) Let $\eta \in (1, \infty)$ and $\rho \in (1, \infty)$. From Proposition* 3.8 *and Proposition* 3.9*, it follows that the definition of the space* $\mathrm{RBLO}\,(\mu)$ *is independent of the choices of η and ρ. From now on, we will always assume $\eta = \rho = 2$ when we consider* $\mathrm{RBLO}\,(\mu)$*.*

(ii) From Lemma 2.8 *in* [16]*, Proposition* 3.8 *and Proposition* 3.9*, it is easy to see that* $\mathrm{RBLO}\,(\mu) \subset \mathrm{RBMO}\,(\mu)$*.*

We next recall the notion of the *natural maximal operator*, which is a variant in the non-doubling context of the so-called natural maximal operator on $\mathbb{R}^d$ in [1, 15] and was introduced by Jiang in [7]. For any locally integrable function f and $x \in \mathbb{R}^d$, define

$$\mathcal{M}(f)(x) := \sup_{\substack{Q \ni x \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y).$$

Recall that the non centered doubling maximal operator $N(f) := \mathcal{M}(|f|)$ is defined in [16, p. 126].

The following theorem is an improvement of Theorem 2 of [7] by proving that (3.45) below holds automatically under the assumption that $f \in \mathrm{RBMO}\,(\mu)$ and $\mathcal{M}(f)$ is finite almost everywhere.

Theorem 3.3. *Let $f \in \mathrm{RBMO}\,(\mu)$. Then $\mathcal{M}(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case, there exists a positive constant C independent of f such that*

$$\|\mathcal{M}(f)\|_{\mathrm{RBLO}\,(\mu)} \le C\|f\|_{\mathrm{RBMO}\,(\mu)}.$$

Proof. Suppose that $f \in \mathrm{RBMO}\,(\mu)$ and there exists a point $x_0 \in \mathbb{R}^d$ such that $\mathcal{M}(f)(x_0) < \infty$. It then follows from Lemma 2 in [7] that there exists a positive constant C independent of f such that for any doubling cube $Q \ni x_0$,

$$\frac{1}{\mu(Q)} \int_Q \mathcal{M}(f)(x)\, d\mu(x) \le C\|f\|_{\mathrm{RBMO}\,(\mu)} + \operatorname*{essinf}_Q \mathcal{M}(f). \tag{3.44}$$

By (3.44) and Proposition 3.10, Theorem 3.3 is reduced to proving that for any doubling cubes $Q \subset R$,

$$m_Q[\mathcal{M}(f)] - m_R[\mathcal{M}(f)] \lesssim [1 + \delta(Q,R)]\|f\|_{\mathrm{RBMO}\,(\mu)}. \tag{3.45}$$

To prove (3.45), for any point $x \in Q$, we further set

$$\mathcal{M}_1(f)(x) := \sup_{\substack{P \ni x,\, P \text{ doubling} \\ l(P) \le 4l(R)}} \frac{1}{\mu(P)} \int_P f(y)\, d\mu(y),$$

$$\mathcal{M}_2(f)(x) := \sup_{\substack{P \ni x,\, P \text{ doubling} \\ l(P) > 4l(R)}} \frac{1}{\mu(P)} \int_P f(y)\, d\mu(y),$$

$$\mathcal{U}_{1,Q} := \{x \in Q : \mathcal{M}_1(f)(x) \ge \mathcal{M}_2(f)(x)\} \quad \text{and} \quad \mathcal{U}_{2,Q} := Q \setminus \mathcal{U}_{1,Q}.$$

Then for any $x \in Q$, $\mathcal{M}(f)(x) = \max(\mathcal{M}_1(f)(x), \mathcal{M}_2(f)(x))$. By writing

$$f = [f - m_R(f)]\chi_{\frac{4}{3}Q} + [f - m_R(f)]\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q} + m_R(f)$$

and using the fact that $m_R(f) \le m_R[\mathcal{M}(f)]$, we see that

$$\begin{aligned} m_Q[\mathcal{M}(f)] - m_R[\mathcal{M}(f)] \le\ & \frac{1}{\mu(Q)} \int_{\mathcal{U}_{1,\,Q}} \mathcal{M}_1([f - m_R(f)]\chi_{\frac{4}{3}Q})(x)\, d\mu(x) \\ & + \frac{1}{\mu(Q)} \int_{\mathcal{U}_{1,\,Q}} \mathcal{M}_1([f - m_R(f)]\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q})(x)\, d\mu(x) \\ & + \frac{1}{\mu(Q)} \int_{\mathcal{U}_{2,\,Q}} \{\mathcal{M}_2(f)(x) - m_R[\mathcal{M}(f)]\}\, d\mu(x) \\ =\ & :\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned}$$

Notice that $\mathcal{M}(|f|) = N(f)$, which is bounded on $L^2(\mu)$ (see [16, p. 126]). From this, the Hölder inequality, Corollary 3.5 in [16], the doubling property of Q and Lemma 2.1, it follows that

$$\begin{aligned} \mathrm{I}_1 &\le \left\{ \frac{1}{\mu(Q)} \int_Q \left\{ N\left([f - m_R(f)]\chi_{\frac{4}{3}Q} \right)(x) \right\}^2 d\mu(x) \right\}^{1/2} \\ &\lesssim \left\{ \frac{1}{\mu(Q)} \int_{\frac{4}{3}Q} |f(x) - m_R(f)|^2\, d\mu(x) \right\}^{1/2} \\ &\lesssim \left\{ \frac{1}{\mu(Q)} \int_{\frac{4}{3}Q} \left| f(x) - m_{\widetilde{\frac{4}{3}Q}}(f) \right|^2 d\mu(x) \right\}^{1/2} + \left| m_{\widetilde{\frac{4}{3}Q}}(f) - m_Q(f) \right| \\ &\quad + |m_Q(f) - m_R(f)| \\ &\lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{RBMO}\,(\mu)}. \end{aligned}$$

To estimate I_2, we will prove that for any point $x \in Q$ and any doubling cube $P \ni x$ with $l(P) \le 4l(R)$,

$$\begin{aligned} \mathrm{J} :&= \frac{1}{\mu(P)} \int_P |f(y) - m_R(f)| \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}(y)\, d\mu(y) \\ &\lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{RBMO}\,(\mu)}. \end{aligned} \tag{3.46}$$

If $P \subset \frac{4}{3}Q$, then $\mathrm{J} \equiv 0$ and (3.46) holds automatically. Assume that $P \not\subset \frac{4}{3}Q$. We then have that $l(P) \ge \frac{1}{6} l(Q)$, which together with the fact that $l(P) \le 4l(R)$ implies that $Q \subset 13P \subset 57R$. Thus, Lemma 2.1 together with (2.10) in [16] yields that

$$\begin{aligned} \mathrm{J} \le\ & \frac{1}{\mu(P)} \int_P |f(y) - m_P(f)|\, d\mu(y) + \left| m_P(f) - m_{\widetilde{13P}}(f) \right| \\ & + \left| m_{\widetilde{13P}}(f) - m_Q(f) \right| + |m_Q(f) - m_R(f)| \lesssim [1 + \delta(Q, R)] \|f\|_{\mathrm{RBMO}\,(\mu)}. \end{aligned}$$

Now we estimate I_3. Notice that for any $x \in Q$ and any doubling cube P containing x with $l(P) > 4l(R)$, $R \subset \frac{3}{2}P$. Then from the fact that $m_{\widetilde{\frac{3}{2}P}}(f) \le m_R[\mathcal{M}(f)]$, it follows that

$$m_P(f) - m_R[\mathcal{M}(f)] \le \left| m_P(f) - m_{\widetilde{\frac{3}{2}P}}(f) \right| + m_{\widetilde{\frac{3}{2}P}}(f) - m_R[\mathcal{M}(f)] \\ \lesssim \|f\|_{\mathrm{RBMO}\,(\mu)}.$$

Taking the supremum over all doubling cubes P containing x with $l(P) > 4l(R)$, we have that for any $x \in Q$,

$$\mathcal{M}_2(f)(x) - m_R[\mathcal{M}(f)] \lesssim \|f\|_{\mathrm{RBMO}\,(\mu)}.$$

This implies that $\mathrm{I}_3 \lesssim \|f\|_{\mathrm{RBMO}\,(\mu)}$.

Combining the estimates for I_1 through I_3 leads to (3.45), which together with (3.44) implies that $\mathcal{M}$ is bounded from RBMO (μ) to RBLO (μ) and hence completes the proof of Theorem 3.3. ■

Remark 3.10. *(i) If μ is the d-dimensional Lebesgue measure, Theorem* 3.3 *was obtained by Bennett in* [1].

(ii) From Theorem 3.3 *and the facts that for any $f \in$* RBMO (μ), *$|f| \in$* RBMO (μ) *with $\||f|\|_{\mathrm{RBMO}\,(\mu)} \lesssim \|f\|_{\mathrm{RBMO}\,(\mu)}$ (see* [16, Proposition 2.5]) *and that $N(f) = \mathcal{M}(|f|)$, it follows that if $f \in$* RBMO (μ), *then $N(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case, $N(f) \in$* RBLO (μ) *and there exists a positive constant C independent of f such that*

$$\|N(f)\|_{\mathrm{RBLO}\,(\mu)} \le C\|f\|_{\mathrm{RBMO}\,(\mu)}.$$

If μ is the d-dimensional Lebesgue measure and RBLO (μ) *is replaced by* RBMO (μ), *this conclusion was obtained by Bennett, DeVore and Sharpley in* [2].

It was proved in [7] that $f \in$ RBLO (μ) if and only if $\mathcal{M}(f) - f \in L^\infty(\mu)$ and f satisfies (3.43), and moreover, $\|f\|_{\mathrm{RBLO}\,(\mu)} \sim \|\mathcal{M}(f) - f\|_{L^\infty(\mu)}$. As a corollary of this fact and Theorem 3.3, we can improve Theorem 3 of [7] by removing the regularity assumption on $\mathcal{M}(f)$ as in (3.45). We omit the details here.

Theorem 3.4. *A locally integrable function f belongs to* RBLO (μ) *if and only if there exist $h \in L^\infty(\mu)$ and $g \in$* RBMO (μ) *with $\mathcal{M}(g)$ finite μ-a. e. such that*

$$f = \mathcal{M}(g) + h. \tag{3.47}$$

Furthermore, $\|f\|_{\mathrm{RBLO}\,(\mu)} \sim \inf(\|g\|_{\mathrm{RBMO}\,(\mu)} + \|h\|_{L^\infty(\mu)})$, where the infimum is taken over all representations of f as in (3.47).

Now we introduce the definition of the space rblo$\,(\mu)$.

Definition 3.4. *Let* $\eta \in (1, \infty)$, $\rho \in [\eta, \infty)$ *and* $\beta_\rho := \rho^{d+1}$. *A function* $f \in L^1_{\rm loc}(\mu)$ *is said to belong to the space* rblo$\,(\mu)$ *if there exists a nonnegative constant* $\widetilde{C}$ *such that for any cube* $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q \Big[f(x) - \operatorname*{essinf}_{\widetilde{Q}^\rho} f \Big] \, d\mu(x) \le \widetilde{C},$$

that for any two (ρ, β_ρ)*-doubling cubes* $Q \subset R$ *with* $Q \notin \mathcal{D}$,

$$\operatorname*{essinf}_Q f - \operatorname*{essinf}_R f \le \widetilde{C}[1 + \delta(Q,\, R)], \tag{3.48}$$

that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q |f(y)| \, d\mu(y) \le \widetilde{C}, \tag{3.49}$$

and that for any cube $Q \in \mathcal{D}$,

$$\Big| \operatorname*{essinf}_{\widetilde{Q}^\rho} f \Big| \le \widetilde{C}. \tag{3.50}$$

Moreover, we define the rblo$\,(\mu)$ *norm of* f *by the minimal constant* $\widetilde{C}$ *as above and denote it by* $\|f\|_{{\rm rblo}\,(\mu)}$.

We now prove that the definition of the space rblo$\,(\mu)$ is independent of the chosen constants η and ρ. To this end, let $\eta \in (1, \infty)$. Suppose that for a given $f \in L^1_{\rm loc}(\mu)$, there exists a nonnegative constant $\widetilde{C}$ and a collection of numbers $\{f_Q\}_Q$ such that

$$\sup_{Q \notin \mathcal{D}} \frac{1}{\mu(\eta Q)} \int_Q [f(y) - f_Q] \, d\mu(y) \le \widetilde{C}, \tag{3.51}$$

that for any two cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|f_Q - f_R| \le \widetilde{C}[1 + \delta(Q, R)], \tag{3.52}$$

that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q |f(y)| \, d\mu(y) + |f_Q| \le \widetilde{C}, \tag{3.53}$$

and that for any cube Q,

$$f_Q \le \operatorname*{essinf}_Q f. \tag{3.54}$$

We then define the *norm* $\|f\|_{\star,\eta} := \inf\{\widetilde{C}\}$, where the infimum is taken over all the constants $\widetilde{C}$ as above and all the numbers $\{f_Q\}_Q$ satisfying (3.51) through (3.54).

Similar to Proposition 3.8, with a slight modification of the proof for Proposition 3.2, we have the following property on the norm $\|\cdot\|_{\star,\eta}$ and we leave the details to the reader.

Proposition 3.11. *The norm $\|\cdot\|_{\star,\eta}$ is independent of the choice of the constant $\eta \in (1,\infty)$.*

Based on Proposition 3.11, from now on, we write $\|\cdot\|_\star$ *instead of* $\|\cdot\|_{\star,\eta}$. The proofs of the following two propositions are slight modifications of the proofs for Proposition 3.3 and Proposition 3.4. We leave the details to the reader.

Proposition 3.12. *Let η, ρ and β_ρ be as in Definition 3.4. Then the norms $\|\cdot\|_\star$ and $\|\cdot\|_{\mathrm{rblo}\,(\mu)}$ are equivalent.*

Proposition 3.13. *Let $\eta \in (1,\infty)$, $\rho \in [\eta,\infty)$ and $\beta_\rho := \rho^{d+1}$. For $f \in L^1_{\mathrm{loc}}(\mu)$, the following statements are equivalent:*

(i) $f \in \mathrm{rblo}\,(\mu)$.

(ii) *There exists a nonnegative constant C_5 satisfying (3.48) through (3.50) and that for any (ρ,β_ρ)-doubling cube $Q \notin \mathcal{D}$,*

$$\frac{1}{\mu(Q)}\int_Q \left[f(x) - \operatorname*{essinf}_Q f\right] d\mu(x) \le C_5. \tag{3.55}$$

(iii) *There exists a nonnegative constant C_6 satisfying (3.49), (3.50), (3.55) and that for any (ρ,β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,*

$$m_Q(f) - m_R(f) \le C_6[1 + \delta(Q,R)].$$

Moreover, the minimal constants C_5 and C_6 as above are equivalent to $\|f\|_{\mathrm{rblo}\,(\mu)}$.

Remark 3.11. *Let $\eta \in (1,\infty)$ and $\rho \in [\eta,\infty)$. From Proposition* 3.11 *and Proposition* 3.12*, it follows that the definition of the space* $\mathrm{rblo}\,(\mu)$ *is independent of the choices of η and ρ. From now on, we will always assume $\eta = \rho = 2$ when we consider* $\mathrm{rblo}\,(\mu)$.

From Definition 2.9 and Definition 3.4 together with Proposition 3.2 and Proposition 3.11, it is easy to see that $\mathrm{rblo}\,(\mu) \subset \{\mathrm{RBLO}\,(\mu) \cap \mathrm{rbmo}\,(\mu)\}$. Therefore, as a consequence of Corollary 3.2, we have the following result.

Corollary 3.3. *Let $k \in \mathbb{N}$ and S_k be as in Section 2. Then*

$$\mathrm{rblo}\,(\mu) \subset \{b \in \mathrm{RBLO}\,(\mu) : \ S_k(b) \in L^\infty(\mu)\}.$$

We now establish the relation between the space RBLO$\,(\mu)$ and the space rblo$\,(\mu)$ and some characterizations of the space rblo$\,(\mu)$ by certain maximal function.

Proposition 3.14. *Let $k \in \mathbb{N}$ and S_k be as in Section 2. If $f \in \mathrm{RBLO}\,(\mu)$, then $f - S_k f \in \mathrm{rblo}\,(\mu)$ and*

$$\|f - S_k(f)\|_{\mathrm{rblo}\,(\mu)} \le C\|f\|_{\mathrm{RBLO}\,(\mu)},$$

where C is a positive constant independent of k and f.

Proof. Without loss of generality, we may assume that $\|f\|_{\mathrm{RBLO}\,(\mu)} = 1$. We first show that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(2Q)} \int_Q |f(x) - S_k(f)(x)|\, d\mu(x) \lesssim 1. \tag{3.56}$$

To do so, let us consider the following two cases:

Case (i) There exists some $x_0 \in Q \cap \mathrm{supp}\,(\mu)$ such that $Q \subset Q_{x_0,\,k-2}$. In this case, we have $H_Q^{x_0} \ge k-2$. On the other hand, by the fact that $Q \in \mathcal{D}$ and Lemma 2.2 (d), we see that $H_Q^{x_0} \le 1$, which in turn implies that $1 \le k \le 3$. Moreover, from the facts that $-2 \le H_Q^{x_0} - k \le 0$ and that $Q \subset Q_{x_0,\,k-2}$, Lemma 2.2 (c) and Lemma 4.2 in [17], it follows that for any $x \in \mathrm{supp}\,(\mu) \cap Q$, $-1 \le H_Q^x - k + 2 \le 3$ and $Q \subset Q_{x,\,k-3}$. By this fact, (3.41), Lemma 2.1 and Lemma 2.2 (g), we have that for any $x \in \mathrm{supp}\,(\mu) \cap Q$,

$$\left|\operatorname*{essinf}_{\widetilde{Q}} f - \operatorname*{essinf}_{Q_{x,\,k}} f\right| \le \left|\operatorname*{essinf}_{\widetilde{Q}} f - \operatorname*{essinf}_{Q_{x,\,k-3}} f\right| + \left|\operatorname*{essinf}_{Q_{x,\,k-3}} f - \operatorname*{essinf}_{Q_{x,\,k}} f\right| \lesssim 1. \tag{3.57}$$

For each $x \in Q \cap \mathrm{supp}\,(\mu)$, write

$$\begin{aligned} &|f(x) - S_k(f)(x)| \\ &\quad \le f(x) - \operatorname*{essinf}_{\widetilde{Q}} f + \left|\operatorname*{essinf}_{\widetilde{Q}} f - \operatorname*{essinf}_{Q_{x,\,k}} f\right| + \left|\operatorname*{essinf}_{Q_{x,\,k}} f - S_k(f)(x)\right|. \end{aligned}$$

Notice that an easy argument involving (A-2) through (A-4) in Section 2 yields that for any $x \in \mathrm{supp}\,(\mu)$,

$$\left|\operatorname*{essinf}_{Q_{x,\,k}} f - S_k(f)(x)\right| \lesssim 1. \tag{3.58}$$

Then (3.56) follows from the combination of (3.57), (3.58) and the following trivial fact that

$$\frac{1}{\mu(2Q)}\int_Q \left[f(x)-\operatorname*{essinf}_{\widetilde{Q}} f\right]\,d\mu(x)\le 1.$$

Case (ii) For any $x\in Q\cap \operatorname{supp}(\mu)$, $Q\not\subset Q_{x,\,k-2}$. In this case, notice that by Lemma 2.2 (b), for any $x\in Q$, $Q_{x,\,k-1}\subset \frac{7}{5}Q$. Then from (A-1), (A-2), the Tonelli theorem and Proposition 3.8, it follows that

$$\begin{aligned}
&\frac{1}{\mu(2Q)}\int_Q |f(x)-S_k(f)(x)|\,d\mu(x)\\
&\le \frac{1}{\mu(2Q)}\int_Q \left|f(x)-\operatorname*{essinf}_{\widetilde{\frac{7}{5}Q}} f\right|d\mu(x)+\frac{1}{\mu(2Q)}\int_Q\left|\operatorname*{essinf}_{\widetilde{\frac{7}{5}Q}} f-S_k(f)(x)\right|d\mu(x)\\
&\le \frac{2}{\mu(2Q)}\int_{\frac{7}{5}Q}\left[f(y)-\operatorname*{essinf}_{\widetilde{\frac{7}{5}Q}} f\right]d\mu(y)\lesssim 1.
\end{aligned}$$

Now we prove that for any doubling cube Q,

$$m_Q\left(f-S_k(f)\right)-\operatorname*{essinf}_Q\,[f-S_k(f)]\lesssim 1. \tag{3.59}$$

From Proposition 3.10 and (3.58), it follows that

$$\begin{aligned}
&m_Q\left(f-S_k(f)\right)-\operatorname*{essinf}_Q\,[f-S_k(f)]\\
&\quad\le \frac{1}{\mu(Q)}\int_Q\left\{[f(x)-S_k(f)(x)]-\operatorname*{essinf}_Q f-\operatorname*{essinf}_Q[-S_k(f)]\right\}d\mu(x)\\
&\quad\lesssim 1+\frac{1}{\mu(Q)}\int_Q\left\{\left[-S_k(f)(x)+\operatorname*{essinf}_{Q_{x,\,k}} f\right]\right.\\
&\qquad\left.+\left[-\operatorname*{essinf}_{Q_{x,\,k}} f-\operatorname*{essinf}_Q[-S_k(f)]\right]\right\}d\mu(x)\lesssim 1.
\end{aligned}$$

Thus (3.59) holds.

By (3.56) and (3.59), for any cube $Q\in\mathcal{D}$,

$$\begin{aligned}
&\left|\operatorname*{essinf}_{\widetilde{Q}}[f-S_kf]\right|\\
&\quad\le\left|\operatorname*{essinf}_{\widetilde{Q}}[f-S_kf]-m_{\widetilde{Q}}\left(f-S_k(f)\right)\right|+\left|m_{\widetilde{Q}}\left(f-S_k(f)\right)\right|\lesssim 1.
\end{aligned}$$

From this together with (3.56), (3.59) and Proposition 3.13, to complete the proof of Proposition 3.14, it remains to prove that for any two doubling cubes $Q\subset R$ with $Q\notin\mathcal{D}$,

$$m_Q(f-S_k(f))-m_R(f-S_k(f))\lesssim 1+\delta(Q,R).$$

By Proposition 3.10, we first write

$$\begin{aligned}&m_Q(f-S_k(f))-m_R(f-S_k(f))\\&\quad=m_Q(f)-m_R(f)-m_Q(S_k(f))+m_R(S_k(f))\\&\quad\lesssim[1+\delta(Q,R)]-m_Q(S_k(f))+m_R(S_k(f)).\end{aligned}$$

As in the proof of (3.56), we consider the following three cases.

Case (1) There exists some $x_0\in R\cap\operatorname{supp}(\mu)$ such that $R\subset Q_{x_0,k-2}$. In this case, Lemma 4.2 in [17] and Lemma 2.1 yield that for any $x\in Q\cap\operatorname{supp}(\mu)$ and $y\in R\cap\operatorname{supp}(\mu)$, $R\subset Q_{x_0,k-2}\subset Q_{x,k-3}\subset Q_{y,k-4}\subset Q_{x,k-5}$ with $\delta(Q_{y,k-4},Q_{x,k-5})\lesssim 1$. This implies that

$$\begin{aligned}&\left|\operatorname*{essinf}_{Q_{x,k}}f-\operatorname*{essinf}_{Q_{y,k}}f\right|\\&\quad\le\left|\operatorname*{essinf}_{Q_{x,k}}f-\operatorname*{essinf}_{Q_{x,k-5}}f\right|+\left|\operatorname*{essinf}_{Q_{x,k-5}}f-\operatorname*{essinf}_{Q_{y,k-4}}f\right|+\left|\operatorname*{essinf}_{Q_{y,k-4}}f-\operatorname*{essinf}_{Q_{y,k}}f\right|\lesssim 1,\end{aligned}$$

which together with (3.58) yields that

$$\begin{aligned}&-m_Q(S_k(f))+m_R(S_k(f))\\&\quad\le\frac{1}{\mu(Q)}\frac{1}{\mu(R)}\int_Q\int_R\left\{\left|S_k(f)(x)-\operatorname*{essinf}_{Q_{x,k}}f\right|+\left|\operatorname*{essinf}_{Q_{x,k}}f-\operatorname*{essinf}_{Q_{y,k}}f\right|\right.\\&\qquad\left.+\left|\operatorname*{essinf}_{Q_{y,k}}f-S_k(f)(y)\right|\right\}d\mu(x)\,d\mu(y)\lesssim 1.\end{aligned}$$

Case (2) For any $x\in R\cap\operatorname{supp}(\mu)$, $Q\not\subset Q_{x,k-2}$. In this case, for any $x\in R\cap\operatorname{supp}(\mu)$, $R\not\subset Q_{x,k-2}$. It then follows from Lemma 2.2 (b) that for any $x\in Q\cap\operatorname{supp}(\mu)$ and $y\in R\cap\operatorname{supp}(\mu)$, $Q_{x,k}\subset Q_{x,k-1}\subset\frac{7}{5}Q$ and $Q_{y,k}\subset\frac{7}{5}R$. By the Tonelli theorem and Lemma 2.1,

$$\begin{aligned}&-m_Q(S_k(f))+m_R(S_k(f))\\&\quad\le\frac{1}{\mu(Q)}\frac{1}{\mu(R)}\int_Q\int_R\left\{\left|S_k(f)(x)-\operatorname*{essinf}_{\widetilde{\frac{7}{5}Q}}f\right|+\left|\operatorname*{essinf}_{\widetilde{\frac{7}{5}Q}}f-\operatorname*{essinf}_{\widetilde{\frac{7}{5}R}}f\right|\right.\\&\qquad\left.+\left|\operatorname*{essinf}_{\widetilde{\frac{7}{5}R}}f-S_k(f)(y)\right|\right\}d\mu(x)\,d\mu(y)\lesssim 1+\delta(Q,R).\end{aligned}$$

Case (3) For any $x\in R\cap\operatorname{supp}(\mu)$, $R\not\subset Q_{x,k-2}$ and there exists some $x_0\in R\cap\operatorname{supp}(\mu)$ such that $Q\subset Q_{x_0,k-2}$. In this case, Lemma 4.2 in [17] implies that for any $x\in Q\cap\operatorname{supp}(\mu)$, $Q\subset Q_{x_0,k-2}\subset Q_{x,k-3}$, and Lemma 2.2(b) implies that for any $x\in Q\cap\operatorname{supp}(\mu)$, $Q_{x,k}\subset Q_{x,k-1}\subset\frac{7}{5}R$.

By these facts, Lemma 2.1 (e) and (3.41), we have that for any $x \in Q \cap \operatorname{supp}(\mu)$,

$$\left|\operatorname*{essinf}_{Q_{x,k}} f - \operatorname*{essinf}_{\widetilde{\frac{7}{5}R}} f\right| \le 1 + \delta\left(Q_{x,k}, \frac{7}{5}R\right) \lesssim 1 + \delta(Q, R).$$

From this, (3.58) and the Tonelli theorem, we deduce that

$$\begin{aligned}
&-m_Q(S_k(f)) + m_R(S_k(f)) \\
&\quad \le \frac{1}{\mu(Q)}\frac{1}{\mu(R)}\int_Q\int_R \left\{ \left|S_k(f)(x) - \operatorname*{essinf}_{Q_{x,k}} f\right| + \left|\operatorname*{essinf}_{Q_{x,k}} f - \operatorname*{essinf}_{\widetilde{\frac{7}{5}R}} f\right| \right. \\
&\qquad \left. + \left|\operatorname*{essinf}_{\widetilde{\frac{7}{5}R}} f - S_k(f)(y)\right|\right\} d\mu(x)\, d\mu(y) \lesssim 1 + \delta(Q, R),
\end{aligned}$$

which completes the proof of Proposition 3.14. ■

We next define the *local natural maximal operator*, which is a local variant of $\mathcal{M}$. For any locally integrable function f and $x \in \mathbb{R}^d$, let

$$\mathcal{M}_l(f)(x) := \sup_{\substack{Q \ni x,\, Q \notin \mathcal{D} \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y).$$

Lemma 3.11. *Let f be a locally integrable function. Then $f \in \operatorname{rblo}(\mu)$ if and only if $\mathcal{M}_l(f) - f \in L^\infty(\mu)$ and f satisfies* (3.48), (3.49) *and* (3.50). *Furthermore, $\|\mathcal{M}_l(f) - f\|_{L^\infty(\mu)} \sim \|f\|_{\operatorname{rblo}(\mu)}$.*

Proof. Assuming that $f \in \operatorname{rblo}(\mu)$, we then see that (3.48) through (3.50) hold. For μ-a. e. $x \in \mathbb{R}^d$, there exists a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $l(Q_k) \to 0$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f(y)\, d\mu(y) = f(x); \tag{3.60}$$

see [16, p. 96]. Let x be any point satisfying (3.60) and Q containing x be a doubling cube with $Q \notin \mathcal{D}$. Then we obtain that $f(x) \ge \operatorname{essinf}_Q f$ and so $m_Q(f) - f(x) \lesssim \|f\|_{\operatorname{rblo}(\mu)}$. Taking the supremum over all doubling cubes containing x in the complement of $\mathcal{D}$, we have $\mathcal{M}_l(f)(x) - f(x) \lesssim \|f\|_{\operatorname{rblo}(\mu)}$.

Conversely, assume that f satisfies (3.48) through (3.50) and $\mathcal{M}_l(f) - f \in L^\infty(\mu)$. Then it is easy to see that for any doubling cube $Q \notin \mathcal{D}$ and μ-a. e. $x \in Q$, $f(x) \ge m_Q(f) - \|\mathcal{M}_l(f) - f\|_{L^\infty(\mu)}$. This yields that

$$\operatorname*{essinf}_Q f \ge m_Q(f) - \|\mathcal{M}_l(f) - f\|_{L^\infty(\mu)},$$

which together with (3.48) through (3.50) and Proposition 3.13 implies that $f \in \operatorname{rblo}(\mu)$ and $\|f\|_{\operatorname{rblo}(\mu)} \lesssim \|\mathcal{M}_l(f) - f\|_{L^\infty(\mu)}$. Therefore, the proof of Lemma 3.11 is completed. ■

Lemma 3.12. *If $f \in \mathrm{rbmo}\,(\mu)$, then there exists a nonnegative constant $\widetilde{C}$ independent of f such that for any doubling cube $Q \notin \mathcal{D}$,*

$$\frac{1}{\mu(Q)}\int_Q \mathcal{M}_l(f)(x)\,d\mu(x) \le \widetilde{C}\|f\|_{\mathrm{rbmo}\,(\mu)} + \operatorname*{essinf}_Q \mathcal{M}_l(f).$$

Moreover, if $\mathcal{M}_l(f)$ is μ-a. e. finite, then

$$\frac{1}{\mu(Q)}\int_Q \mathcal{M}_l(f)(x)\,d\mu(x) - \operatorname*{essinf}_Q \mathcal{M}_l(f) \le \widetilde{C}\|f\|_{\mathrm{rbmo}\,(\mu)}.$$

Proof. Fix $f \in \mathrm{rbmo}\,(\mu)$. Without loss of generality, we may assume that $\|f\|_{\mathrm{rbmo}\,(\mu)} = 1$. For any doubling cube $Q \notin \mathcal{D}$, write

$$f = [f - m_Q(f)]\chi_{\frac{4}{3}Q} + m_Q(f)\chi_{\frac{4}{3}Q} + f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}.$$

Obviously, $\mathcal{M}_l$ is bounded on $L^2(\mu)$ since $\mathcal{M}_l(f) \le N(f)$. Then by this fact, the Hölder inequality, the doubling property of Q, Corollary 3.1 and Lemma 2.1,

$$\begin{aligned}
&\int_Q \mathcal{M}_l\left[(f - m_Q(f))\chi_{\frac{4}{3}Q}\right](x)\,d\mu(x)\\
&\lesssim \left\{\int_{\frac{4}{3}Q} |f(x) - m_Q(f)|^2\,d\mu(x)\right\}^{1/2}[\mu(Q)]^{1/2}\\
&\lesssim \left\{\int_{\frac{4}{3}Q} \left|f(x) - m_{\widetilde{\frac{4}{3}Q}}(f)\right|^2\,d\mu(x)\right\}^{1/2}[\mu(Q)]^{1/2} + \mu(Q)\left|m_Q(f) - m_{\widetilde{\frac{4}{3}Q}}(f)\right|\\
&\lesssim \mu(Q).
\end{aligned}$$

From this, it follows that

$$\frac{1}{\mu(Q)}\int_Q \mathcal{M}_l\left[(f - m_Q(f))\chi_{\frac{4}{3}Q}\right](x)\,d\mu(x) \lesssim 1.$$

Therefore, with the aid of Proposition 3.13, Lemma 3.12 is reduced to proving that there exists a positive constant C such that for μ-a. e. $x \in Q$,

$$\mathcal{M}_l\left[m_Q(f)\chi_{\frac{4}{3}Q} + f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\right](x) \le C + \operatorname*{essinf}_Q \mathcal{M}_l(f). \tag{3.61}$$

For any doubling cube R containing x with $R \notin \mathcal{D}$ and any $y \in Q$, if $R \subset \frac{4}{3}Q$, then

$$\begin{aligned}
\mathrm{E} :&= \frac{1}{\mu(R)}\int_R \left[m_Q(f)\chi_{\frac{4}{3}Q}(z) + f(z)\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}(z)\right]\,d\mu(z) - \mathcal{M}_l(f)(y)\\
&\le m_Q(f) - \mathcal{M}_l(f)(y) \le 0.
\end{aligned}$$

Assume that $R \not\subset \frac{4}{3}Q$ now. We then see that $l(R) \geq \frac{1}{6}l(Q)$. There exist two cases.

Case (i) $l(R) \leq 4l(Q)$. In this case, it is easy to see that $Q \subset 13R \subset 57Q$ with the aid of the fact that $l(R) \geq \frac{1}{6}l(Q)$. From this fact, Proposition 3.4, (3.7) and Lemma 2.1, it follows that

$$\begin{aligned}
\mathrm{E} &= \frac{1}{\mu(R)} \int_R [f(z) - m_Q(f)]\, \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}(z)\, d\mu(z) + m_Q(f) - \mathcal{M}_l(f)(y) \\
&\leq \frac{1}{\mu(R)} \int_R \Big[|f(z) - m_R(f)| + \big|m_R(f) - m_{\widetilde{13R}}(f)\big| \\
&\quad + \big|m_{\widetilde{13R}}(f) - m_Q(f)\big|\Big]\, d\mu(z) \\
&\lesssim 1 + \delta(Q, 13R) \lesssim 1.
\end{aligned}$$

Case (ii) $l(R) > 4l(Q)$. In this case, Lemma 2.2 (a) and (d) imply that $Q \subset \frac{3}{2}R \subset Q_{z,\, H_R^z - 1}$ and $H_R^z \geq 0$ for any $z \in R \cap \operatorname{supp}(\mu)$. Let $R_1 := \{z \in R : Q_{z,\, H_R^z - 1} \notin \mathcal{D}\}$ and $R_2 := R \setminus R_1$. Then we can write

$$\begin{aligned}
\mathrm{E} &= \frac{1}{\mu(R)} \int_{R_1} \Big[m_Q(f)\chi_{\frac{4}{3}Q}(z) + f(z)\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}(z) - \mathcal{M}_l(f)(y)\Big]\, d\mu(z) \\
&\quad + \frac{1}{\mu(R)} \int_{R_2} \cdots \\
&=: \mathrm{E}_1 + \mathrm{E}_2.
\end{aligned}$$

If $z \in R_1$, then $m_{Q_{z,\, H_R^z - 1}}(f) \leq \mathcal{M}_l(f)(y)$ since $y \in Q_{z,\, H_R^z - 1}$. Therefore, by (3.7), Proposition 3.4, Lemma 2.1, Lemma 2.2 (g) and the doubling property of Q and R,

$$\begin{aligned}
\mathrm{E}_1 &\leq \frac{1}{\mu(R)} \int_{R_1} \Big[\Big|m_Q(f) - m_{\widetilde{\frac{3}{2}R}}(f)\Big|\, \chi_{\frac{4}{3}Q}(z) + \Big|f(z) - m_{\widetilde{\frac{3}{2}R}}(f)\Big|\, \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}(z) \\
&\quad + \Big|m_{\widetilde{\frac{3}{2}R}}(f) - m_{Q_{z,\, H_R^z - 1}}(f)\Big| + m_{Q_{z,\, H_R^z - 1}}(f) - \mathcal{M}_l(f)(y)\Big]\, d\mu(z) \\
&\lesssim \frac{\mu(\frac{4}{3}Q)}{\mu(R)} \Big|m_Q(f) - m_{\widetilde{\frac{3}{2}R}}(f)\Big| + \frac{1}{\mu(R)} \int_R \left[1 + \delta\left(\frac{3}{2}R, Q_{z,\, H_R^z - 1}\right)\right] d\mu(z) \\
&\lesssim \frac{1}{\mu(R)} \int_{\frac{3}{2}R} \Big|f(z) - m_{\widetilde{\frac{3}{2}R}}(f)\Big|\, d\mu(z) + 1 \lesssim 1.
\end{aligned}$$

On the other hand, for $z \in R_2$, Lemma 2.2 (d) implies that $H_R^z \leq 2$, from which it follows that $Q_{y,\,2} \subset Q_{y,\, H_R^z - 1}$. The fact that $y \in (Q_{y,\, H_R^z - 1} \cap Q_{z,\, H_R^z - 1})$ together with Lemma 4.2 in [17] implies that $Q_{y,\, H_R^z - 1} \subset Q_{z,\, H_R^z - 2} \subset Q_{y,\, H_R^z - 3}$. Moreover, the fact that $H_R^z \geq 0$ yields that $Q_{y,\, H_R^z - 3} \subset Q_{y,\, -3}$ (Recall that by Definition 2.4, if $\mathbb{R}^d$ is an initial cube, then for any $y \in \operatorname{supp}(\mu)$ and

$k \le 0$, $Q(y, k) = \mathbb{R}^d$). Thus, from the fact that $m_{Q_{y,2}}(f) \le \mathcal{M}_l(f)(y)$, (3.7), Proposition 3.4, Lemma 2.1, Lemma 2.2 (g) and the doubling property of Q and R, it follows that

$$\begin{aligned}
\mathrm{E}_2 &\le \frac{1}{\mu(R)} \int_{R_2} \Big[\Big| m_Q(f) - m_{\widetilde{\frac{3}{2}R}}(f) \Big| \chi_{\frac{4}{3}Q}(z) + \Big| f(z) - m_{\widetilde{\frac{3}{2}R}}(f) \Big| \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}(z) \\
&\quad + \Big| m_{\widetilde{\frac{3}{2}R}}(f) - m_{Q_{z,\, H_R^z - 2}}(f) \Big| + \Big| m_{Q_{z,\, H_R^z - 2}}(f) - m_{Q_{y,\,2}}(f) \Big| \Big] \, d\mu(z) \\
&\lesssim \frac{1}{\mu(R)} \int_R [1 + \delta(Q_{y,\,2}, Q_{z,\, H_R^z - 2})] \, d\mu(z) \lesssim 1.
\end{aligned}$$

Then (3.61) holds, which completes the proof of Lemma 3.12. ■

The following Theorem 3.5 and Theorem 3.6 are local variants of Theorem 3.3 and Theorem 3.4; see also [7, 1, 15]. We point out that unlike the case RBLO$\,(\mu)$, if $f \in \mathrm{rbmo}\,(\mu)$, then $\mathcal{M}_l(f)$ is finite almost everywhere.

Theorem 3.5. *$\mathcal{M}_l$ is bounded from* rbmo$\,(\mu)$ *to* rblo$\,(\mu)$*, namely, there exists a positive constant C such that for all $f \in \mathrm{rbmo}\,(\mu)$,*

$$\|\mathcal{M}_l(f)\|_{\mathrm{rblo}\,(\mu)} \le C \|f\|_{\mathrm{rbmo}\,(\mu)}.$$

Proof. Fix $f \in \mathrm{rbmo}\,(\mu)$. By the homogeneity of $\mathcal{M}_l$, we only need to prove the conclusion of the theorem for $\|f\|_{\mathrm{rbmo}\,(\mu)} = 1$. We first prove that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(2Q)} \int_Q |\mathcal{M}_l(f)(x)| \, d\mu(x) \lesssim 1. \tag{3.62}$$

Write $f = f\chi_{\frac{4}{3}Q} + f\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}$. Then by the Hölder inequality, the boundedness of $\mathcal{M}_l$ in $L^2(\mu)$ and Corollary 3.1, we deduce that

$$\frac{1}{\mu(2Q)} \int_Q \left| \mathcal{M}_l \left(f\chi_{\frac{4}{3}Q} \right)(x) \right| \, d\mu(x) \lesssim \left\{ \frac{1}{\mu(2Q)} \int_{\frac{4}{3}Q} |f(x)|^2 \, d\mu(x) \right\}^{1/2} \lesssim 1.$$

On the other hand, for any $x \in Q$ and any doubling cube $P \ni x$ with $P \notin \mathcal{D}$ and $P \cap (\mathbb{R}^d \setminus \frac{4}{3}Q) \neq \emptyset$, it is easy to see that $l(P) \ge \frac{1}{6} l(Q)$. This implies that $Q \subset 13P$ and hence $\widetilde{13P} \in \mathcal{D}$. Therefore, Proposition 3.4 and Lemma 2.1 yield that

$$\begin{aligned}
&\frac{1}{\mu(P)} \int_P |f(z)| \, d\mu(z) \\
&\quad \le \frac{1}{\mu(P)} \int_P |f(z) - m_P(f)| \, d\mu(z) + \left| m_P(f) - m_{\widetilde{13P}}(f) \right| + \left| m_{\widetilde{13P}}(f) \right| \\
&\quad \lesssim 1.
\end{aligned}$$

This further implies that (3.62) holds, and hence $\mathcal{M}_l(f)$ is finite almost everywhere.

Now we prove that for any doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$m_Q[\mathcal{M}_l(f)] - m_R[\mathcal{M}_l(f)] \lesssim 1 + \delta(Q, R). \tag{3.63}$$

Let

$$Q_1 := \{x \in Q : \text{ for any doubling cube } P \text{ containing } x, \\ \text{if } l(P) > 4l(R), \text{ then } P \in \mathcal{D}\}$$

and $Q_2 := Q \setminus Q_1$. Moreover, for any $x \in Q$, set

$$\mathcal{M}_l^1(f)(x) := \sup_{\substack{P \ni x,\, P \text{ doubling} \\ P \notin \mathcal{D} \text{ and } l(P) \le 4l(R)}} \frac{1}{\mu(P)} \int_P f(y)\, d\mu(y),$$

and for any $x \in Q_2$, set

$$\mathcal{M}_l^2(f)(x) := \sup_{\substack{P \ni x,\, P \text{ doubling} \\ P \notin \mathcal{D} \text{ and } l(P) > 4l(R)}} \frac{1}{\mu(P)} \int_P f(y)\, d\mu(y),$$

$$\mathcal{U}_{1,\,Q} := \{x \in Q_2 : \mathcal{M}_l^1(f)(x) \ge \mathcal{M}_l^2(f)(x)\} \quad \text{and} \quad \mathcal{U}_{2,\,Q} := Q_2 \setminus \mathcal{U}_{1,\,Q}.$$

Then for any $x \in (Q_1 \cup \mathcal{U}_{1,\,Q})$, $\mathcal{M}_l(f)(x) = \mathcal{M}_l^1(f)(x)$ and for any $x \in \mathcal{U}_{2,\,Q}$, $\mathcal{M}_l(f)(x) = \mathcal{M}_l^2(f)(x)$. By writing

$$f = [f - m_R(f)]\chi_{\frac{4}{3}Q} + [f - m_R(f)]\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q} + m_R(f)$$

and using the fact that $m_R \le m_R[\mathcal{M}_l(f)]$, we see that

$$\begin{aligned} &m_Q[\mathcal{M}_l(f)] - m_R[\mathcal{M}_l(f)] \\ &\quad \le \frac{1}{\mu(Q)} \int_{(Q_1 \cup \mathcal{U}_{1,\,Q})} \mathcal{M}_l^1([f - m_R(f)]\chi_{\frac{4}{3}Q})(x)\, d\mu(x) \\ &\qquad + \frac{1}{\mu(Q)} \int_{(Q_1 \cup \mathcal{U}_{1,\,Q})} \mathcal{M}_l^1([f - m_R(f)]\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q})(x)\, d\mu(x) \\ &\qquad + \frac{1}{\mu(Q)} \int_{\mathcal{U}_{2,\,Q}} \{\mathcal{M}_l^2(f)(x) - m_R[\mathcal{M}_l(f)]\}\, d\mu(x) \\ &\quad =: \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3. \end{aligned}$$

Using the estimate for I_1 in the proof of Theorem 3.3, we see that $\mathrm{F}_1 \lesssim 1 + \delta(Q, R)$. On the other hand, an argument similar to (3.46) yields that for any point $x \in (Q_1 \cup \mathcal{U}_{1,\,Q})$ and any doubling cube $P \ni x$ with $P \notin \mathcal{D}$ and $l(P) \le 4l(R)$,

$$\frac{1}{\mu(P)} \int_P |f(y) - m_R(f)| \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}(y)\, d\mu(y) \lesssim 1 + \delta(Q, R).$$

This implies that $\mathrm{F}_2 \lesssim 1 + \delta(Q, R)$.

To estimate F_3, it suffices to prove that for any doubling cube P containing x with $P \notin \mathcal{D}$ and $l(P) > 4l(R)$, $m_P(f) - m_R[\mathcal{M}_l(f)] \lesssim 1$. For any $z \in P \cap \operatorname{supp}(\mu)$, Lemma 2.2 (a) and (d) imply that $R \subset \frac{3}{2}P \subset Q_{z,\,H_P^z-1}$ and $H_P^z \geq 0$. Set

$$R_1 := \{y \in R: \text{ there exists a point } z_y \in P \text{ such that } Q_{z_y,\,H_P^{z_y}-1} \notin \mathcal{D}\}$$

and $R_2 := R \setminus R_1$. Then we have

$$\begin{aligned}&m_P(f) - m_R[\mathcal{M}_l(f)]\\&= \frac{1}{\mu(R)}\int_{R_1}[m_P(f) - \mathcal{M}_l(f)(y)]\,d\mu(y) + \frac{1}{\mu(R)}\int_{R_2}\cdots.\end{aligned}$$

Observe that for any $y \in R_1$, $m_{Q_{z_y,\,H_P^{z_y}-1}}(f) \leq \mathcal{M}_l(f)(y)$. This together with (3.7), Lemma 2.1 and Lemma 2.2 (g) implies that

$$\begin{aligned}m_P(f) - \mathcal{M}_l(f)(y) &\leq \left|m_P(f) - m_{Q_{z_y,\,H_P^{z_y}-1}}(f)\right| + m_{Q_{z_y,\,H_P^{z_y}-1}}(f) - \mathcal{M}_l(f)(y)\\&\leq \left|m_P(f) - m_{\widetilde{\frac{3}{2}P}}(f)\right| + \left|m_{\widetilde{\frac{3}{2}P}}(f) - m_{Q_{z_y,\,H_P^{z_y}-1}}(f)\right| \lesssim 1.\end{aligned}$$

On the other hand, for any $y \in (R_2 \cap \operatorname{supp}(\mu))$, Lemma 2.2 (d) implies that $H_P^z \leq 2$ for any $z \in (P \cap \operatorname{supp}(\mu))$. Moreover, an easy argument involving the facts that $y \in (Q_{y,\,H_P^z-1} \cap Q_{z,\,H_P^z-1})$ and that $H_P^z \geq 0$ and Lemma 4.2 in [17] yields that

$$Q_{y,\,2} \subset Q_{y,\,H_P^z-1} \subset Q_{z,\,H_P^z-2} \subset Q_{y,\,H_P^z-3} \subset Q_{y,\,-3}.$$

Thus, from Lemma 2.2 (g) and the fact that $\mathcal{M}_l(f)(y) \geq m_{Q_{y,\,2}}(f)$, it follows that

$$\begin{aligned}m_P(f) - \mathcal{M}_l(f)(y) &\leq \left|m_P(f) - m_{Q_{z,\,H_P^z-2}}(f)\right| + \left|m_{Q_{z,\,H_P^z-2}}(f) - m_{Q_{y,\,2}}(f)\right|\\&\lesssim 1 + \delta(P, Q_{z,\,H_P^z-2}) + \delta(Q_{y,\,2}, Q_{z,\,H_P^z-2}) \lesssim 1.\end{aligned}$$

Therefore, combining these estimates above concludes that for any doubling cube $P \ni x$ with $P \notin \mathcal{D}$ and $l(P) > 4l(R)$, $m_P(f) - m_R[\mathcal{M}_l(f)] \lesssim 1$, which implies that $F_3 \lesssim 1$. The combination of estimates for F_1 through F_3 implies (3.63).

By Lemma 3.12, to finish the proof of Theorem 3.5, we need to verify that for any cube $Q \in \mathcal{D}$,

$$\left|\operatorname*{essinf}_{\widetilde{Q}} \mathcal{M}_l(f)\right| \lesssim 1. \tag{3.64}$$

If $\operatorname{essinf}_{\widetilde{Q}} \mathcal{M}_l(f) \geq 0$, then (3.62) implies (3.64). Assume that

$$\operatorname*{essinf}_{\widetilde{Q}} \mathcal{M}_l(f) < 0.$$

Then we see that $|\operatorname{essinf}_{\widetilde{Q}} \mathcal{M}_l(f)| = \operatorname{esssup}_{\widetilde{Q}}\{-\mathcal{M}_l(f)\}$. Recall that for any $x \in \operatorname{supp}(\mu)$, $Q_{x,2} \notin \mathcal{D}$ and $Q_{x,1} \in \mathcal{D}$ (see Lemma 2.2 (d) and (e)). By these facts and Proposition 3.1, for all $x \in \operatorname{supp}(\mu)$,

$$\begin{aligned} -\mathcal{M}_l(f)(x) &\le \inf_{\substack{P \ni x,\, P \notin \mathcal{D} \\ P \text{ doubling}}} m_P(|f|) \\ &\le \left| m_{Q_{x,2}}(|f|) - m_{Q_{x,1}}(|f|) \right| + m_{Q_{x,1}}(|f|) \lesssim 1, \end{aligned}$$

which completes the proof of (3.64) and hence the proof of Theorem 3.5. ■

Theorem 3.6. *A locally integrable function f belongs to* $\operatorname{rblo}(\mu)$ *if and only if there exist $h \in L^\infty(\mu)$ and $g \in \operatorname{rbmo}(\mu)$ such that*

$$f = \mathcal{M}_l(g) + h. \tag{3.65}$$

Furthermore, $\|f\|_{\operatorname{rblo}(\mu)} \sim \inf(\|g\|_{\operatorname{rbmo}(\mu)} + \|h\|_{L^\infty(\mu)})$, *where the infimum is taken over all representations of f as in* (3.65).

Proof. If there exist g and h satisfying (3.65), then by Theorem 3.5, $\mathcal{M}_l(g) \in \operatorname{rblo}(\mu)$, which implies $f \in \operatorname{rblo}(\mu)$ and

$$\|f\|_{\operatorname{rblo}(\mu)} \lesssim \|\mathcal{M}_l(g)\|_{\operatorname{rblo}(\mu)} + \|h\|_{L^\infty(\mu)} \lesssim \|g\|_{\operatorname{rbmo}(\mu)} + \|h\|_{L^\infty(\mu)}.$$

To see the converse, suppose that $f \in \operatorname{rblo}(\mu)$. By Theorem 3.5 again, we see $\mathcal{M}_l(f) \in \operatorname{rblo}(\mu)$. Set $h := f - \mathcal{M}_l(f)$ and $g := f$. Then Theorem 3.6 follows from Lemma 3.11, which completes the proof of Theorem 3.6. ■

4. Boundedness of inhomogeneous Littlewood-Paley g-function

This section is devoted to establishing the boundedness of the inhomogeneous Littlewood-Paley g-function in $h^{1,\infty}_{\rm atb}(\mu)$ and $\operatorname{rbmo}(\mu)$.

Theorem 4.1. *There exists a positive constant C such that for all $f \in h^{1,\infty}_{\rm atb}(\mu)$,*

$$\|g(f)\|_{L^1(\mu)} \le C\|f\|_{h^{1,\infty}_{\rm atb}(\mu)}.$$

Proof. By the Fatou lemma, to prove the theorem, it suffices to show that for any ∞-atomic block or ∞-block $b := \sum_{j=1}^2 \lambda_j a_j$ as in Definition 2.7 or Definition 3.2, we have $\|g(b)\|_{L^1(\mu)} \lesssim \sum_{j=1}^2 |\lambda_j|$.

Since for any $x \in \operatorname{supp}(\mu)$,

$$g(b)(x) \le |S_1(b)(x)| + \left\{ \sum_{k=2}^\infty |D_k(b)(x)|^2 \right\}^{\frac{1}{2}},$$

by (3.34), Theorem 4.1 is reduced to showing

$$\int_{\mathbb{R}^d}\left\{\sum_{k=2}^{\infty}|D_k(b)(x)|^2\right\}^{\frac{1}{2}}d\mu(x)\lesssim\sum_{j=1}^{2}|\lambda_j|. \tag{4.1}$$

Assume that b is an ∞-atomic block with $\operatorname{supp}(b)\subset R\notin\mathcal{D}$. By an argument similar to that used in the proof of Theorem 3.1 in [27], we obtain the estimate (4.1).

If b is an ∞-block with $\operatorname{supp}(b)\subset R\in\mathcal{D}$, we write

$$\begin{aligned}&\int_{\mathbb{R}^d}\left\{\sum_{k=2}^{\infty}|D_k(b)(x)|^2\right\}^{\frac{1}{2}}d\mu(x)\\&=\int_{4R}\left\{\sum_{k=2}^{\infty}|D_k(b)(x)|^2\right\}^{\frac{1}{2}}d\mu(x)+\int_{\mathbb{R}^d\setminus 4R}\left\{\sum_{k=2}^{\infty}|D_k(b)(x)|^2\right\}^{\frac{1}{2}}d\mu(x)\\&=:\mathrm{I}+\mathrm{II}.\end{aligned}$$

Using the boundedness of the g-function $g(f)$ in $L^2(\mu)$ and an argument similar to the estimates for I_1 and I_2 in the proof of [27, Theorem 3.1] again, we also obtain $\mathrm{I}\lesssim\sum_{j=1}^{2}|\lambda_j|$.

To estimate II, choose any point $x_0\in R\cap\operatorname{supp}(\mu)$. By the Hölder inequality, the fact that for any x, $y\in\mathbb{R}^d$ with $x\neq y$,

$$\left[\sum_{k=2}^{\infty}|D_k(x,y)|^2\right]^{1/2}\lesssim\frac{1}{|x-y|^n}\quad\text{(see [17, p. 82])}$$

and (2.3), it follows that for $j=1,\,2$, and any $x\in Q_{x_0,\,H_R^{x_0}-2}\setminus(4R)$,

$$\begin{aligned}\left\{\sum_{k=2}^{\infty}|D_k(a_j)(x)|^2\right\}^{\frac{1}{2}}&\le\left[\int_{Q_j}\sum_{k=2}^{\infty}|D_k(x,y)|^2|a_j(y)|^2\,d\mu(y)\right]^{\frac{1}{2}}[\mu(Q_j)]^{\frac{1}{2}}\\&\lesssim\left[\int_{Q_j}\frac{|a_j(y)|^2}{|x-y|^{2n}}\,d\mu(y)\right]^{\frac{1}{2}}[\mu(Q_j)]^{\frac{1}{2}}\lesssim\frac{1}{|x-x_0|^n},\end{aligned}$$

where x_j is the center of Q_j, and in the last step, we used the fact that $|x-x_j|\sim|x-x_0|$ for any $x\in Q_{x_0,\,H_R^{x_0}-2}\setminus(4R)$. On the other hand, since $R\in\mathcal{D}$, by Lemma 2.2 (d), we have $H_R^{x_0}\le 1$. Thus for any $k\ge 2$ and $y\in R\cap\operatorname{supp}(\mu)$, by (A-3) in Section 2 and Lemma 4.2 in [17], we have $Q_{y,\,k-2}\subset Q_{y,\,H_R^{x_0}-1}\subset Q_{x_0,\,H_R^{x_0}-2}$, and so $\operatorname{supp}(D_k(b))\subset Q_{x_0,\,H_R^{x_0}-2}$.

Therefore, by Lemma 2.2 (a),

$$\begin{aligned}\mathrm{II} &= \int_{Q_{x_0,\,H_R^{x_0}-2}\setminus 4R} \left\{\sum_{k=2}^{\infty} |D_k(b)(x)|^2\right\}^{\frac{1}{2}} d\mu(x)\\ &\lesssim \sum_{j=1}^{2} |\lambda_j| \int_{Q_{x_0,\,H_R^{x_0}-2}\setminus Q_{x_0,\,H_R^{x_0}+1}} \frac{1}{|x-x_0|^n}\, d\mu(x) \lesssim \sum_{j=1}^{2} |\lambda_j|,\end{aligned}$$

which completes the proof of Theorem 4.1. ■

The following conclusion is a local variant of Lemma 9.3 in [16].

Lemma 4.1. *There exists some constant P_0 (big enough) depending on C_0 and n such that if $x \in \mathbb{R}^d$ is some fixed point and $\{f_Q\}_{Q\ni x}$ is a collection of numbers such that $f_Q - f_R \le [1+\delta(Q,R)]C_x$ for all doubling cubes $Q \subset R$ with $x \in Q$ and $Q \notin \mathcal{D}$ such that $1+\delta(Q,R) \le P_0$, and $|f_R| \le C_x$ for all doubling cubes $R \in \mathcal{D}$ with $R \ni x$, then*

$$f_Q - f_R \le C[1+\delta(Q,R)]C_x$$

for all doubling cubes $Q \subset R$ with $x \in Q$ and $Q \notin \mathcal{D}$, where C depends on C_0, n and P_0.

Proof. Let $x \in \mathbb{R}^d$ be as in the lemma, $Q \subset R$ be two doubling cubes in $\mathbb{R}^d$ with $x \in Q$ and $Q \notin \mathcal{D}$. If $R \notin \mathcal{D}$, then Lemma 4.1 can be proved by a slight modification of the proof for Lemma 9.3 [16]. Thus we only consider the case that $R \in \mathcal{D}$. Let $Q_0 := Q$ and Q_1 be the first cube of the form $2^k Q$, $k \ge 0$, such that $1+\delta(Q, Q_1) > P$, where P (big enough) is the constant as in [16, Lemma 9.2]. Since $1+\delta(Q, 2^{-1}Q_1) \le P$, by (d) and (a) of Lemma 2.1, we have $\delta(Q,Q_1) \le P + C$. Therefore, by Lemma 2.1 (b), for the doubling cube $\widetilde{Q_1}$, we have $\delta(Q, \widetilde{Q_1}) \le \widetilde{C}$, where $\widetilde{C}$ is a positive constant.

In general, for any $\widetilde{Q_i}$, we denote by Q_{i+1} the first cube of the form $2^k\widetilde{Q_i}$, $k \ge 0$, such that $1+\delta(\widetilde{Q_i}, Q_{i+1}) > P$, and we consider the cube $\widetilde{Q_{i+1}}$. Then, by (b) and (d) of Lemma 2.1, we have $\delta(\widetilde{Q_i}, \widetilde{Q_{i+1}}) \le \widetilde{C}$ and

$$1+\delta(\widetilde{Q_i}, \widetilde{Q_{i+1}}) \ge 1+\delta(\widetilde{Q_i}, Q_{i+1}) > P.$$

Let k_0 be the smallest integer k such that $\widetilde{Q_k} \in \mathcal{D}$. Notice that $\widetilde{Q}_0 = Q_0 = Q \notin \mathcal{D}$. By the increasing property of $\{\widetilde{Q_i}\}_i$, we know that $k_0 \in \mathbb{N}$. We then have that

$$f_Q - f_R \le \sum_{i=0}^{k_0-1} [f_{\widetilde{Q_i}} - f_{\widetilde{Q_{i+1}}}] + \left| f_{\widetilde{Q_{k_0}}} - f_R \right|.$$

Let N be the smallest integer such that $\widetilde{Q_{N+1}}$ is the first cube of the sequence $\{\widetilde{Q_i}\}_i$ such that $R \subset \widetilde{Q_{N+1}}$. Then we see that $k_0 \le N+1$ by the assumption that $R \in \mathcal{D}$. From the fact that $\widetilde{Q_N} \subset (3R)$ and Lemma 2.1, it follows that $\delta(R, \widetilde{Q_{N+1}}) \lesssim 1$. By this observation and [16, Lemma 9.2] together with the assumptions of the lemma, if we take $P_0 := \widetilde{C}$, then

$$\begin{aligned} f_Q - f_R &\lesssim \sum_{i=0}^{k_0-1} [1 + \delta(\widetilde{Q_i}, \widetilde{Q_{i+1}})] C_x + 2C_x \lesssim [1 + \delta(Q, \widetilde{Q_{k_0}})] C_x \\ &\le [1 + \delta(Q, \widetilde{Q_{N+1}})] C_x \lesssim [1 + \delta(Q, R)] C_x, \end{aligned}$$

which completes the proof of Lemma 4.1. ■

Analogous to Theorem 3.2 in [27] for the boundedness of the homogeneous Littlewood-Paley g-function $\dot{g}(f)$ in $\mathrm{RBMO}\,(\mu)$, we have Theorem 4.2 below for the boundedness of the inhomogeneous Littlewood-Paley g-function $g(f)$ in $\mathrm{rbmo}\,(\mu)$. However, unlike Theorem 3.2 there, if $f \in \mathrm{rbmo}\,(\mu)$, then $g(f)$ is finite almost everywhere.

Theorem 4.2. *There exists a positive constant C such that for all $f \in \mathrm{rbmo}\,(\mu)$,*

$$\left\| [g(f)]^2 \right\|_{\mathrm{rblo}\,(\mu)} \le C \|f\|^2_{\mathrm{rbmo}\,(\mu)}.$$

Proof. By the homogeneity of $g(f)$, we may assume that $\|f\|_{\mathrm{rbmo}\,(\mu)} = 1$. We first consider the case that $\mathbb{R}^d$ *is an initial cube*. In this case, to show Theorem 4.2, we first verify that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(2Q)} \int_Q [g(f)(x)]^2 \, d\mu(x) \lesssim 1. \tag{4.2}$$

For any $x \in Q \cap \mathrm{supp}\,(\mu)$, we write

$$[g(f)(x)]^2 = |S_1(f)(x)|^2 + \sum_{k=2}^{H_Q^x+3} |D_k(f)(x)|^2 + \left[g^{H_Q^x}(f)(x) \right]^2,$$

where

$$\left[g^{H_Q^x}(f)(x) \right]^2 := \sum_{k=H_Q^x+4}^{\infty} |D_k(f)(x)|^2.$$

By the fact that $Q_{x,k} \subset \frac{7}{5}Q$ for $k \ge H_Q^x + 2$ (see Lemma 2.2 (b)), the boundedness of the g-function $g(f)$ in $L^2(\mu)$ (see [17, Theorem 6.1]) and

Corollary 3.1,

$$\tag{4.3} \begin{aligned} \frac{1}{\mu(2Q)}\int_Q \left[g^{H_Q^x}(f)(x)\right]^2 d\mu(x) &\le \frac{1}{\mu(2Q)}\int_Q \left[g\left(f\chi_{\frac{7}{5}Q}\right)(x)\right]^2 d\mu(x)\\ &\lesssim \frac{1}{\mu(2Q)}\int_{\frac{7}{5}Q} |f(x)|^2\, d\mu(x) \lesssim 1. \end{aligned}$$

Moreover, for any $f \in \mathrm{rbmo}\,(\mu)$, $k \ge 2$ and $z \in \mathrm{supp}\,(\mu)$,

$$\tag{4.4} |D_k(f)(z)| \lesssim 1.$$

Indeed, since $\mathrm{supp}\,(D_k(z,\cdot)) \subset Q_{z,\,k-2}$, by the vanishing moment of D_k, (A-4) and Lemma 3.10, we have

$$|D_k(f)(z)| \lesssim \int_{Q_{z,\,k-2}} \frac{|f(y) - m_{Q_{z,\,k}}(f)|}{[|z-y| + l(Q_{z,\,k})]^n}\, d\mu(y) \lesssim 1.$$

Thus, (4.4) holds, which together with the fact that $0 \le H_Q^x \le 1$ (Lemma 2.2 (d)) implies that

$$\tag{4.5} \sum_{k=2}^{H_Q^x+3} |D_k(f)(x)|^2 \lesssim 1.$$

The estimates (4.3) and (4.5) together with (3.36) imply that the estimate (4.2) holds.

From (4.2), it follows immediately that for any doubling cube Q that $\mathrm{essinf}_Q[g(f)]^2 \lesssim 1$. Therefore, to complete the proof of Theorem 4.2, by Proposition 3.13, (4.2) and Lemma 4.1, we still need to show that for any doubling cube $Q \notin \mathcal{D}$ and any $y \in Q$,

$$\tag{4.6} \frac{1}{\mu(Q)}\int_Q \left\{[g(f)(x)]^2 - [g(f)(y)]^2\right\} d\mu(x) \lesssim 1,$$

and for any doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$\tag{4.7} m_Q\left[g(f)^2\right] - m_R\left[g(f)^2\right] \lesssim [1 + \delta(Q,R)]^4.$$

We first establish (4.6). For any doubling cube $Q \notin \mathcal{D}$ and $x \in Q \cap \mathrm{supp}\,(\mu)$, if $0 \le H_Q^x \le 5$, we use the following trivial estimate that

$$\tag{4.8} \begin{aligned} [g(f)(x)]^2 - [g(f)(y)]^2 &\le [g(f)(x)]^2\\ &= |S_1(f)(x)|^2 + \sum_{k=2}^{H_Q^x+3} |D_k(f)(x)|^2 + \left[g^{H_Q^x}(f)(x)\right]^2. \end{aligned}$$

If $\frac{7}{5}Q \in \mathcal{D}$, then (4.6) can be deduced from (4.8), (4.3), (4.4) and (3.36) directly. If $\frac{7}{5}Q \notin \mathcal{D}$, notice that $\int_{\mathbb{R}^d} D_k(x,y)\,d\mu(y) = 0$ when $0 \le H_Q^x \le 5$ and $k \ge H_Q^x + 4$. Moreover, by Lemma 2.2 (b), we have $\operatorname{supp}(D_k(x,\cdot)) \subset Q_{x,k-2} \subset \frac{7}{5}Q$. These facts imply that

$$D_k f(x) = D_k \left[\left(f - m_{\widetilde{\frac{7}{5}Q}}(f)\right)\chi_{\frac{7}{5}Q}\right](x).$$

On the other hand, since $g^{H_Q^x}(f) \le g(f)$, the doubling property of Q together with the boundedness of the g-function $g(f)$ in $L^2(\mu)$ (see [17, Theorem 6.1]) and Corollary 3.1 yields that

$$\begin{aligned}(4.9)\qquad &\frac{1}{\mu(Q)}\int_Q \left[g^{H_Q^x}(f)(x)\right]^2 d\mu(x)\\ &\le \frac{1}{\mu(Q)}\int_Q \left[g\left(\left(f - m_{\widetilde{\frac{7}{5}Q}}(f)\right)\chi_{\frac{7}{5}Q}\right)(x)\right]^2 d\mu(x)\\ &\lesssim \frac{1}{\mu(2Q)}\int_{\frac{7}{5}Q}\left|f(x) - m_{\widetilde{\frac{7}{5}Q}}(f)\right|^2 d\mu(x) \lesssim 1.\end{aligned}$$

This together with (4.8), (3.36) and (4.4) implies (4.6).

Now suppose that $H_Q^x \ge 6$. We then have

$$\begin{aligned}[g(f)(x)]^2 - [g(f)(y)]^2 &\le |S_1(f)(x)|^2 + \sum_{k=2}^{H_Q^x-3}\left[|D_k(f)(x)|^2 - |D_k(f)(y)|^2\right]\\ &\quad + \sum_{k=H_Q^x-2}^{H_Q^x+3}|D_k(f)(x)|^2 + \left[g^{H_Q^x}(f)(x)\right]^2.\end{aligned}$$

Using (4.3), (4.9), (4.4) and (3.36) again, we see that the estimate (4.6) is reduced to showing that for any x, $y \in Q \cap \operatorname{supp}(\mu)$,

$$(4.10)\qquad \sum_{k=2}^{H_Q^x-3}\left[|D_k(f)(x)|^2 - |D_k(f)(y)|^2\right] \lesssim 1.$$

An application of (4.4) implies that

$$\sum_{k=2}^{H_Q^x-3}\left[|D_k(f)(x)|^2 - |D_k(f)(y)|^2\right] \lesssim \sum_{k=2}^{H_Q^x-3}|D_k(f)(x) - D_k(f)(y)|.$$

For $y \in Q \cap \operatorname{supp}(\mu)$ and $2 \le k \le H_Q^x - 3$, we have that $Q \subset Q_{x,k}$, which together with Lemma 4.2 (c) in [17] implies that $Q_{y,k-2} \subset Q_{x,k-3}$. This fact together with the vanishing moment of D_k, (A-5) and Lemma 3.10 further yields that

$$\begin{aligned}
&|D_k(f)(x) - D_k(f)(y)| \\
&\quad = \left| \int_{Q_{x,\,k-3}} [D_k(x,z) - D_k(y,z)] \left[f(z) - m_{Q_{x,\,k}}(f) \right] d\mu(z) \right| \\
&\quad \lesssim \int_{Q_{x,\,k-3}} \frac{|x-y| \left| f(z) - m_{Q_{x,\,k}}(f) \right|}{l(Q_{x,\,k})[|x-z| + l(Q_{x,\,k})]^n} \, d\mu(z) \lesssim \frac{|x-y|}{l(Q_{x,\,k})}.
\end{aligned}$$

Therefore from the fact that $|x-y| \lesssim l(Q)$ and Lemma 3.4 in [17], it follows that

$$\sum_{k=2}^{H_Q^x-3} \left[|D_k(f)(x)|^2 - |D_k(f)(y)|^2 \right] \lesssim \sum_{k=2}^{H_Q^x-3} \frac{|x-y|}{l(Q_{x,\,k})} \lesssim \frac{l(Q)}{l(Q_{x,\,H_Q^x-3})} \lesssim 1,$$

since $Q \subset Q_{x,\,H_Q^x-3}$.

We now prove (4.7). For any doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$, and any $x \in Q \cap \operatorname{supp}(\mu)$ and $y \in R \cap \operatorname{supp}(\mu)$, we first consider the case that $H_Q^x \geq H_R^x + 10$. In this case, if $H_R^x \geq 6$, we write

$$\begin{aligned}
[g(f)(x)]^2 - [g(f)(y)]^2 \leq |S_1(f)(x)|^2 &+ \sum_{k=2}^{H_R^x+3} \left[|D_k(f)(x)|^2 - |D_k(f)(y)|^2 \right] \\
&+ \sum_{k=H_R^x+4}^{H_Q^x+3} |D_k(f)(x)|^2 + \left[g^{H_Q^x}(f)(x) \right]^2.
\end{aligned}$$

Observe that for $k \geq H_R^x + 4$ and $x \in \operatorname{supp}(\mu)$, $\operatorname{supp}(D_k(x,\cdot)) \subset Q_{x,\,k-2}$ and by Lemma 2.2 (e), $Q_{x,\,k-2} \notin \mathcal{D}$. Therefore, using (4.4) and repeating the argument of (3.13) in [27], we obtain

$$\sum_{k=H_R^x+4}^{H_Q^x+3} |D_k(f)(x)| \lesssim 1 + \sum_{k=H_R^x+4}^{H_Q^x-1} |D_k(f)(x)| \lesssim [1 + \delta(Q,R)]^2. \tag{4.11}$$

Consequently, (4.7) follows from (4.9), (3.36), (4.10) and (4.11).

While if $0 \leq H_R^x \leq 5$, by writing

$$\begin{aligned}
&[g(f)(x)]^2 - [g(f)(y)]^2 \\
&\quad \leq |S_1(f)(x)|^2 + \sum_{k=2}^{H_R^x+3} |D_k(f)(x)|^2 + \sum_{k=H_R^x+4}^{H_Q^x+3} |D_k(f)(x)|^2 + \left[g^{H_Q^x}(f)(x) \right]^2,
\end{aligned}$$

we see that (4.7) follows from (4.9), (4.4), (3.36) and (4.11).

Similarly, if $H_R^x \le H_Q^x \le H_R^x + 9$, (4.7) follows from (4.9), (4.4), (3.36) and (4.10), which completes the proof of the case that $\mathbb{R}^d$ is an initial cube.

If $\mathbb{R}^d$ is not an initial cube, we can also show that (4.2), (4.6) and (4.7) hold. We omit the details here; see [27]. This finishes the proof of Theorem 4.2. ■

An argument similar to the proof of [27, Corollary 3.1] yields the following conclusion. We omit the details.

Corollary 4.1. *There exists a positive constant C such that for all $f \in$ rbmo(μ),*

$$\|g(f)\|_{\mathrm{rblo}\,(\mu)} \le C\|f\|_{\mathrm{rbmo}\,(\mu)}.$$

Remark 4.1. *We point out that if we define the inhomogeneous Littlewood-Paley g-function $\widetilde{g}(f)$ as follows,*

$$\widetilde{g}(f)(x) := \left[|S_0(f)(x)|^2 + \sum_{k=1}^{\infty} |D_k(f)(x)|^2\right]^{1/2},$$

then Theorem 4.1, Theorem 4.2 and Corollary 4.1 are still true. Notice that when $\mathbb{R}^d$ is an initial cube, then $S_0 \equiv 0$ and $\widetilde{g}(f)$ degenerates into $g(f)$.

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