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Cyclic Blaschke products for composition operators

Eva A. Gallardo-Gutiérrez and Pamela Gorkin

Abstract

In this work, cyclic Blaschke products for composition operators induced by disc automorphisms are studied. In particular, we obtain interpolating Blaschke products that are cyclic for nonelliptic automorphisms and we obtain a new characterization of Blaschke products that are not finite products of interpolating Blaschke products.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane and $\partial\mathbb{D}$ its boundary. Recall that the Hardy space $\mathcal{H}^2$ is the Hilbert space consisting of holomorphic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_2 = \left(\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, \frac{d\theta}{2\pi} \right)^{1/2}$$

is finite. A classical result due to Fatou states that every Hardy function f has radial limit at $e^{i\theta} \in \partial\mathbb{D}$, except possibly on a set Lebesgue measure zero (see [9], for instance). Throughout this work, $f(e^{i\theta})$ will denote the radial limit of f at $e^{i\theta}$.

If φ is an analytic function on $\mathbb{D}$ that takes $\mathbb{D}$ into itself, the Littlewood Subordination Principle [23] ensures that the composition operator induced by φ

$$C_\varphi f = f \circ \varphi, \qquad \big(f \in \mathcal{H}^2\big)$$

takes the Hardy space $\mathcal{H}^2$ boundedly into itself.

Composition operators have attracted the attention of many operator theorists in the last decades, due, in part, to their ability to link classical branches of mathematics such as function theory and operator theory (see

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the monographs [6] and [29] for more about the subject). In this sense, one of the most interesting features of composition operators is their connection with the well-known problem of Hilbert space theory: *The Invariant Subspace Problem.* This problem, which remains open in the context of infinite-dimensional separable Hilbert spaces, consists of determining whether or not every bounded linear operator T on a Hilbert space $\mathcal{H}$ has a proper, closed (nontrivial) invariant subspace. For a good source of references and results on invariant subspaces we refer the reader to Chalendar and Esterle's survey [5].

In the eighties, Nordgren, Rosenthal, and Wintrobe [27] (see also [28]) proved that the *Invariant Subspace Problem* is equivalent to the fact that every minimal non-zero invariant subspace for a composition operator induced by a hyperbolic automorphism of $\mathbb{D}$ is one dimensional. Recall that an automorphism φ of $\mathbb{D}$ can be expressed in the form

$$\varphi(z) = e^{i\theta}\,\frac{p-z}{1-\overline{p}z} \qquad \big(z \in \mathbb{D}\big),$$

where $p \in \mathbb{D}$ and $-\pi < \theta \le \pi$. Recall that φ is called *hyperbolic* if $|p| > \cos(\theta/2)$ (thus, φ fixes two points on $\partial\mathbb{D}$), *parabolic* if $|p| = \cos(\theta/2)$ (so, φ fixes just one point, located on $\partial\mathbb{D}$) and *elliptic* if $|p| < \cos(\theta/2)$ (therefore, φ fixes two points, one in $\mathbb{D}$, see [29, Chapter 0], for example).

Nordgren, Rosenthal, and Wintrobe's result may be restated as follows: if C_φ is a composition operator induced by a hyperbolic automorphism, any linear bounded operator T on a Hilbert space $\mathcal{H}$ has a closed (nontrivial) invariant subspace if and only if for any Hardy function f that is not an eigenvector of C_φ, there exists a nonzero function g in $\overline{\operatorname{span}}\{f, C_\varphi f, C_\varphi^2 f, \dots\}$, such that

$$\overline{\operatorname{span}}\big\{g, C_\varphi g, C_\varphi^2 g, \dots\big\} \neq \overline{\operatorname{span}}\big\{f, C_\varphi f, C_\varphi^2 f, \dots\big\}.$$

When viewed from this perspective, the study of the orbit of Hardy functions is seen to be of great value in the study of invariant subspaces. In addition, it is clear that the study of invariant subspaces leads to the natural concept of cyclicity: a linear bounded operator T on a Hilbert space $\mathcal{H}$ is said to be cyclic if there is a vector $f \in \mathcal{H}$ (called *a cyclic vector for* T) such that the linear span generated by its orbit, $\operatorname{span}\{T^n f\}_{n\ge 0}$, is dense in $\mathcal{H}$. Here T^0 denotes the identity operator I.

In this work, we focus on the study of the C_φ-orbits of Hardy functions whenever φ is an automorphism of the unit disc $\mathbb{D}$. We are primarily interested in understanding when Blaschke products are cyclic for these operators. To this end, we observe that if φ_n is the n-th iterate of the map φ, that is,

$$\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi \qquad (n \text{ times}),$$

then $C_\varphi^n = C_{\varphi_n}$ for any $n \geq 0$, where φ_0 denotes the identity function. Therefore, the behavior of the sequence of iterates $\{\varphi_n\}$ will play an important role in determining the localization of the zeroes of those Blaschke products that are cyclic.

In fact, as a preliminary result we show in Section 3 that if B is a finite Blaschke product (that is, B has a finite number of zeroes), then B is not cyclic for any composition operator induced by a nonelliptic automorphism, though such operators are always cyclic operators on $\mathcal{H}^2$ (see [4] and [12], for instance). This reveals a difference with respect to other linear fractional composition operators that are cyclic for which the iterates $\{\varphi_n\}$ tend to the boundary uniformly on compacta. For instance, $B(z) = z$ is a cyclic vector for C_φ on $\mathcal{H}^2$ whenever φ is a hyperbolic nonautomorphism (see [4] and [12], for instance).

If φ is an elliptic automorphism, then C_φ is cyclic whenever φ is conjugated to a rotation through an irrational multiple of π (see [4] or [12]). In this case, we prove the following stronger fact: For the automorphism $\varphi(z) = e^{i\theta} z$ a Hardy function F is a cyclic vector for C_φ in $\mathcal{H}^2$ if and only if $F^{(m)}$ does not vanish at 0 for any m. Hence, in what follows our task will be reduced to the study of those Blaschke products that are cyclic for composition operators induced by nonelliptic automorphisms.

We begin by recalling important results about interpolating Blaschke products; information that will be essential to our study of cyclic vectors. Recall that a Blaschke product B is said to be interpolating if the zero sequence of B is an interpolating sequence for the algebra $\mathcal{H}^\infty$ of bounded analytic functions on $\mathbb{D}$; that is, the zero sequence $\{z_n\}$ has the property that given any bounded sequence of complex numbers $\{w_n\}$, there exists a function $f \in \mathcal{H}^\infty$ such that $f(z_n) = w_n$ for all n. Such Blaschke products are easier to handle than general Blaschke products, and they are surprisingly flexible. In fact, the biggest open question in this area is whether or not every Blaschke product can be uniformly approximated by an interpolating Blaschke product (see [13, p. 430, Problem 5.4], and [18] for the history and most recent work on this problem). Also, it is well known that the problem reduces to understanding the behavior of finite products of interpolating Blaschke products or the so-called *Carleson-Newman Blaschke products*.

Our main result about interpolating Blaschke products appears in Section 4 and states that there exist cyclic interpolating Blaschke products for composition operators induced by nonelliptic automorphisms in $\mathcal{H}^2$. The proof is accomplished by means of a construction, which we call our *basic construction*. Note that the behavior of the iterates of a hyperbolic disc automorphism is different from the corresponding one of a parabolic automorphism, since the iterates of the former one tend to the boundary of $\mathbb{D}$

more quickly than those of the latter. In any case, given the very special nature of interpolating Blaschke products, this seems to be a surprising result.

In Section 5 we will present a new characterization of nonelliptic automorphisms of $\mathbb{D}$ in terms of Blaschke products satisfying a strong form of cyclicity. In addition, we present a new characterization of Blaschke products that are Carleson-Newman. This representation depends on a factorization of the Blaschke product B in the algebra $\mathcal{H}^\infty + \mathcal{C}$, which is the closed subalgebra of L^∞ generated by $\mathcal{H}^\infty$ and $\overline{z}$. Both proofs depend, in part, on our basic construction.

Finally, in Section 6, we use a recent result due to Dyakonov and Nicolau [10] to simplify the construction of universal Blaschke products that appeared in [15] and obtain a stronger form of the aforementioned result.

2. Preliminaries

In this section, we collect some useful facts needed throughout this paper.

2.1. Automorphisms of the unit disc

As mentioned in the introduction, the disc automorphisms, like general linear fractional maps, can be classified according to their fixed points. One of the most interesting features in this sense has to do with the *normal forms*.

If φ is parabolic, that is, φ has only one fixed point α, conjugating by the linear fractional map $T(z) = \frac{1}{z-\alpha}$ it follows that $\varphi = T^{-1}\psi T$, where

$$\psi(z) = z + \tau \qquad (\tau \neq 0).$$

On the other hand, if φ has two distinct fixed points α and β, conjugating by

$$Tz = \frac{z-\alpha}{z-\beta},$$

we deduce that $\varphi = T^{-1}\psi T$, with $\psi(z) = \mu z$. In this case, φ is called elliptic if $|\mu| = 1$ and hyperbolic if $\mu > 0$. Note that no other value of μ is allowed because φ is an automorphism. In all of the preceding cases, the map ψ is called the *normal form*.

Note that if φ is a hyperbolic automorphism of $\mathbb{D}$, both fixed points of φ are located on $\partial\mathbb{D}$, while an elliptic automorphism φ has one of the fixed points in $\mathbb{D}$. Parabolic automorphisms have their unique fixed point on $\partial\mathbb{D}$.

Using normal forms, it can be checked easily that the iterates of any nonelliptic automorphism $\{\varphi_n\}$ converge uniformly on compact subsets of $\mathbb{D}$ to one fixed point, which is called *attractive fixed point*. We refer to Ahlfors' book [1] for more details.

Blaschke products

Given a sequence of (not necessarily distinct) points $\{z_k\}$ in $\mathbb{D}\backslash\{0\}$ satisfying the *Blaschke condition*

$$\sum_{k=1}^{\infty}(1-|z_k|)<\infty,$$

the infinite product

$$B(z)=\prod_{k=1}^{\infty}\frac{|z_k|}{z_k}\frac{z_k-z}{1-\overline{z_k}z},$$

converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function B with the following properties:

i) B vanishes precisely at the points $\{z_k\}$, with the corresponding multiplicities (that is, $\{z_k\}$ is the zero sequence of B).

ii) $|B(z)|<1$ for every $z\in\mathbb{D}$.

iii) $|B(e^{i\theta})|=1$ almost everywhere on $\partial\mathbb{D}$.

The holomorphic function B is called the *Blaschke product with zero sequence* $\{z_k\}$. A general expression for a Blaschke product is given by

$$e^{i\theta}\,z^N\prod_{k=1}^{\infty}\frac{|z_k|}{z_k}\frac{z_k-z}{1-\overline{z_k}z}$$

where $N\geq 0$ is an integer. Recall that B is said to be normalized if $B(0)>0$. For more properties about Blaschke products, we refer to Garnett's book [13].

The algebra $\mathcal{H}^\infty+\mathcal{C}$

Recall that the closed subalgebra of L^∞ generated by $\mathcal{H}^\infty$ and the conjugate $\overline{z}$ of z is the smallest subalgebra of L^∞ containing the algebra $\mathcal{H}^\infty$ properly. This algebra will be denoted by $\mathcal{H}^\infty+\mathcal{C}$. It has the very important property that its maximal ideal space, $M(\mathcal{H}^\infty+\mathcal{C})$, is the same as the maximal ideal space of $\mathcal{H}^\infty$, $M(\mathcal{H}^\infty)$, minus the open unit disc. Thus, $M(\mathcal{H}^\infty+\mathcal{C})$ is the part of $M(\mathcal{H}^\infty)$ that is not well understood. In this subsection, we recall briefly some of the main results on division in this algebra that will be useful in Section 4. The first result can be found in Hoffman's classic paper [22] and [11].

Lemma 2.1. *A Blaschke product B is not a finite product of interpolating Blaschke products if and only if there exists a sequence $\{z_n\}$ in $\mathbb{D}$ such that $\left\{B\circ\left(\frac{z+z_n}{1+\overline{z_n}z}\right)\right\}$ tends to zero uniformly on compacta as $n\to\infty$.*

Division and multiplication in $\mathcal{H}^\infty + \mathcal{C}$ is well understood ([2], [19], and [20]). The two primary results that we will need here can be found in [2] and [20].

Lemma 2.2. *Let $h \in \mathcal{H}^\infty + \mathcal{C}$ and let b be interpolating. If $h(z_n) \to 0$ on the zero sequence of b, then $h/b = h\bar{b} \in \mathcal{H}^\infty + \mathcal{C}$.*

Lemma 2.3. *Let $h \in \mathcal{H}^\infty + \mathcal{C}$ and let u be an inner function. If*

$$\lim_{|z|\to 1} \big|h(z)\big|\big(1 - |u(z)|\big) = 0,$$

then for each n there is a function $c_n \in \mathcal{H}^\infty + \mathcal{C}$ such that $h = u^n c_n$.

Thus, u^n divides h in $\mathcal{H}^\infty + \mathcal{C}$ for each positive integer n. We note that the Poisson kernel is asymptotically multiplicative on $\mathcal{H}^\infty + \mathcal{C}$ $\big([8, \text{p. } 169]\big)$ in the following sense: if we have two functions u and v in $\mathcal{H}^\infty + \mathcal{C}$ defined on the unit circle and write u and v again for the Poisson extension to $\mathbb{D}$, then given $\epsilon > 0$, there exists $r > 0$ such that $0 < r < 1$ and

$$\big|u(z)v(z) - (uv)(z)\big| < \epsilon$$

for $|z| > r$. Therefore, saying $h = u^n c_n$ implies that $|h(z) - u^n(z)c_n(z)| \to 0$ as $|z| \to 1$. Thus, u^n acts like a divisor of h, asymptotically. We use this frequently in what follows.

3. Cyclic Blaschke products

We begin this section by noting that in the study of cyclic Blaschke products it suffices to study composition operators induced by concrete disc automorphisms. In fact, if ψ is any hyperbolic automorphism of $\mathbb{D}$, it is not hard to see that ψ can be conjugated under a disc automorphism T to a hyperbolic automorphism φ that fixes 1 and -1; that is,

$$\varphi(z) = \frac{z + r}{rz + 1} \tag{3.1}$$

with $0 < r < 1$. Since $\psi = T \circ \varphi \circ T^{-1}$, the operators C_ψ and C_φ are similar operators. The same reasoning applies to parabolic automorphisms, and we deduce that it is enough to consider those that fix the point 1; that is,

$$\varphi(z) = \frac{(2 - a)z + a}{-az + 2 + a} \tag{3.2}$$

with a a nonzero complex number such that the real part satisfies $\Re a = 0$. Finally, if ψ is an elliptic automorphism, ψ is conjugated to a rotation $\varphi(z) = e^{i\theta} z$.

We remark that while any parabolic or hyperbolic automorphism induces a cyclic composition operator on $\mathcal{H}^2$, elliptic automorphisms induce cyclic composition operators on $\mathcal{H}^2$ if and only if they can be conjugated to a rotation $e^{i\theta}z$ where θ is not a rational multiple of π (see [4], for instance).

The next result follows from the observation that if ψ and φ are conjugated disc automorphisms, $\psi = T \circ \varphi \circ T^{-1}$, then $C_{T^{-1}} = C_T^{-1}$ is invertible on $\mathcal{H}^2$ and, therefore, has dense range.

Proposition 3.1. *Let ψ and φ be disc automorphisms conjugated under T. A Hardy function f is cyclic for C_ψ if and only if $C_T f$ is cyclic for C_φ.*

Note that in the particular case that f is a Blaschke product, the function $C_T f$ is also a Blaschke product, since T is an automorphism of the unit disc.

Our next result is concerned with cyclic composition operators induced by elliptic disc automorphisms.

Theorem 3.2. *Let φ be an elliptic automorphism conjugated to a rotation through an irrational multiple of π. Let T conjugate φ to $\hat{\varphi}(z) = e^{i\theta}z$. A Hardy function F is a cyclic vector for C_φ in $\mathcal{H}^2$ if and only if $(F \circ T)^{(m)}$ does not vanish at 0 for any m.*

Proof. We begin by noting that $F \circ T$ is cyclic for $C_{\hat{\varphi}}$ and if F vanishes at the fixed point p of φ in $\mathbb{D}$, then every function $f \in \overline{\operatorname{span}}\{F \circ T \circ \hat{\varphi}_n : n \geq 0\}$ vanishes at 0. The same is true for $(F \circ T)^{(m)}(0)$. In this case, then, it is easy to see that F cannot be cyclic.

We turn now to the other direction. By Proposition 3.1, it is enough to prove the result when $\varphi(z) = e^{i\theta}z$, where θ is an irrational multiple of π. In this case, let F be a Hardy function such that $F^{(m)}$ does not vanish at 0. Without loss of generality we may assume that $\|F\|_2 \leq 1$.

Claim: *If $f \in \mathcal{H}^2$ is orthogonal to $\{C_{\varphi_n}F : n \geq 0\}$, then $f(0) = 0$.*

Assume, for the moment, that the claim has been established. Let $f \in \mathcal{H}^2$ satisfy

$$\langle f,\, F \circ \varphi_n \rangle = 0 \tag{3.3}$$

for every n. We will show that f is the zero function, and therefore it will follow that F is a cyclic vector for C_φ.

Using the claim, we may write $f(z) = zg(z)$. Note that $g \in \mathcal{H}^2$ and

$$0 = \langle zg,\, F \circ \varphi_n \rangle = \langle g,\, M_z^\star C_{\varphi_n} F \rangle, \tag{3.4}$$

where $M_z^\star$ denotes the adjoint of the multiplication operator M_z on $\mathcal{H}^2$. Since

$$M_z^\star h(z) = \frac{h(z) - h(0)}{z} \qquad (z \in \mathbb{D}),$$

for every $h \in \mathcal{H}^2$, a quick computation shows that

$$M_z^\star C_{\varphi_n} F = e^{in\theta} C_{\varphi_n} M_z^\star F$$

for every $n \geq 0$. Hence, using equation (3.4), we see that

$$\left\langle g,\, C_{\varphi_n} M_z^\star F \right\rangle = 0$$

for every $n \geq 0$. Thus, g is orthogonal to the orbit of the Hardy function $M_z^\star F$ under C_φ. Because of the claim and since $(M_z^\star F)^{(m)}(0) \neq 0$, we deduce that $g(0) = 0$.

Continuing in this fashion and taking into account that for all $k \geq 1$, we have

$$M_z^{\star k} C_{\varphi_n} F = e^{ink\theta} C_{\varphi_n} M_z^{\star k} F,$$

for every $n \geq 0$, we deduce that f is the zero function. We proceed to the proof of the claim.

Proof of the claim. Assume that $f \in \mathcal{H}^2$ satisfies

$$\left\langle f,\, F \circ \varphi_n \right\rangle = 0 \tag{3.5}$$

for every n. Since $F \not\equiv 0$ on $\mathbb{D}$, we may fix $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|F(z_0)| = R_0 \neq 0$. Let us fix $\varepsilon > 0$. Since the reproducing kernel, $K_{z_0}(z) = 1/(1 - \overline{z_0} z)$, is cyclic for C_φ (see [4]), there exist positive integers $n_1, \ldots, n_N$ and complex numbers $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ (both sequences depending on z_0) such that

$$\left\| \sum_{j=1}^{N} \lambda_j K_{z_0} \circ \varphi_{n_j} - f \right\|_2 < \varepsilon. \tag{3.6}$$

From (3.5), (3.6) and the fact that $\|F\|_2 \leq 1$, we deduce

$$\left| \left\langle \sum_{j=1}^{N} \lambda_j K_{z_0} \circ \varphi_{n_j},\, F \circ \varphi_n \right\rangle \right| < \varepsilon \tag{3.7}$$

for every n. Equivalently, we obtain that for every n

$$\left| \left\langle C_{\varphi_n}^\star \Big(\sum_{j=1}^{N} \lambda_j K_{z_0} \circ \varphi_{n_j} \Big),\, F \right\rangle \right| < \varepsilon, \tag{3.8}$$

where $C_{\varphi_n}^\star$ denotes the adjoint of C_{φ_n}. A computation shows that $C_{\varphi_n}^\star$ is the composition operator induced by $\varphi_{-n}(z) = e^{-in\theta} z$. Moreover, $K_{z_0} \circ \varphi_{n_j - n}$ is the reproducing kernel at $e^{i(n-n_j)\theta}\, z_0$. Hence, from (3.8) we have

$$\left| \sum_{j=1}^{N} \lambda_j F\big(e^{i(n-n_j)\theta}\, z_0\big) \right| < \varepsilon$$

for every n. Since θ is an irrational multiple of π, the set $\{e^{in\theta} z_0\}_{n\geq 0}$ is dense in the circle $\{z : z = |z_0|\}$. Therefore, for each $\lambda \in \partial\mathbb{D}$ there exists a subsequence $\{e^{in_k\theta} z_0\}_{k\geq 0}$ such that $e^{in_k\theta} z_0 \to \lambda z_0$ as $k \to \infty$. Thus,

$$\left|\sum_{j=1}^{N} \lambda_j F(0)\right| \leq \max_{|\lambda|=1}\left|\sum_{j=1}^{N} \lambda_j F(\lambda e^{-in_j\theta} z_0)\right| < \varepsilon.$$

By our choice of F, it follows that $\left|\sum_{j=1}^{N} \lambda_j\right| < \varepsilon/|F(0)|$. Now, since the topology induced by the $\mathcal{H}^2$-norm implies the topology of uniform convergence on compacta, we deduce from (3.6) that

$$|f(0)| \leq \left|\sum_{j=1}^{N} \lambda_j - f(0)\right| + \left|\sum_{j=1}^{N} \lambda_j\right| < \left(1 + \frac{1}{|F(0)|}\right)\varepsilon.$$

Since ε is arbitrary, it follows that $f(0) = 0$. This proves the claim, and therefore, Theorem 3.2. ∎

As an immediate corollary we deduce

Corollary 3.3. *Let φ be an elliptic automorphism conjugated to a rotation through an irrational multiple of π. Let p be the fixed point of φ in $\mathbb{D}$. Then every Blaschke product B such that $(B \circ T)^m$ does not vanish at 0 is a cyclic vector for C_φ in $\mathcal{H}^2$.*

Now we turn to nonelliptic automorphic composition operators.

Proposition 3.4. *Let φ be a nonelliptic disc automorphism. Then no finite Blaschke product is a cyclic vector for C_φ on $\mathcal{H}^2$.*

In the proof below, we denote by $\rho(z, w) = \left|\frac{w-z}{1-\overline{w}z}\right|$ the pseudohyperbolic distance between two points w, z in $\mathbb{D}$.

Proof. Assume that B is a finite Blaschke product with zero sequence $\{z_k\}_{k=1}^{N}$. Note that for each $n \geq 0$, $B \circ \varphi_n$ is also a finite Blaschke product with zero sequence $\{\varphi_{-n}(z_k)\}_{k=1}^{N}$.

We claim that the sequence $\{\varphi_{-n}(z_k)\}_{\{1\leq k\leq N, n\geq 0\}}$ is a Blaschke sequence. In fact, the key observation is that for any point $p \in \mathbb{D}$, $\{\varphi_{-n}(p)\}_{n\geq 0}$ is a Blaschke sequence. To check this, fix $p \in \mathbb{D}$. Since the pseudohyperbolic distance ρ is invariant under disc automorphisms, we deduce

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(1 - |\varphi_{-n}(p)|^2\right) &= \sum_{n=0}^{\infty} 1 - \rho(\varphi_{-n}(p), 0)^2 = \sum_{n=0}^{\infty} 1 - \rho(p, \varphi_n(0))^2 \\
&= \sum_{n=0}^{\infty} \frac{(1 - |p|^2)(1 - |\varphi_n(0)|^2)}{|1 - \overline{p}\,\varphi_n(0)|^2} \\
&\leq \frac{1 + |p|}{1 - |p|} \sum_{n=0}^{\infty} (1 - |\varphi_n(0)|^2). \qquad (3.9)
\end{aligned}$$

Since φ is a nonelliptic automorphism, it is not hard to check (using (3.1) or (3.2)) that $\{\varphi_n(0)\}$ is a Blaschke sequence, and therefore the series in (3.9) converges. Since k ranges over a finite set, $\{\varphi_{-n}(z_k)\}_{\{1\leq k\leq N,\, n\geq 0\}}$ is a Blaschke sequence, establishing our claim.

From this we may conclude that the linear span generated by the C_φ-orbit of B is not dense as follows: Consider the Blaschke product F that vanishes at $\{0\}\cup\{\varphi_{-n}(z_k)\}_{\{1\leq k\leq N,\, n\geq 0\}}$. It is orthogonal to $C_{\varphi_n}B$ for every $n\geq 0$, and therefore to its linear span. ∎

A closer look at the proof of Proposition 3.4 yields the following necessary condition for an infinite Blaschke product to be cyclic.

Proposition 3.5. *Let φ be a nonelliptic disc automorphism. Let B be a cyclic Blaschke product for C_φ with zero sequence $\{z_k\}$. Then there exists a subsequence $\{z_{k_j}\}$ tending to the φ-attractive fixed point as $j\to\infty$.*

Proof. Without loss of generality, we may assume that 1 is the fixed point of φ. Suppose to the contrary that there exists a positive constant $\alpha>0$ such that $|1-z_k|>\alpha$ for every $k\geq 1$. If we show that $\{\varphi_{-n}(z_k)\}_{k\geq 1, n\geq 0}$ is a Blaschke sequence, we may construct a function F as in Proposition 3.4 that is orthogonal to $C_{\varphi_n}B$ for every $n\geq 0$. This would contradict the fact that B is a cyclic vector and, thereby establish our claim.

Now,

$$\sum_{n,\,k\geq 0} 1-\left|\varphi_{-n}(z_k)\right|^2=\sum_{n,\,k\geq 0}\frac{(1-|z_k|^2)(1-|\varphi_n(0)|^2)}{|1-\overline{z_k}\,\varphi_n(0)|^2}.$$

Note that for any n and k we have

$$\left|1-\overline{z_k}\,\varphi_n(0)\right|\geq\left|1-z_k\right|-\left|z_k\right|\,\left|1-\varphi_n(0)\right|.$$

From here, taking into account that $\{\varphi_n(0)\}$ tends to the φ-attractive fixed point 1, we deduce that the series above converges. Therefore, the sequence $\{\varphi_{-n}(z_k)\}_{\{k\geq 1,\, n\geq 0\}}$ is Blaschke. ∎

Note that Proposition 3.5 implies, in particular, that the behavior of a Blaschke product with zeroes far from $\{\varphi_n(0)\}$ is irrelevant. In other words, if B is cyclic and we multiply B by a Blaschke product C that is continuous at the cluster point of the sequence $\{\varphi_n(0)\}$, then $C\circ\varphi_n$ will converge to the unimodular constant $C(1)$ uniformly on compacta and the function CB will behave just like the function B. Thus, any attempt to characterize cyclic Blaschke products must focus on the behavior of subproducts of the Blaschke product. This observation will play an important role in the next two sections.

4. Cyclic interpolating Blaschke products

In this section, we exhibit interpolating Blaschke products that are cyclic for any composition operator induced by nonelliptic automorphisms. Because of Proposition 3.5, we have to exhibit a Blaschke product B so that a subsequence of the sequence of its zeroes tend to the attractive fixed point of φ.

Given a point $z_n \in \mathbb{D}$ we let α_n denote the automorphism on $\mathbb{D}$ given by $\alpha_n(z) = (z + z_n)/(1 + \overline{z_n}z)$. In order to state the main result of this section, we introduce the following definition.

Definition 4.1. Let B be a Blaschke product. We say that b is an *approximate subfactor* of B if b divides B in $\mathcal{H}^\infty + \mathcal{C}$.

The key to this section and Section 5 is the following result.

The Basic Construction. Let B be a Blaschke product that is not Carleson-Newman and $\{c_n\}$ a sequence of finite Blaschke products. Then there exist a sequence $\{z_n\}$ of points in $\mathbb{D}$, a sequence of pseudohyperbolic discs $D(n,m) := D(z_{n,m}, r_{n,m})$ and an approximate subfactor b of B with zeroes $\{w_{n,m}\}$ such that

i) $b = \prod_{n,m} b_{n,m}$ where $b_{n,m} \circ \alpha_{n,m} = \lambda_{n,m} c_n$ for some unimodular constant $\lambda_{n,m}$ and $\prod_{(j,k)\neq(n,m)} \left|b_{j,k} \circ \alpha_{n,m}\right|$ tends to 1 uniformly on compacta as $\max(n,m) \to \infty$.

ii) $w_{n,m} \in D(n,m)$;

iii) There exists a unimodular constant λ_n such that (some subsequence of) $b \circ \alpha_{n,m} \to \lambda_n c_n$ as $m \to \infty$;

iv) $\lim_{|z|\to 1} |B(z)|(1 - |b(z)|) = 0$.

Before proving the result, we recall that the pseudohyperbolic disc $D(z_0,r)$ is the inverse image of $D(0,r)$ under $\tau(z) = \frac{z-z_0}{1-\overline{z_0}z}$. Thus, since $\alpha_n(z) = \frac{z+z_n}{1+\overline{z_n}z}$, it follows that $D(z_n, r) = \alpha_n(D(0,r))$. Recall ([13, p. 3]) that the pseudohyperbolic disc $D(z_0, r)$ is a Euclidean disc with center

$$c = \frac{1 - r^2}{1 - r^2|z_0|^2}\, z_0$$

and radius

$$R = r\frac{1 - |z_0|^2}{1 - r^2|z_0|^2}.$$

In particular, given a sequence $\{z_n\}$ of points tending to 1, and an increasing sequence of positive real numbers $\{r_j\}$ we can choose subsequences of $\{z_n\}$ and $\{r_j\}$ converging to 1 so that the pseudohyperbolic discs are disjoint.

We now recall the following lemma that will be useful in the rest of this paper (see, for instance, [14]).

Lemma 4.1. *Let $\{u_n\}$ be a sequence of inner functions. If there exists a constant γ of modulus 1 so that $u_n(0) \to \gamma$, then $u_n \to \gamma$ uniformly on compact subsets of $\mathbb{D}$.*

We include the proof of Lemma 4.1 for the sake of completeness.

Proof of Lemma 4.1. Note that every subsequence of $\{u_n\}$ satisfies the hypotheses. A normal families argument shows that each subsequence of $\{u_n\}$ has a subsequence that converges uniformly on compact subsets to a bounded analytic function f with sup norm at most one. The Maximum Modulus Theorem and the fact that $u_n(0) \to \gamma$ imply that f is the constant function γ. Since each subsequence has a subsequence converging, in the topology of uniform convergence on compacta, to the constant function γ, it follows that u_n converges to γ as well. This proves the statement of the lemma. ∎

Proof of the basic construction. Let $\{c_n\}$ be a sequence of finite Blaschke products and let m_n denote the number of zeroes of c_n. Since B is not a finite product of interpolating Blaschke products, Lemma 2.1 implies that there exists a sequence $\{z_n\}$ such that $B \circ \alpha_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Now, we will choose sequences $\{r_{n,m}\}$ and $\{r'_{n,m}\}$ such that

$$\min\{\prod_{n,m} r'_{n,m}, \prod_{n,m} r_{n,m}\} > \delta > 0$$

and a doubly-indexed sequence $\{\delta_{(j,k),(n,m)}\}$ such that

$$\prod_{(j,k)\neq(n,m)} (1 - \delta_{(j,k),(n,m)}) \to 1,$$

as $\max(n, m) \to \infty$.

We will choose a subsequence of pseudohyperbolic discs that will become the discs $D(n, m)$, with centers $z_{n,m}$ and radii $r_{n,m}$ such that

$$\alpha_{n,m}(D(0, r_{n,m})) = D(z_{n,m}, r_{n,m}).$$

We note that by choosing our $z_{n,m}$ very close to the boundary, we may choose our $r_{n,m}$ as close to 1 as we wish.

Stage 1. Choose $D(1, 1)$ so close to the boundary that the normalized Blaschke product $b_{1,1} = \lambda_{1,1} c_1 \circ \alpha_{1,1}{}^{-1}$, where $\lambda_{1,1} \in \partial\mathbb{D}$, has the property that its zeroes satisfy

$$1 - |z_{1,1,n}| < 1/(2^{1+1} m_1),$$

for $n = 1, \ldots, m_1$. We may increase $r_{1,1}$, if necessary, in order to assume that the zeroes of $b_{1,1}$ are in $D(1, 1)$ and $|b_{1,1}|_{D(1,1)^c} > r'_{1,1}$.

Stage 2. Choose $D(1,2)$ and $\alpha_{1,2}$ so that

- $D(1,1) \cap D(1,2) = \emptyset$;
- $|b_{1,1}|_{D(1,2)} > \max(r_{1,2}, 1 - \delta_{(1,1),(1,2)})$;
- $\|B\|_{D(1,2)} = \sup_{\{z \in D(0,r_{1,2})\}} |B \circ \alpha_{1,2}(z)| < 1/2^{1+2}$;
- The Blaschke product $b_{1,2} = \lambda_{1,2} c_1 \circ \alpha_{1,2}{}^{-1}$ for some $\lambda_{1,2} \in \partial\mathbb{D}$, and the zeroes, $\{z_{1,2,n}\}_{n=1}^{m_1}$, of $b_{1,2}$ satisfy

$$1 - |z_{1,2,n}| < 1/(2^{1+2} m_1)$$

  for $n = 1, \ldots, m_1$;
- $|b_{1,2}|_{D(1,1)} > \max(r_{1,2}, 1 - \delta_{(1,2),(1,1)})$ and $|b_{1,2}|_{D(1,2)^c} > r'_{1,2}$.

(We note that the second bulleted statement above can be satisfied because $b_{1,1}$ is a finite Blaschke product and $D(n,m)$ tends to the boundary as $\max(n,m) \to \infty$. The third bullet can be satisfied because $|\alpha_n(0)| \to 1$ as $n \to \infty$ and $B \circ \alpha_n \to 0$ uniformly on compacta.)

Now choose $D(2,1)$ disjoint from the previous discs so that

- $|b_{1,j}|_{D(2,1)} > \max(r_{2,1}, 1 - \delta_{(1,j),(2,1)})$ for $j = 1, 2$;
- $\|B\|_{D(2,1)} = \sup_{\{z \in D(0,r_{2,1})\}} |B \circ \alpha_{2,1}(z)| < 1/2^{2+1}$;
- The Blaschke product $b_{2,1} = \lambda_{2,1} c_2 \circ \alpha_{2,1}{}^{-1}$ for some $\lambda_{2,1} \in \partial\mathbb{D}$, where the zeroes $\{z_{2,1,n}\}$ of $b_{2,1}$ satisfy

$$1 - |z_{2,1,n}| < 1/(2^{2+1} m_2);$$

- $|b_{2,1}|_{D(1,1) \cup D(1,2)} > \max(r_{1,1}, r_{1,2}, 1 - \delta_{(2,1),(1,1)}, 1 - \delta_{(2,1),(1,2)})$.

Again, we may increase $r_{1,2}$ if necessary, so that we may assume that the zeroes of $b_{1,2}$ are in $D(1,2)$ and $|b_{2,1}|_{D(2,1)^c} > r'_{2,1}$.

General construction. Having constructed the sequence up to this point, we choose our discs $D(j,k)$, radii $\{r_{j,k}\}$, sequences $\{1 - \delta_{(j,k),(n,m)}\}$ and Blaschke factors $b_{n,m}$ as above. Enumerating our discs diagonally (in order $(1,1), (1,2), (2,1), (1,3)$, and so on), we choose the next Blaschke product $b_{j,k}$ and disc $D(j,k)$ in our sequence so that $D(j,k)$ is disjoint from the previous discs and

a) $|b_{n,m}|_{D(j,k)} > \max(r_{(n,m)}, 1 - \delta_{(n,m),(j,k)})$ for all prior Blaschke products (if (n,m) comes before (j,k));

b) $\|B\|_{D(j,k)} = \sup_{\{z \in D(0,r_{j,k})\}} |B \circ \alpha_{j,k}(z)| < 1/2^{j+k}$;

c) The Blaschke product $b_{j,k} = \lambda_{j,k} c_j \circ {\alpha_{j,k}}^{-1}$ for some $\lambda_{j,k} \in \partial\mathbb{D}$, and its zeroes satisfy
$$1 - |z_{j,k,n}| < 1/(2^{j+k} m_j)$$
for $n = 1, \ldots, m_j$;

d) $b_{n,m}$ satisfies $|b_{n,m}|_{D(n,m)^c} > r'_{n,m}$ and
$$|b_{n,m}|_{D(j,k)} > \max(r_{j,k}, 1 - \delta_{(n,m),(j,k)}),$$
on all discs $D(j,k)$ prior to $D(n,m)$ (if (j,k) comes before (n,m)).

By condition (c) above, we know that $\{z_{j,k,n}\}_{j,k,n}$ forms a Blaschke sequence. Now form the normalized Blaschke product b with these zeroes.

Now we check facts $(i) - (iv)$ of the basic construction.

First we check (i). Note that by (a) and (d) above, $|b_{r,s}|_{D(j,k)} > 1 - \delta_{(r,s),(j,k)}$ on $D(j,k)$ if $(r,s) \neq (j,k)$. Now consider $b \circ \alpha_{p,q}$. Because the convergence is uniform on compacta, as well as absolute, we may rearrange terms so that
$$\left|b \circ \alpha_{p,q}\right| = \prod_{n,m} \left|b_{n,m} \circ \alpha_{p,q}\right| = \left|(b_{p,q} \circ \alpha_{p,q})\right| \prod_{(n,m)\neq(p,q)} \left|(b_{n,m} \circ \alpha_{p,q})\right|.$$

Now
$$\prod_{(n,m)\neq(p,q)} \left|b_{n,m} \circ \alpha_{p,q}(0)\right| \geq \prod_{(n,m)\neq(p,q)} \inf_{D(p,q)} \left|b_{n,m}\right|_{D(p,q)} > \prod \left(1 - \delta_{(n,m),(p,q)}\right) \to 1,$$
as $\max{(p,q)} \to \infty$.

Since these functions are bounded by 1 and the Blaschke products are normalized, it follows that
$$\prod_{(n,m)\neq(p,q)} \left|(b_{n,m} \circ \alpha_{p,q})\right| \to 1$$
uniformly on compacta as $\max(p,q) \to \infty$. Since $b_{p,q} \circ \alpha_{p,q} = \lambda_{p,q} c_p$, we have established (i).

Our construction guarantees that (ii) occurs.

Now $\{\lambda_{p,q}\}$ is bounded, so we may choose a subsequence $\{p, q_n\}$ that is in turn a subsequence of $\{p-1, q_n\}$ and such that
$$\lambda_{p,q_n} \to \lambda_p \in \partial\mathbb{D}$$
as $n \to \infty$. Thus, as $n \to \infty$, there exists $\lambda'_p \in \partial\mathbb{D}$ with
$$b \circ \alpha_{p,q_n} \to \lambda'_p c_p$$
uniformly on compacta, establishing (iii).

To see that (iv) holds, let $\varepsilon > 0$ be fixed. Note that if $z \notin \cup_{j,k} D(j,k)$ then $|b_{j,k}(z)| \geq r'_{j,k}$. In addition,

$$\left|b(z)\right| = \prod_{\substack{1\leq j\leq N,\\ 1\leq k\leq N}} \left|b_{j,k}(z)\right| \prod_{\max\{j,k\}\geq N} \left|b_{j,k}(z)\right| \geq \prod_{\substack{1\leq j\leq N,\\ 1\leq k\leq N}} \left|b_{j,k}(z)\right| \prod_{\max\{j,k\}\geq N} r'_{j,k}.$$

By our choice of $r'_{j,k}$, we know that $\prod_{\max\{j,k\}\geq N} r'_{j,k} \to 1$ as $N \to \infty$. Thus, we may choose N sufficiently large so that $\prod_{\max\{j,k\}\geq N} r'_{j,k} \geq \sqrt{1-\varepsilon}$.

Now the finite product $\prod_{1\leq j\leq N,\, 1\leq k\leq N} |b_{j,k}(z)| \to 1$ as $|z| \to 1$, so there exists $\delta_1 > 0$ such that $|z| > \delta_1$ implies $|b(z)| > 1 - \epsilon$ if $z \notin \cup_{n,k} D(n,k)$. Because of the construction, we have that $\|B\|_{D(j,k)} < 1/2^{j+k}$. Hence, if $z \in \cup_{j,k} D(j,k)$, then $|B(z)| < 1/2^{j+k}$. As $|z| \to 1$, we see that $\max(j,k) \to \infty$. Thus, there exists $\delta_2 > 0$ such that $|B(z)| < \epsilon$ if $|z| > \delta_2$. Let $\delta = \max(\delta_1, \delta_2)$. Then $|z| > \delta$ implies

$$|B(z)|(1 - |b(z)|) < \varepsilon,$$

which establishes (iv). ∎

We turn to showing that there exist cyclic interpolating Blaschke products for composition operators induced by nonelliptic automorphisms.

Note that once we have one cyclic interpolating Blaschke B for C_φ we have, at least, a denumerable set: if B is a cyclic interpolating Blaschke product for C_φ, then $C_\varphi B$ is also an interpolating Blaschke product. In fact, the set

$$\operatorname{span}\{C_{\varphi_n} B :\ n \geq 1\}$$

is the image of the dense set $\operatorname{span}\{C_{\varphi_n} B :\ n \geq 0\}$ under the operator C_φ. Since C_φ has dense range, $\operatorname{span}\{C_{\varphi_n} B : n \geq 1\}$ is also a dense set. This means that $C_\varphi B$ is also cyclic.

Proposition 4.2. *Let φ be a nonelliptic disc automorphism. There exists a cyclic interpolating Blaschke B product for C_φ. Moreover, $C_{\varphi_n} B$ is also a cyclic interpolating Blaschke product for every $n \geq 1$.*

Remark 4.3. Note that not all the interpolating Blaschke products with a subsequence of zeroes tending to the attractive fixed point of a nonelliptic disc automorphism φ are cyclic for C_φ. In particular, if φ is a hyperbolic automorphism of $\mathbb{D}$, the Blaschke product with zeros at $\{\varphi_n(0)\}_{n\in\mathbb{Z}}$ is an interpolating Blaschke satisfying the required condition but it is no longer a cyclic vector, since it is an eigenfunction of C_φ (for the characterization of Blaschke products that are eigenfunctions of composition operators induced by hyperbolic automorphisms, we refer to [24] and [25]).

Proof. If φ is a nonelliptic automorphism, it is clear that the iterates φ_n may be expressed by

$$\varphi_n(z) = \frac{z + z_n}{1 + \overline{z_n} z}$$

where $z_n = \varphi_n(0)$ tend to the φ-attractive fixed point on $\partial\mathbb{D}$ (note that from (3.1) and (3.2) we know that $z_n = \frac{1-\mu^n}{1+\mu^n}$ where $\mu = \frac{1-r}{1+r}$ if φ is a hyperbolic automorphism and $z_n = \frac{na}{2+na}$ if φ is parabolic).

For each n, let $a_n = 1 - 1/n$, and choose the sequence (c_n) so that $c_1 = 1$ and let $c_n(z) = (z - a_n)/(1 - \overline{a_n} z)$. By the basic construction, there exists a Blaschke product $B = \prod b_{n,m}$ with properties $(i) - (iv)$ above. We must show that B is interpolating. To see this, note that B has precisely one zero in $D(n, m)$ for each n and m. So

$$\prod_{(j,k)\neq(n,m)} \rho(w_{j,k}, w_{n,m}) = \prod_{(j,k)\neq(n,m)} \left|b_{n,m}(w_{j,k})\right| \geq \prod r_{n,m} > \delta > 0,$$

by property (a) in the basic construction above. Thus B is interpolating.

Now we claim that B is cyclic.

First, note that by our choice of c_1 there is a subsequence of $\{B \circ \varphi_{n,k}\}$ converging to a constant function λ. Further, by the basic construction, there exists $\lambda = \lambda(k) \in \partial\mathbb{D}$ such that $B \circ \varphi_{n,k}$ converges to $\lambda(z - a_n)/(1 - \overline{a_n} z)$ on compacta as $k \to \infty$. Since a sequence in $\mathcal{H}^2$ converges weakly to a function g if and only if it it is bounded in $\mathcal{H}^2$ and converges uniformly on compact sets (see, for example, [30, p. 189]), we conclude that

$$\left\langle B \circ \varphi_{n,k}, \lambda \frac{(z - a_n)}{(1 - \overline{a_n} z)} \right\rangle \to 1 \qquad \text{as } k \to \infty.$$

Thus

$$\left\| B \circ \varphi_{n,k} - \lambda \frac{z - a_n}{1 - \overline{a_n} z} \right\|_2 \to 0 \qquad \text{as } k \to \infty.$$

Therefore, the closure of $\operatorname{span}\{B \circ \varphi_{n,k}\}$ contains $(z - a_n)/(1 - \overline{a_n} z)$ for each n and the constants. It follows that the reproducing kernel at a_n in $\mathcal{H}^2$, that is,

$$K_{a_n}(z) = \frac{1}{1 - \overline{a_n} z}$$

also belongs to the closure of $\operatorname{span}\{B \circ \varphi_{n,k}\}$. Since $\{K_{a_n} : n \geq 1\}$ is a spanning set in $\mathcal{H}^2$, it follows that B is cyclic. ■

5. Strong forms of cyclicity and Carleson–Newman Blaschke products

In this section, we characterize nonelliptic automorphisms of the unit disc as well as those Blaschke products that are finite products of interpolating Blaschke products.

To this end, recall that given a bounded linear operator T on a Banach space $\mathcal{B}$, a function f is said to be *supercyclic* if its *projective orbit* $\{\lambda T^n f : \lambda \in \mathbb{C},\ n \geq 0\}$ is dense in $\mathcal{B}$. If the projective orbit of f is dense in the weak topology of $\mathcal{B}$, f is said to be *weakly supercyclic*. Note that, by Mazur's Theorem (see [7]), every weakly supercyclic vector is cyclic.

Theorem 5.1. *Let $\{\varphi_n\}_{n\geq 0}$ denote the iterates of an automorphism of $\mathbb{D}$. The following are equivalent:*

1. *φ is a nonelliptic automorphism;*
2. *There exists a Blaschke product B with zeroes at a subsequence of $\{\varphi_n(0)\}$ (with zeroes repeated) such that B is not a finite product of interpolating Blaschke products.*
3. *There exists a Blaschke product b such that b is weakly supercyclic for C_φ.*

Proof. First suppose that φ is nonelliptic. Then $|\varphi_n(0)| \to 1$ as $n \to \infty$. For each k choose n_k so that $1 - |\varphi_{n_k}(0)| < 1/k^3$. Thus $\sum k(1 - |\varphi_{n_k}(0)|) < \infty$. Let B be the Blaschke product with zeroes $\{\varphi_{n_k}(0)\}$ and the zero $\varphi_{n_k}(0)$ repeated k times. Writing $z_k = \varphi_{n_k}(0)$, we note that if $c < 1$ and $|z_0| < c$, then

$$\left|B \circ \left(\frac{z_0 + z_k}{1 + \overline{z_k} z_0}\right)\right| \leq \left|\frac{(z_0 + z_k)/(1 + \overline{z_k} z_0) - z_k}{1 - \overline{z_k}(z_0 + z_k)/(1 + \overline{z_k} z_0)}\right|^k = |z_0|^k \leq c^k.$$

Since $c < 1$, we see that a subsequence of $B \circ (z + z_k)/(1 + \overline{z_k} z)$ tends to zero uniformly on compacta as $k \to \infty$. By Lemma 2.1, we see that B is not a finite product of interpolating Blaschke products.

Now suppose that B exists. Let $\{c_n\}$ be a sequence of finite Blaschke products so that $\{c_n\}$ is dense in the ball of $\mathcal{H}^\infty$ in the topology of local uniform convergence.

Property (iii) of the basic construction ensures that there exists a Blaschke product b such that that $b \circ \varphi_{n,m}$ converges locally uniformly to $\lambda_n c_n$ as $m \to \infty$, for some $\lambda_n \in \partial\mathbb{D}$. Since pointwise bounded convergence coincides with weak convergence in $\mathcal{H}^2$, the result follows.

Now we show that the existence of a weakly supercyclic Blaschke product for C_φ implies that φ is nonelliptic. Our proof is based on an argument that appears in [12, Theorem 5.2].

Suppose that φ is an elliptic automorphism and let $p \in \mathbb{D}$ denote the fixed point of φ. Since b is assumed to be a weakly supercyclic Blaschke product, we see that $b(p) \neq 0$. Since $\varphi \in \mathcal{H}^2$ and b is weakly supercyclic, there exists a net such that

$$\lambda_\alpha(b \circ \varphi_\alpha) \to \varphi.$$

In particular,

$$\lambda_\alpha b(p) = \lambda_\alpha(b \circ \varphi_\alpha)(p) \to \varphi(p).$$

Thus, $\lambda_\alpha \to p/b(p)$ and

$$b \circ \varphi_\alpha \to (p/b(p))\varphi.$$

By extracting a subnet, if necessary, we may assume that there is an elliptic automorphism ψ such that

$$\varphi_\alpha \to \psi.$$

Then $b \circ \psi = (p/b(p))\varphi$. So b must be a nonconstant multiple of an automorphism. By Hurwitz's theorem [1, p. 178] we see that b cannot be weakly supercyclic. This contradiction completes the proof of this theorem. ■

We present our first characterization of Carleson-Newman Blaschke products.

Theorem 5.2. *A Blaschke product B is not a finite product of interpolating Blaschke products if and only if there is a sequence $\{\alpha_n\} = \{(z + z_n)/(1 + \overline{z_n}z)\}$ of automorphisms with $|\alpha_n(0)| \to 1$ and a Blaschke product b such that b divides B in $\mathcal{H}^\infty + \mathcal{C}$ and $\{\lambda(b \circ \alpha_n) : n \geq 0, \lambda \in \mathbb{C}\}$ is dense in the topology of local uniform convergence.*

Proof. First suppose that B is not a finite product of interpolating Blaschke products. By the basic construction, there exists a Blaschke product b with properties $(i) - (iv)$. By Lemma 2.3 and property (iv), B is divisible by b in $\mathcal{H}^\infty + \mathcal{C}$. The density follows from property (iii).

For the other direction, note that $B = bu$ for some $u \in \mathcal{H}^\infty + \mathcal{C}$. Now, $b \circ \alpha_{n_k} \to 0$ uniformly on compacta for some subsequence $\{\alpha_{n_k}\}$. Thus

$$B \circ \alpha_{n_k}(z) = (bu) \circ \alpha_{n_k}(z).$$

Since $|\alpha_n(0)| \to 1$, passing to a subsequence if necessary, we may apply Lemma 4.1 and assume that there exists $\lambda \in \partial\mathbb{D}$ such that $\alpha_{n_k} \to \lambda$ uniformly on compact subsets of $\mathbb{D}$. So $|\alpha_{n_k}(z)| \to 1$ as well. Using the asymptotic multiplicative property of the Poisson kernel, we conclude that $B \circ \alpha_{n_k} \to 0$ pointwise on $\mathbb{D}$, and therefore some subsequence converges to zero uniformly on compacta. By Lemma 2.1, B is not a finite product of interpolating Blaschke products. ■

6. Universal Blaschke products

The study of cyclicity is related to the study of universality, a subject we now investigate in a particular situation. In the more general context, given two topological vector spaces X and Y and bounded linear maps $T_l : X \to Y$, an element $x \in X$ is universal for $\{T_l\}$ if $\{T_l x : l \in I\}$ is dense in Y. We will study universal Blaschke products; that is, a Blaschke product b is said to be universal for the ball of $\mathcal{H}^\infty$ with respect to a sequence $\{\alpha_n\}$ if $\{b\circ\alpha_n : n \in \mathbb{N}\}$ is dense in the ball of $\mathcal{H}^\infty$ in the local uniform topology. Universality, like cyclicity, has received a lot of attention and the interested reader is referred to [3], [15], [21], and [26] for results specific to universal Blaschke products or inner functions, as well as the survey works [16] and [17].

To obtain a result on universal Blaschke products, we apply a theorem of Dyakonov and Nicolau [10]. For the reader's convenience, we quote the results we need below. Recall that an interpolating sequence is said to be thin if

$$\lim_{k\to\infty} \prod_{n\neq k} \rho(z_n, z_k) = 1.$$

Theorem 6.1 (Theorem 1.5, [10]). *Given a thin sequence $\{z_n\}$, there is a sequence $\{m_j\}$ with $m_j \to 1$ such that for every sequence $\{w_j\}$ satisfying $|w_j| \le m_j$ there is a solution $F \in \mathcal{H}^\infty$ such that $\|F\|_\infty \le 1$ and $F(z_j) = w_j$ for all j.*

Dyakonov and Nicolau note that the function that is actually constructed is, in fact, a Blaschke product with thin zero sequence, and there exists a sequence $\{\tau_j\}$ with $0 < \tau_j < 1$ and $\tau_j \to 1$ such that the zeroes of the Blaschke product, denoted $\{\zeta_j\}$, can be chosen so that $\rho(\zeta_j, z_j) \le \tau_j$ for all j. They note that this can be verified in the same way as Earl's proof ([11], [13, p. 309], [10, Lemma 2.3]).

We will also need the following lemma, known as Hoffman's lemma (see [22] and [13, p. 404]).

Lemma 6.1 (Hoffman's Lemma). *Let B be an interpolating Blaschke product with zeroes $\{z_n\}$ satisfying*

$$\inf_n (1 - |z_n|^2)|B'(z_n)| \ge \delta > 0.$$

Then there exist $\lambda > 0$ and $r > 0$ such that $2\lambda/(1 + \lambda^2) < \delta$ and the set $\{z : |B(z)| < r\}$ is the union of pairwise disjoint domains V_n with $z_n \in V_n$ and

$$V_n \subset \{z : \rho(z, z_n) < \lambda\}.$$

We remark that if $\{z : \rho(z, z_k) \leq \lambda\} \cap \{z : \rho(z, z_m) \leq \lambda\} \neq \emptyset$, then for some $z \in \mathbb{D}$ we have [13, p. 4]

$$\rho(z_m, z_k) \leq \frac{\rho(z_m, z) + \rho(z, z_k)}{1 + \rho(z_m, z)\rho(z, z_k)} \leq \frac{2\lambda}{1 + \lambda^2} < \delta.$$

But

$$\prod_{n \neq k} \rho(z_n, z_k) = (1 - |z_k|^2)|B'(z_k)| \geq \delta,$$

a contradiction.

Theorem 6.2. *A Blaschke product B is not a finite product of interpolating Blaschke products if and only if there is a Blaschke product D and a sequence of automorphisms $\{\alpha_n\}$, where $\alpha_n(z) = (z + z_n)/(1 + \overline{z_n} z)$ and $|\alpha_n(0)| \to 1$, such that D divides B in $\mathcal{H}^\infty + \mathcal{C}$ and D is universal for the ball of $\mathcal{H}^\infty$ with respect to $\{\alpha_n\}$.*

Proof. First suppose that there is a Blaschke product D and a sequence of automorphisms $\{\alpha_n\}$ with $|\alpha_n(0)| \to 1$ such that D is universal for the ball of $\mathcal{H}^\infty$ with respect to $\{\alpha_n\}$ and $B = Du$ for some $u \in \mathcal{H}^\infty + \mathcal{C}$. Since D is universal, there exists a subsequence of α_n (also denoted α_n) such that $D \circ \alpha_n \to 0$ locally uniformly on compact subsets of $\mathbb{D}$. Now $|\alpha_n(0)| \to 1$, so (passing to a subsequence, if necessary) there is a unimodular constant λ such that $\alpha_n \to \lambda$ uniformly on compact subsets of $\mathbb{D}$. Since the Poisson kernel is asymptotically multiplicative on $\mathcal{H}^\infty + \mathcal{C}$, we see that $B = Du$ implies that $B \circ \alpha_n \to 0$ uniformly on compacta. By Lemma 2.1, we know that B is not a finite product of interpolating Blaschke products, completing the proof in this direction.

Now let $\{d_n\}$ denote a sequence of finite Blaschke products that is dense in $\mathcal{H}^\infty$ in the local uniform topology.

Choose a thin sequence $\{z_n\} = \{z_{j,k}\}$ converging to a point of $\partial\mathbb{D}$ so that $B \circ ((z + z_{j,k})/(z + \overline{z_{j,k}} z))$ tends to zero uniformly on compact sets as $\max(j, k) \to \infty$. Let $\{\tau_{j,k}\}$ be the sequences we obtain from Theorem 6.1 and the comments following it. We use the basic construction to choose (successively) sequences $D(j, k)$, $(r_{j,k})$, and $(1 - \delta_{(j,k),(l,m)})$ so that

1. $|B(z)| < 1/(j + k)$ on $D(j, k)$ for all j, k.
2. $\prod_{j,k} r_{j,k} > 0$;
3. $r_{j,k} > \tau_{j,k}$;
4. $\prod_{j,k} r_{j,k} \to 1$ as $k \to \infty$, for fixed j;
5. The Blaschke product d with zeroes $\{w_{j,k}\}$ has a subsequence such that $d \circ \alpha_{j,k} \to \lambda_j d_j$, where $|\lambda_j| = 1$ and $k \to \infty$;
6. $\lim_{|z| \to 1} |B(z)|(1 - |d(z)|) = 0$.

Using Theorem 6.1 and noting that $m_{j,k} \to 1$, we may choose $w'_{j,k} \to \overline{\lambda_j}$ as $k \to \infty$ and $|w'_{j,k}| \le m_{j,k}$. Let b_1 denote the Blaschke product we obtain from Theorem 6.1 with zeroes $\{\zeta_{j,k}\}$ satisfying $\rho(\zeta_{j,k}, z_{j,k}) < \tau_{j,k} < r_{j,k}$ and $b_1(z_{j,k}) = w'_{j,k}$ for all j and k.

Now consider the Blaschke product $D_1 = b_1 d$. Let α_n denote the automorphism given by $\alpha_n(z) = (z + z_n)/(1 + \overline{z_n} z)$ (where $\{z_n\} = \{z_{j,k}\}$ is chosen so that $B \circ \alpha_n \to 0$). Then $(b_1 \circ \alpha_{j,k})(0) = b_1(z_{j,k}) \to \overline{\lambda_j}$ as $k \to \infty$ uniformly on compacta. Since $|\lambda_j| = 1$, $b_1 \circ \alpha_{j,k}$ tends to $\overline{\lambda_j}$ uniformly on compacta. Thus

$$D_1 \circ \alpha_{j,k} = (b_1 \circ \alpha_{j,k})(d \circ \alpha_{j,k}) \to \overline{\lambda_j}(\lambda_j d_j) = d_j$$

as $k \to \infty$. Since d_j is dense in the ball of $\mathcal{H}^\infty$, we see that D_1 is a universal Blaschke product for $\{\alpha_n\}$.

In addition, we have established that b_1 is interpolating, $b_1(z_{j,k}) = w_{j,k}$, and since the zeroes $\{\zeta_{j,k}\}$ of b_1 are in $\overline{D(j,k)}$, we see that $|B(\zeta_{j,k})| \le 1/(k + j)$. By Lemma 2.2, we know that $B\overline{b_1} \in \mathcal{H}^\infty + \mathcal{C}$. We claim that

$$\lim_{|z| \to 1} |B\overline{b_1}(z)|(1 - |d(z)|) = 0.$$

Choose a sequence $\{v_n\}$ in $\mathbb{D}$ with $|v_n| \to 1$ and a positive number α such that $\sup |d(v_n)| \le \alpha < 1$. We claim that $(B\overline{b_1})(v_n) \to 0$.

By Schwarz's lemma, there exist $\delta_1 > 0$ and $r_1 > \alpha$ such that $|d(v)| \le r_1$ if $v \in D(v_n, \delta_1)$. Let $\gamma_n(z) := (z + \zeta_n)/(1 + \overline{\zeta_n} z)$. Passing to a subsequence, we may suppose that there exists $\lambda \in \partial\mathbb{D}$ such that $v_n \to \lambda$ and the normal family $B \circ \gamma_n$ converges uniformly on compacta to a bounded analytic function. Since $B\overline{b_1} \in \mathcal{H}^\infty + \mathcal{C}$, there exist $h \in \mathcal{H}^\infty$ and $c \in \mathcal{C}$ such that

$$B\overline{b_1} = h + c.$$

Since c is continuous, we know that $\{c(v_n)\}$ converges to $c(\lambda)$. Adding and subtracting a constant, if necessary, we may assume that $c(\lambda) = 0$.

Suppose first that $b_1(v_n) \to 0$. Then, by [22], there exists a subsequence of ζ_n (which we denote by ζ_n again) such that $\rho(v_n, \zeta_n) \to 0$. Since b_1 is interpolating, Hoffman's lemma implies that there exist δ and r such that the set $\{z : |b_1(z)| < r\}$ is the union of pairwise disjoint domains V_n with $\zeta_n \in V_n$ and

$$V_n \subset \{z : \rho(z, \zeta_n) < \delta\}.$$

Now $B = (B\overline{b_1})b_1$ and the Poisson kernel is asymptotically multiplicative. Since $\sup |d(v_n)| \le \alpha < 1$ and $\rho(\zeta_n, v_n) \to 0$, there exists $\beta < 1$ such that

$$\sup_{|z| = \delta} |d \circ \gamma_n(z)| \le \beta$$

for n sufficiently large. By property (iv), $B\circ\gamma_n$ tends to zero on $\{z : |z| = \delta\}$. Since $b_1 \circ \gamma_n$ is bounded away from zero, we see that $\{(B\overline{b_1}) \circ \alpha_n\}$ tends to zero on $\{z : |z| = \delta\}$. In other words,

$$(h + c) \circ \gamma_n = (B\overline{b_1}) \circ \gamma_n \to 0$$

pointwise on $\{z : |z| = \delta\}$. Now c is continuous and we assume that $c(v_n) \to 0$, so $c(w_n) \to 0$ whenever $w_n \to \lambda$. Thus $h\circ\gamma_n \to 0$ on $\{z : |z| = \delta\}$ and, by the maximum modulus theorem, this also holds on $\{z : |z| \leq \delta\}$. Thus $B\overline{b_1} \circ \gamma_n \to 0$ uniformly on $\{z : |z| \leq \delta\}$. But $\rho(\zeta_n, v_n) \to 0$, so for n sufficiently large $\rho(\zeta_n, v_n) < \delta$. Thus $B\overline{b_1}(v_n) \to 0$, as desired.

If $\{b_1(v_n)\}$ does not converge to zero, then we can find a subsequence of v_n (which we denote again by v_n) on which b_1 is bounded away from zero. But

$$B(v_n) = (B\overline{b_1}b_1)(v_n).$$

Since $B\overline{b_1} \in \mathcal{H}^\infty + \mathcal{C}$ and $|v_n| \to 1$, we use the fact that the Poisson kernel is multiplicative to conclude that $(B\overline{b_1})(v_n)b_1(v_n) \to 0$. Consequently $(B\overline{b_1})(v_n) \to 0$. Since every subsequence of $\{B\overline{b_1}(v_n)\}$ has a subsequence that converges to 0, we may conclude that $B\overline{b_1}(v_n) \to 0$. Now we may apply Lemma 2.3 to conclude that $B\overline{b_1 d} \in \mathcal{H}^\infty + \mathcal{C}$, completing the proof. ■

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Eva A. Gallardo-Gutiérrez
Departamento de Matemáticas
Universidad de Zaragoza e IUMA
Plaza San Francisco s/n
50009 Zaragoza, Spain
eva@unizar.es

Pamela Gorkin
Department of Mathematics
Bucknell University
PA 17837, Lewisburg, USA
pgorkin@bucknell.edu

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